Framed Torsion-Free Sheaves on $\mathbb{C}P^2$, Hilbert Schemes, and Representations of Infinite Dimensional Lie Algebras

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1 Introduction

Denote by $\mathfrak{M}(r,n)$ be the moduli space of rank r torsion free sheaves on $\mathbb{CP}^2$, framed at the $\mathbb{CP}^1$ at ∞, with second Chern class equal to n. $\mathfrak{M}(r,n)$ is a partial compactification of the moduli space of $U(r)$ instantons on $\mathbb{R}^4$ with instanton number n. An r dimensional torus $T = (\mathbb{C}^*)^r$ acts on $\mathfrak{M}(r,n)$, and the T-equivariant cohomology of $\mathfrak{M}(r,n)$ has a natural subspace, denoted by $H_T^{mid}(\mathfrak{M}(r,n))$, of dimension equal to the Euler characteristic $\chi(M(r,n))$. The generating function

$$\sum_{q=0}^{\infty} \chi(\mathfrak{M}(r,n))q^n = \sum_{q=0}^{\infty} dim\big(H_T^{mid}(\mathfrak{M}(r,n))\big)q^n = \prod_{i=1}^{r}\prod_{m=1}^{\infty} \frac{1}{(1-q^m)}$$

is the graded dimension of the r-colored bosonic Fock space representation of an r-dimensional Heisenberg algebra. For each $\vec{l} \in \mathbb{Z}^r$, there is twist the action of T on $\mathfrak{M}(r,n)$, and we denote the space $\mathfrak{M}(r,n)$ equipped with the $\vec{l}$-twisted T action by $\mathfrak{M}_{\vec{l}}(r,n)$. Considering a generating function for all $\mathfrak{M}_{\vec{l}}(r,n)$ together yields the graded dimension of the r-colored fermionic Fock space representation of an r-dimesional Clifford algebra. In this paper we give a geometric realization of these Heisenberg and Clifford representations on the equivariant cohomology of the moduli space of framed, rank r torsion-free sheaves. When $r = 1$, the moduli space $\mathfrak{M}(1,n)$ is isomorphic to the Hilbert scheme ${\mathbb{C}^2}^{[n]}$of n points on $\mathbb{C}^2$; thus, our construction is a natural generalization of the original constructions of Nakajima [Nak99] and Grojnowski [Gro96], as modified by Vasserot [Vas01], of one-dimensional Heisenberg algebra actions on the cohomology of the Hilbert schemes ${\mathbb{C}^2}^{[n]}$.

Denote by X_r the resolution of the simple singularity $\mathbb{C}^2/\mathbb{Z}_r$. Parallel to our constructions on $\mathfrak{M}(r,n)$, we give a different construction of the same r-colored bosonic and fermionic Fock spaces using the equivariant cohomology of the Hilbert scheme $X_r^{[n]}$ of n points on X_r. For the Heisenberg algebra, this construction has also been considered by Qin-Wang, [QW], and is a second natural generalization of the original Nakajima/Grojnowski construction on the Hilbert scheme of points on $\mathbb{C}^2$ (which coincides with X_r when $r = 1$.) Thus there are two natural generalizations of the same construction- in the first, one replaces the surface $\mathbb{C}^2$ by the surface X_r, and in the second one replaces rank one torsion-free sheaves by rank r torsion-free sheaves.

Our representations are constructed by exhibiting explicit correspondences inside products of T-stable subvarieties of the spaces $\mathfrak{M}(r,n)$, respectively $\mathbb{C}^*$-stable subvarieties of $X_r^{[n]}$, and using equivariant localization to prove that our correspondences satisfy the defining relations of Heisenberg and Clifford algebras. Representations of Heisenberg and Clifford algebras are very closely related; in fact, starting from repesentations of a Clifford algebra, one can construct representations of a Heisenberg algebra, and vice-versa. The translation between the language of bosonic and fermionic operators, which was initially discovered by physicists, is known in the mathematics literature as the "boson-fermion correspondence." Our constructions provide a geometric interpretation of this correspondence. This extends results of Savage [Sav], who relates the Heisenberg algebra action on the cohomology of the Hilbert Schemes ${\mathbb{C}^2}^{[n]}$ to a geometric realization of level one representations of the Lie algebra $sl(\infty)$.

The connection between the representation theory of affine Lie algebras and instanton geometry was discovered by H. Nakajima in the remarkable work [Nak94], [Nak98]. In this work, Nakajima constructed representations of infinite dimensional Lie algebras on the homology of quiver varieties, which are generalizations of $U(r)$-instanton moduli spaces. In particular, Nakajima constructs level r representations of an affine Lie algebra $\widehat{g}$ on the homology of moduli spaces of $U(r)$ instantons on $\widetilde{\mathbb{C}^2/\Gamma}$, where $\Gamma \subset SL(2, \mathbb{C})$ is a finite subgroup and Γ and $\widehat{g}$ are related by the McKay correspondence. In his construction the finite subgroup Γ determines the Lie algebra being represented and and the gauge group $U(r)$ determines the level of the representation.

It seems equally natural, however, to expect algebraic objects associated to G (rather than to the finite subgroup Γ) to be related to the topology of moduli spaces of G-instantons on $\widetilde{\mathbb{C}^2/\Gamma}$. When $\Gamma = \mathbb{Z}_k$ is a cyclic group, a conjecture of I. Frenkel says that the homology of the moduli space of G-instantons on $\widetilde{\mathbb{C}^2/\mathbb{Z}_k}$ should carry level k representations of the affine Lie algebra $\widehat{g}$ associated to G. This second construction of affine Lie algebra representations is different from the Nakajima construction, in that the gauge group G determines the algebra being represented, and the finite subgroup $\mathbb{Z}_k$ determines the level of the representation.

To summarize, fix a simply-laced complex affine Lie algebra $\widehat{g}$, corresponding to both a compact Lie group G and a finite subgroup Γ. There should be two different constructions of representations of the Lie algebra $\widehat{g}$ on the homology of instanton moduli spaces:

(a) $\widehat{g}$ acts on the homology of moduli spaces of $U(k)$-instantons on $\widetilde{\mathbb{C}^2/\Gamma}$. The level of the representation is determined by the gauge group $U(k)$; and

(b) $\widehat{g}$ acts on the homology of moduli spaces of G-instantons on $\widetilde{\mathbb{C}^2/\mathbb{Z}_k}$. The level of the representation is determined by the finite subgroup $\mathbb{Z}_k$.

The construction (a) is contained in [Nak94], [Nak98]; when $k = 1$ it is also a special case of a construction of Baranovksy [Bar00]. The constructions of Heisenberg and Clifford modules in this paper essentially give the construction (b) when $G = U(r)$ is of type A and $k = 1$. Algebraically, this passage to representations of the affine Lie algebra $\widehat{gl(r)}$ from Fock space representations of Heisenberg/Clifford algebras uses vertex operators, [FK81] , [Seg81],[KKLW81], and it is an interesting problem to interpret all of the vertex operators used in this passage geometrically. We should also emphasize that when $G \neq U(r)$, the construction (b) is still conjectural. If $G \neq U(r)$ and $\Gamma \neq \mathbb{Z}_k$, it is an open problem to determine what sort of exotic representations can be realized using the corresponding instanton moduli spaces.

When both G and Γ are of type A, the above moduli spaces all have descriptions as Nakajima quiver varieties, and we expect the constructions (a) and (b) together to give a geometric interpretation of level-rank duality in the representation theory of $\widehat{gl(r)}$, first studied algebraically by Frenkel in [Fre82].

The existence of two geometric constructions of the same representation, one using the varieties $\mathfrak{M}(r,n)$ and the other using the varieties $X_r^{[n]}$ suggests a close relationship between these two varieties. We begin to consider this relationship in the last section of the paper, where we exhibit a curious numerical duality between the cohomologies of $\mathfrak{M}(r,n)$ and $X_r^{[n]}$ coming from the decomposition theorem. This part of the paper owes its existence to discussions with N. Proudfoot.

Equivariant cohomology and localization have been used before in order to study the topology of the moduli space $\mathfrak{M}(r,n)$. Of particular note are the papers [NY05], [NY04] which contain much of the equivariant topology used in this paper. The aim of [NY05], [NY04], which is to study instanton counting, does not require a geometric realization of representations, and it would be interesting to relate instanton counting on surfaces to geometric realizations of affine Lie algebra representations. We also thank H. Nakajima for drawing our attention to the reference [Nak01], where a very similar construction of representations of affine Lie algebras is obtained using equivariant localization and the moduli spaces $\mathfrak{M}(r,n)$.

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2 The Clifford Algebra and the Heisenberg Algebra

2.1 Partitions and Symmetric Functions

Let $\lambda = (\lambda_0 \geq \ldots \geq \lambda_m > 0)$ be a partition of n, which we write using the notation $\lambda \vdash n$.
We may associate to λ its Young diagram which we view as a subset of the first quadrant of $\mathbb{Z}^2$, as in [NY05],[NY04]. Given a pair of partitions $\lambda_\alpha, \lambda_\beta$, and a point $s \in \mathbb{Z}^2$, Nakajima and Yoshioka define the *relative hook lengh* $h_{\alpha,\beta}(s)$, as a function of the arm and leg lengths of s relative to the partitions λ_α and λ_β. We refer to [NY05] for the precise definition, but note here that if $\lambda_\alpha = \lambda_\beta$ and s is a point in the Young diagram of the partition, then the relative hook length of s is equal to its ordinary hook length.

Let *Sym* be the $\mathbb{C}$-vector space of symmetric functions. Of the many important bases of this space, will have occasion to use the following [Mac95]:
the monomial symmetric functions m_λ
the power sum symmetric functions p_λ
the Schur functions s_λ
the elementary symmetric functions e_λ
the homogeneous symmetric functions h_λ.
$Sym \simeq \mathbb{C}[p_1, p_2, \ldots]$ is a polynomial algebra in the power-sum symmetric functions $\{p_n\}_{n>0}$.

2.2 The Clifford Algebra

Let Cl be Clifford Algebra generated by $\psi(k), \psi^*(k)$, $k \in \mathbb{Z}$, and a central element c, with anti-commutation relations

$$\{\psi(k), \psi(l)\} = \{\psi^*(k), \psi^*(l)\} = 0, \qquad \{\psi(k), \psi^*(l)\} = \delta_{kl} c$$

Define the spin module $\mathcal{F}$ to be the unique irreducible Clifford module which admits a vector ν_0 such that

$$c\nu_0 = \nu_0$$

$$\psi(k)\nu_0 = 0 \quad \forall k \leq 0$$

$$\psi^*(k)\nu_0 = 0 \quad \forall k > 0$$

The Spin Module $\mathcal{F}$ is also known as the Fermionic Fock space, and this space has a nice realization in terms of semi-infinite monomials. A semi-infinite monomial is an infinite expression of the form

$$i_0 \wedge i_1 \wedge i_2 \wedge \ldots$$

where $i_0 > i_1 > i_2 > \ldots$ are integers and $i_n = i_{n-1} - 1$ for $n \gg 0$.
For any semi-infinite monomial $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$, there exists $k \in \mathbb{Z}$ such that for $n \gg 0$, $i_n = -n + k$, and we will refer to this k as the charge of the semi-infinite wedge $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$. Put another way, the charge of $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$ is the integer k such that $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$ differs from $k \wedge k-1 \wedge k-2 \wedge \ldots$ at only finitely many places.
Let $\mathcal{F}(m)$ be the $\mathbb{C}$-vector space spanned by all semi-infinite monomials of charge m, and let

$$\mathcal{F} = \bigoplus_m \mathcal{F}(m)$$

The action of $\psi(k), \psi^*(k)$ on $\mathcal{F}$ is defined by wedging and contracting operators:

$$\psi(k)(i_0 \wedge i_1 \wedge \ldots) = \begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge k \wedge i_s \ldots & i_{s-1} > k > i_s \\ 0 & k = i_s \text{ for some } s \end{cases}$$

$$\psi^*(k)(i_0 \wedge i_1 \wedge \ldots) = \begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge i_{s+1} \ldots & k = i_s \\ 0 & k \neq i_s \text{ for all } s \end{cases}$$

If we define an inner product on $\mathcal{F}$ by declaring the semi-infinite monomials to be an orthonormal basis, then $\psi(k)$ and $\psi^*(k)$ are adjoint operators. Note that

$$\psi(k) : \mathcal{F}(m) \longrightarrow \mathcal{F}(m+1)$$

raises charge by one while

$$\psi^*(k) : \mathcal{F}(m) \longrightarrow \mathcal{F}(m-1)$$

lowers charge by one.
There is also an r-tuple version of the Clifford Algebra, denoted by Cl^r; it is generated by $\psi_i(k), \psi_i^*(k)$, $k \in \mathbb{Z}$, $i = 1, \ldots, r$ and a central element c, with anti-commutation relations

$$\{\psi_i(k), \psi_j(l)\} = \{\psi_i^*(k), \psi_j^*(l)\} = 0, \quad \{\psi_i(k), \psi_j^*(l)\} = \delta_{ij}\delta_{kl}c$$

We define the r-colored fermionic Fock space $\mathcal{F}^r$ by taking the tensor product of r copies of the space $\mathcal{F}$,

$$\mathcal{F}^r = \mathcal{F} \otimes \ldots \otimes \mathcal{F}$$

so that Cl^r acts naturally on $\mathcal{F}^r$.

2.3 The Heisenberg Algebra

The Heisenberg Algebra $\mathcal{H}$ is the infinite dimensional Lie algebra generated by c, $p(n)$, $n \in \mathbb{Z}$, with commutation relations

$$[p(n), p(m)] = n\delta_{n+m,0}c$$

$$[p(n), c] = 0$$

Let $\mathcal{B}(k)$ be the unique irreducible $\mathcal{H}$ module which admits a vector ν_0 such that

$$c\nu_0 = \nu_0$$

$$p(n)\nu_0 = 0 \quad \forall n < 0$$

$$p(0)\nu_0 = k\nu_0$$

and let $\mathcal{B} = \bigoplus_{k\in\mathbb{Z}} \mathcal{B}(k)$.
$\mathcal{B}$ is known as the Bosonic Fock space, and this space has a nice realization in terms of symmetric functions.

Let $\mathcal{B}_{alg} = \mathbb{C}[q, q^{-1}, p_1, p_2, \ldots]$, where p_i are the power-sum symmetric functions, and let $\mathcal{B}_{alg}(k) = q^k\mathbb{C}[p_1, p_2, \ldots] = q^k Sym$. Define the action of $\mathcal{H}$ on the space on $\mathcal{B}_{alg}(k)$ by

$$cf = f$$

$$p(n)f = p_n f \quad \forall n > 0$$

$$p(n)f = \frac{\partial}{\partial p_n} f \quad \forall n < 0$$

$$p(0)f = kf$$

Putting together the actions for different k, we have an action of $\mathcal{H}$ on $\mathcal{B}_{alg}$. We define an inner product on $\mathcal{B}_{alg}$ by declaring the elements $q^m s_\lambda$ to be an orthonormal basis. With this inner product, the operators $p(n)$ and $p(-n)$ are adjoints.
There is also an r-colored version of the Heisenberg algebra, denoted by $\mathcal{H}^r$. This is the Lie algebra generated by c, $p_i(n)$, $n \in \mathbb{Z}$, $i = 1, \ldots, r$with commutation relations

$$[p_i(n), p_j(m)] = n\delta_{n+m,0}\delta_{i,j}c$$

$$[p_i(n), c] = 0$$

If we take r copies of $\mathcal{B}_{alg}$ and set $\mathcal{B}^r_{alg} = \mathcal{B}_{alg} \otimes \ldots \otimes \mathcal{B}_{alg}$, then we have a natural action of $\mathcal{H}^r$ on $\mathcal{B}^r_{alg}$. We will refer to $\mathcal{B}^r_{alg}$ as the r-colored bosonic Fock space.

3 The Boson-Fermion Correspondence

We may associate a partition $\lambda = (\lambda_0 \geq \ldots \geq \lambda_k)$ to a semi-infinite monomial $i_0 \wedge i_1 \wedge \ldots$ of charge m by setting

$$\lambda_j = i_j - m + j$$

The correspondence

$$i_0 \wedge i_1 \wedge \ldots \leftrightarrow \lambda$$

allows us to define an isometric vector space isomorphism

$$\phi : \mathcal{F} \longrightarrow \mathcal{B}_{alg}$$

$$\phi(i_0 \wedge i_1 \wedge \ldots) = q^m s_\lambda$$

where $i_0 \wedge i_1 \wedge \ldots$ is a semi-infinite monomial of charge m.
We use this isomorphism to define an action of $\mathcal{H}$ on $\mathcal{F}$, and an action of Cl on $\mathcal{B}_{alg}$. More explicitly, we define the operators $h(k)$, $e(k)$ as homogeneous components of the generating functions

$$exp(\sum_{n=1}^{\infty} \frac{z^n}{n} p(n)) = \sum_{k=1}^{\infty} h(k) z^k$$

$$exp(\sum_{n=1}^{\infty} \frac{z^n}{n} p(-n)) = \sum_{k=1}^{\infty} h(-k) z^{-k}$$

and their inverses

$$exp(-\sum_{n=1}^{\infty} \frac{z^n}{n} p(n)) = \sum_{k=1}^{\infty} e(-k) z^{-k}$$

$$exp(-\sum_{n=1}^{\infty} \frac{z^n}{n} p(n)) = \sum_{k=1}^{\infty} e(k) z^k$$

The operators $h(k), e(k)$ are adjoint to the operators $h(-k), e(-k)$, respectively. As operators on the space of symmetric functions, $h(k)$, $k > 0$ is multiplication by the homogeneous symmetric function h_k. Similarly, $e(k)$, $k > 0$ is multiplicaton by the elementary symmetric funcion e_k [Mac95]
We also define the shift operator

$$q : \mathcal{B}_{alg}(k) \longrightarrow \mathcal{B}_{alg}(k+1)$$

$$q(f) = qf$$

Note that q and q^{-1} are adjoint operators.

Proposition 1. *[Fre85] a) As operators on $\mathcal{B}_{alg}$, the bosons can be written in terms of the fermions:*

$$p(n) = \sum_{j \in \mathbb{Z}} \psi(j+n) \psi^*(j)$$

if $n \neq 0$, and

$$p(0) = \sum_{j>0} \psi(j) \psi^*(j) - \sum_{j \leq 0} \psi^*(j) \psi(j)$$

b) As operators $\mathcal{B}(m) \longrightarrow \mathcal{B}(m \pm 1)$, the fermions can be written in terms of the bosons and the shift operator:

$$\psi(k) = \sum_{n \in \mathbb{Z}} q h(n) e(n - m + k)$$

$$\psi^*(k) = \sum_{n \in \mathbb{Z}} q^{-1} e(n) h(n + m + k)$$

There is an r-colored version of this correspondence, using the isomorphism $\mathcal{F}^r \simeq \mathcal{B}^r$, and r different shift operators $q_0, \ldots q_{r-1}$. We define the operators $h_i(k)$ to be homogeneous components of the generating function

$$exp(\sum_{n=1}^{\infty} \frac{z^n}{n} p_i(n)) = \sum_{k=1}^{\infty} h_i(k) z^k$$

and similarly for the operators $e_i(k)$. In terms of symmetric functions, the operator $h_i(k)$ is multiplication by the ith coordinate homogeneous symmetric function $1 \otimes \ldots \otimes h_k \otimes \ldots 1 \in Sym^r$, and similarly for the $e_i(k)$.

Proposition 2. *a) As operators on $\mathcal{B}^r_{alg}$, the bosons can be contructed from the fermions:*

$$p_i(n) = \sum_{k \in \mathbb{Z}} \psi_i(k+n) \psi_i^*(k)$$

if $n \neq 0$, and

$$p_i(0) = \sum_{j>0} \psi_i(k) \psi_i^*(k) - \sum_{j \leq 0} \psi_i^*(k) \psi_i(k)$$

b) As operators $\mathcal{B}^r_{alg}(\vec{m}) \longrightarrow \mathcal{B}^r(\vec{m} \pm 1_i)$, the fermions can be constructed from the bosons and the shift operators:

$$\psi_i(k) = \sum_{n \in \mathbb{Z}} q_i h_i(n) e_i(n - m_i + k)$$

$$\psi_i^*(k) = \sum_{n \in \mathbb{Z}} q_i^{-1} e_i(n) h_i(n + m_i + k)$$

4 The Affine Lie Algebra $\widehat{gl(r)}$

Let $gl(r)$ denote the Lie algebra of $r \times r$ matrices. Let $e_{i,j}$, $i, j = 0, \ldots, r-1$ denote the matrix units, which form a basis of the vector space $gl(r)$. Let $h_i = e_{ii}$ be the diagonal matrix units. The span of the $\{h_i\}$ (the Cartan subalgebra of diagonal matrices) will be denoted by $\mathfrak{h}$. If n^+ (resp. n^-) are the strictly upper triangular (resp. strictly lower triangular) matrices, then we have a triangular decomposition

$$gl(r) = n^+ \oplus \mathfrak{h} \oplus n^-$$

The $\mathbb{Z}$-lattice inside $\mathfrak{h}^*$ spanned by the dual vectors to the h_i is called the weight lattice, and will be denoted by P.
The affine Lie algebra $\widetilde{gl(r)}$ is the infinite dimensional vector space

$$\widetilde{gl(r)} = gl(r) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with Lie bracket

$$[x \otimes t^k, y \otimes t^l] = [x, y] \otimes t^{k+l} + k \delta_{k+l,0} \langle x, y \rangle c$$

where $\langle x, y \rangle = tr(xy)$ is the invariant bilinear trace form, and c is central.
Let

$$\widehat{gl(r)} = \widetilde{gl(r)} \rtimes d$$

be the semi-direct product of $\widetilde{gl(r)}$ with the derivative $d = t\frac{d}{dt}$. The subalgebra $\overline{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is the Cartan subalgebra of $\widehat{gl(r)}$. If we set

$$\overline{n}^{\pm} = n^{\pm} \oplus (gl(r) \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}])$$

then we have a triangular decomposition

$$\widehat{gl(r)} = \overline{n}^{+} \oplus \overline{\mathfrak{h}} \oplus \overline{n}^{-}.$$

Let $\widehat{P} = P \oplus \mathbb{Z}c \oplus \mathbb{Z}d$ be the weight lattice of $\widehat{gl(r)}$. The subset

$$\widehat{P}^{++} = \{a_{-1}d + a_0 h_0 + \ldots a_{r-1}h_{r-1} \mid a_{-1} \geq a_0 \geq \ldots \geq a_{r-1}\}$$

is called the set of dominant weights.

4.1 Highest Weight representations of $\widehat{gl(r)}$

For $\lambda \in \widehat{P}^{++}$, the irreducible highest weight representation $V(\lambda)$ is by definition the unique irreducible $\widehat{gl(r)}$ with a vector ν_0 satisfying

$$\overline{n}^{+}\nu_0 = 0$$

and

$$h\nu_0 = \langle \lambda, h\rangle \nu_0, \quad h \in \overline{\mathfrak{h}}.$$

The integer $m = \langle \lambda, c\rangle$ is called the *level* of the representation $V(\lambda)$.

4.2 Level One Representations of $\widehat{gl(r)}$ from the Clifford Algebra

One constructs highest weight representations of $\widehat{gl(r)}$ [FK81] , [Seg81],[KKLW81] from the fermionic or bosonic Fock space constructed above by using vertex operators to extend the action of the Clifford or Heisenberg algebra to an action of the entire affine Lie algebra. In order to construct the level one representations of $\widehat{gl(r)}$ inside the fermionic Fock space $\mathcal{F}^r$, we introduce the normal ordering

$$:\psi_i(k)\psi_j^*(l) := \begin{cases} \psi_i(k)\psi_j^*(l) & \text{if } j > 0 \\ -\psi_j^*(l)\psi_i(k) & \text{if } j \leq 0 \end{cases}$$

We can then define an action of $\widehat{gl(r)}$ on $\mathcal{F}^r$ by setting

$$e_{i,j} \otimes t^k \mapsto \sum_{n\in\mathbb{Z}} :\psi_i(n+k)\psi_j^*(n):$$

In particular, the r-dimensional Heisenberg algebra (called the "homogeneous Heisenberg subalgebra")

$$\mathcal{H}^r = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

acts on $\mathcal{F}^r$ as in the Boson-Fermion correspondence above

$$h_i \otimes t^k \mapsto \sum_{n\in\mathbb{Z}} :\psi_i(n+k)\psi_i^*(n):$$

The spaces

$$\mathcal{F}^r(m) = \sum_{m_0+\ldots+m_{r-1}=m} \mathcal{F}(m_0) \otimes \ldots \otimes \mathcal{F}(m_{r-1})$$

are all irreducible level one representations of $\widehat{gl(r)}$, and all of the irreducible level one representations of $\widehat{gl(r)}$ are realized as $\mathcal{F}^r(m)$ for some m. The representation correponding to $m = 0$ is known as the *basic* representation.

4.3 Level One Representations of $\widehat{gl(r)}$ from the Heisenberg Algebra

Alternatively, one can start from the tensor product of the irreducible Heisenberg algebra representation Sym^r and the lattice group algebra $\mathbb{C}[\mathbb{Z}^r]$, and construct the bosonic Fock space

$$\mathcal{B}^r_{alg} = Sym^r \otimes \mathbb{C}[\mathbb{Z}^r]$$

The operators given by the action of the Heisenberg algebra and translation in the lattice can be put together using vertex operators to define an action of $\widehat{gl(r)}$ on the space $\mathcal{B}^r_{alg}$, which decomposes into irreducible level one representations; this construction is known as the Frenkel-Kac construction [FK81]. The isomorphism between the constructions of $\widehat{gl(r)}$ on $\mathcal{F}^r$ and on $\mathcal{B}^r$ essentially follows from the boson-fermion correspondence, and is discussed in [Fre85].

5 Quiver Varieties

5.1 The Moduli space $\mathfrak{M}(r,n)$ of Framed Torsion-free Sheaves on $\mathbb{P}^2$

Let V be a an n-dimensional vector space, let W be an r-dimensional vector space, and define spaces

$$\mathbb{M}(r,n) = \{(A,B,i,j) \in Hom(V,V) \oplus Hom(V,V) \oplus Hom(W,V) \oplus Hom(V,W) \mid [A,B]+ij = 0, stability\}$$

$$\mathfrak{M}(r,n) = \mathbb{M}(r,n)/GL(n,\mathbb{C}).$$

Here "stability" means that we take only those quadruples (A,B,i,j) satisfying the following condition:

If $i(W) \subset V' \subset V$ for some subset V' with $A(V') \subset V'$ and $B(V') \subset V'$, then $V' = V$.

The stability condition guarantees that $GL(n,\mathbb{C})$ acts freely on $\mathbb{M}(r,n)$ [Nak94].

$\mathfrak{M}(r,n)$ is isomorphic to the moduli space of rank r torsion-free sheaves on $\mathbb{P}^2$, framed at the $\mathbb{P}^1$ at infinity, with second Chern class eqaul to n (see [Nak99], Chapter 3.)

5.2 The Moduli space ${X_r}^{[n]}$ of Torsion-free Sheaves on $\widetilde{\mathbb{C}^2/\mathbb{Z}_r}$

For a positive integer r, let $\mathbb{Z}_r \subset SL(2,\mathbb{C})$ be the cyclic subgroup of the diagonal matrices, let $\mathcal{R}_i$, $i = 1,\ldots,r$ be the irreducible represenations of $\mathbb{Z}_r$, and let Q be the two-dimensional $\mathbb{Z}_r$ module defined by the inclusion $\mathbb{Z}_r \hookrightarrow SL(2,\mathbb{C})$. We also allow $r = \infty$, in which case we set $\mathbb{Z}_\infty = \mathbb{C}^*$, embedded in $SL(2,C)$ as the diagonal matrices. A pair of endomporphisms $A, B \in Hom(V,V)$ can be considered as a point $(A,B) \in Hom(Q \otimes V, V)$. Thus, an action of $\mathbb{Z}_r$ on the vector spaces V and W induces an action on $Hom(V,W), Hom(W,V)$, and $Hom(Q \otimes V, V)$. Define varieties

$$\mathbb{M}(\vec{w},\vec{v}) = \{(A,B,i,j) \in Hom_{\mathbb{Z}_k}(Q \otimes V, V) \times Hom_{\mathbb{Z}_k}(W,V) \times Hom_{\mathbb{Z}_l}(V,W) \mid [A,B]+ij = 0, stability\}$$

$$\mathfrak{M}(\vec{w},\vec{v}) = \mathbb{M}(\vec{w},\vec{v}) / \prod_i GL(V_i)$$

Here $\vec{w} = (w_1, \dots w_r), \vec{v} = (v_1, \dots, v_r)$ are the dimension vectors of

$$W = \oplus_i W_i \otimes \mathcal{R}_i$$

and

$$V = \oplus_i V_i \otimes \mathcal{R}_i$$

into irreducbible $\mathbb{Z}_r$-modules, and $Hom_{\mathbb{Z}_r}(X_r, Y)$ denotes the $\mathbb{Z}_r$-invariant part of $Hom(X_r, Y)$.

The spaces $\mathfrak{M}(\vec{w},\vec{v})$ will be called $\widehat{A_{r-1}}$ quiver varieties (or A_∞ quiver varieties in the case $r = \infty$). In the special case that V is a power of the regular representation and W is the trivial representation, so that $\vec{v} = (n,n,\dots,n)$ and $\vec{w} = (1,0,\dots,0)$, the quiver variety $\mathfrak{M}(\vec{v},\vec{w})$ is isomorphic to the Hilbert scheme of n points on the ALE space $X_r = \widetilde{\mathbb{C}^2/\mathbb{Z}_r}$ [Nak99]. We will denote this quiver variety by ${X_r}^{[n]}$.

The $r = \infty$ case will be of particular importance to us, so for future use we record here the following lemma.

Lemma 1. *Let $W = \mathbb{C}$ be the trivial $\mathbb{C}^*$-module, so that $\vec{w} = (\dots,0,1,0,\dots)$, and suppose $dim(V) = n$. Then*

a) The non-empty A_∞ quiver varieties $\mathfrak{M}(\vec{w},\vec{v})$ are all isolated points.

b) The set of all non-empty A_∞ quiver varieties $\{\mathfrak{M}(\vec{w},\vec{v})\}$ is in natural bijection with the set of partitions $\{\lambda \vdash n\}$.

Proof. A proof of this can be found in [Nak99], or [FS03]. □

Note that if $r = 1$, both ${X_r}^{[n]}$ and $\mathfrak{M}(1,n)$ are isomorphic to the Hilbert scheme ${\mathbb{C}^2}^{[n]}$ of n points on $\mathbb{C}^2$.

6 Torus Actions on Quiver Varieties

6.1 Torus actions on $\mathfrak{M}(r,n)$

Let $T' = (\mathbb{C}^*)^r \subset GL(r,\mathbb{C})$ be the maximal torus of diagonal matrices.
Let $\vec{a} = diag(a_1,\dots,a_r) \in T'$, and for $\vec{l} = (l_1,\dots,l_r) \in \mathbb{Z}^r$ let $b_{\vec{l}} = diag(t^{l_1},\dots,t^{l_r})$. We define an action of an $r+1$-dimensional torus $T = \mathbb{C}^* \times T'$ on $\mathfrak{M}(r,n)$ via

$$(t,\vec{a})(A,B,i,j) = (tA, t^{-1}B, i\vec{a}^{-1}b_{\vec{l}}^{-1}, b_{\vec{l}}\vec{a}j)$$

We denote the space $\mathfrak{M}(r,n)$ with the above T-action, which depends on the vector $\vec{l} \in \mathbb{Z}^r$, by $M_{\vec{l}}(r,n)$. The following lemmas address the structure of the the fixed point components of $\mathfrak{M}(r,n)$ under the action of the $r+1$-dimensional torus T and the r-dimensional torus T'.

Lemma 2. *The fixed point components $\mathfrak{M}_{\vec{l}}(r,n)^{T'}$ are products of Hilbert schemes:*

$$\mathfrak{M}_{\vec{l}}(r,n)^{T'} = \coprod_{\sum n_i = n} \mathfrak{M}_{l_1}(1,n_1) \times \ldots \times \mathfrak{M}_{l_r}(1,n_r) \simeq \coprod_{\sum_i n_i = n} \mathbb{C}^{2[n_1]} \times \ldots \times \mathbb{C}^{2[n_r]}$$

Proof. A proof of this lemma can be found in [NY05]. □

The spaces $\mathfrak{M}_l(1,n)$, $l \in \mathbb{Z}$ which occur in the above products are all T-equivariantly isomorphic to the Hilbert scheme $\mathbb{C}^{2[n]}$, but we will think of spaces correponding to different l as different moduli spaces. More precisely, for each $l \in H^2_{\mathbb{C}^*}(\mathbb{C}^2,\mathbb{Z}) \simeq \mathbb{Z}$, we have a rank one vector L_l whose equivariant first Chern class is l. We will think of $\mathfrak{M}_l(1,n)$ as the moduli space of rank 1 framed torsion-free subsheaves $\mathcal{E} \subset L_l$ with $c_2(\mathcal{E}) = n$. The map

$$\otimes L_l : \mathfrak{M}_k(1,n) \longrightarrow \mathfrak{M}_{k+l}(1,n)$$

is a T-equivariant isomorphism.

In order to study the fixed points of $\mathfrak{M}_{\vec{l}}(r,n)$ under the action of the larger torus T, we first consider the case $r = 1$:

Lemma 3. *Let $\vec{w}_l$ be the vector which is 1 in the lth spot and 0 elsewhere. Then the T-fixed point set $\mathfrak{M}_l(1,n)^T$ consists of isolated points, which are naturally identified with the set of A_∞ quiver varieties:*

$$\mathfrak{M}_l(1,n)^T = \coprod_{\sum v_i = n} \mathfrak{M}(\vec{w}_l, \vec{v})$$

Proof. A proof of this lemma can be found in either [Nak99] or [Sav]. □

This implies that for all $k \in \mathbb{Z}$, the collection of A_∞ quiver varieties $\coprod_{\sum v_i = n} \mathfrak{M}(\vec{w}_k, \vec{v})$ can be naturally identified with the set of partitions $\{\lambda \vdash n\}$ of charge k.

Putting these lemmas together, we can identify the T-fixed points $\mathfrak{M}(r,n)^T$:

Lemma 4. *The T fixed points $\mathfrak{M}_{\vec{l}}(r,n)^T$ are isolated and naturally identified with the set of r-tuples of A_∞ quiver varieties,*

$$\mathfrak{M}_{\vec{l}}(r,n)^T = \coprod_{(\vec{v}_1,\ldots,\vec{v}_r) \mid \sum v_{i,j} = n} \mathfrak{M}(\vec{w}_{l_1}, \vec{v}_1) \times \ldots \times \mathfrak{M}(\vec{w}_{l_r}, \vec{v}_r),$$

or, equivalently, with the set of all r-tuples of partitions $(\lambda_1, \ldots, \lambda_r)$ whose total size is n.

Proof. This follows immediately from the previous lemmas. □

The important point to note is that for each $\vec{l} \in \mathbb{Z}^r$ and each $n \geq 0$ we have a space $\mathfrak{M}_{\vec{l}}(r,n)$, equipped with an action of the torus T, and the fixed point components of this action have been identified as r-tuples of A_∞ quiver varieties.

6.2 Torus Actions on ${X_r}^{[n]}$

There is a natural embedding [GK],[IN96]

$$X_r \hookrightarrow ({\mathbb{C}^2}^{[r]})^{\mathbb{Z}_r},$$

where the $\mathbb{Z}_r$ action on ${\mathbb{C}^2}^{[n]}$ is induced from the action of $\mathbb{Z}_r$ on $\mathbb{C}^2$. This $\mathbb{Z}_r$ action commutes with the $\mathbb{C}^*$ action on ${\mathbb{C}^2}^{[n]}$; as a result, both X_r and all of the Hilbert schemes ${X_r}^{[n]}$ inherit $\mathbb{C}^*$ actions. In order to distinguish the situation when $\mathbb{C}^*$ acts on ${X_r}^{[n]}$ from torus actions on $\mathfrak{M}(r, n)$, we will denote the torus which acts on ${X_r}^{[n]}$ by $T^\vee$.

For each $\vec{l} \in H^2_{\mathbb{C}^*}(X_r, \mathbb{Z}) \simeq \mathbb{Z}^r$, we have a line bundle $L_{\vec{l}}$ whose first equivariant Chern class is $\vec{l}$. Given $\vec{l} \in \mathbb{Z}^r$, we define ${X_r}^{[n]}_{\vec{l}}$ to be the moduli space of rank 1 torsion-free subsheaves of $L_{\vec{l}}$. We have isomorphisms

$$\otimes L_{\vec{k}} : {X_r}^{[n]}_{\vec{l}} \longrightarrow {X_r}^{[n]}_{\vec{l}+\vec{k}}.$$

Lemma 5. *The fixed points* $({X_r}^{[n]}_{\vec{l}})^{T^\vee}$ *are isolated, and in natural bijection with* r*-tuples of* A_∞ *quiver varieties with* $dim(V) = n$*, or, equivalently, with the set of all* r*-tuples of partitions* $(\lambda_1, \ldots, \lambda_r)$ *whose total size is* n*.*

Proof. The set ${X_r}^{T^\vee}$ consists of r isolated points, corresponding to the r hook partitions $(k, 1^{n-k})$ of $({\mathbb{C}^2}^{[r]})^T = \{\lambda \vdash r\}$ under the [GK],[IN96] embedding. The fixed points $x_1, \ldots, x_r$ are thus naturally ordered. There is an open cover (constructed explicitly, for example, in [QW]) of X_r by r open sets $U_1, \ldots, U_r$, such that x_i is the origin of $U_i \simeq \mathbb{C}^2$, and the action of $T^\vee$ on U_i in local coordinates is of the form $t(u, v) = (t^r u, t^{-r} v)$. Since any point $y \in ({X_r}^{[n]}_{\vec{l}})^{T^\vee}$ is supported on ${X_r}^{T^\vee}$, we have

$$y \simeq y_1 \oplus \ldots \oplus y_r$$

where $y_i \in (U_i^{[n_i]})$ and $\sum_i n_i = n$. ☐

Alternatively, another proof of this lemma can also be found in [Kuz]. This implies the following corollary, which motivates our parallel treatment of the quiver varieties ${X_r}^{[n]}_{\vec{l}}$ and $\mathfrak{M}_{\vec{l}}(r, n)$.

Corollary 1. *For any* $n \geq 0$ *and any* $\vec{l} \in \mathbb{Z}^r$*, there is a canonical identification of fixed point sets*

$$({X_r}^{[n]}_{\vec{l}})^{T^\vee} \leftrightarrow \mathfrak{M}_{\vec{l}}(r, n)^T$$

Proof. Both sets are in canonical bijection with the set of r-tuples of paritions of total size n, and with the set of all r-tuples of A_∞ quiver varieties with $dim(V) = n$. ☐

6.3 Weight Spaces at $\vec{\lambda} \in \mathfrak{M}_{\vec{l}}(r, n)^T$

Let e_α, $\alpha = 1, \ldots, r$ denote the one-dimensional T-module

$$e_\alpha : (t, e_1, \ldots, e_r) \mapsto e_\alpha.$$

Similarly, we regard t as a one-dimensional t module. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_r)$ be a T-fixed point of $\mathfrak{M}_{\vec{l}}(r, n)$. The tangent space $\mathcal{U}_{\vec{\lambda}}$ to $\mathfrak{M}_{\vec{l}}(r, n)$ at $\vec{\lambda}$ is a T-module.

Proposition 3. *The weight space decomposition of the T-module $\mathcal{U}_{\vec{\lambda}}$ is given by*

$$\mathcal{U}_{\vec{\lambda}} = \sum_{\alpha,\beta=1}^{r} N_{\vec{\lambda}}^{\alpha,\beta}$$

where

$$N_{\vec{\lambda}}^{\alpha,\beta} = e_\beta e_\alpha^{-1} t^{l_\beta - l_\alpha} \times \big(\sum_{s\in\lambda_\alpha} t^{-h_{\beta,\alpha}(s)} + \sum_{s\in\lambda_\beta} t^{h_{\alpha,\beta}(s)} \big).$$

and $h_{\alpha,\beta}(s)$ is the relative hook length of s, relative to the partitions $\lambda_\alpha, \lambda_\beta$.

Proof. This computation can be found in [NY05]. ☐

Since the fixed point components $\mathfrak{M}_{\vec{l}}(r,n)^{T'}$ are products of Hilbert schemes, the tangent space $\mathcal{U}'_{\vec{\lambda}}$ to the subvariety $\mathfrak{M}_{\vec{l}}(r,n)^{T'}$ at $\vec{\lambda}$ can be inferred from the $r = 1$ special case of the above formula:

Proposition 4. *The weight space decomposition of the T-module $\mathcal{U}'_{\vec{\lambda}}$ is given by*

$$\mathcal{U}'_{\vec{\lambda}} = \sum_{\alpha=1}^{r} N_{\vec{\lambda}}^{\alpha}$$

where

$$N_{\vec{\lambda}}^{\alpha} = \sum_{s\in\lambda_\alpha} t^{-h_\alpha(s)} + \sum_{s\in\lambda_\alpha} t^{h_\alpha(s)},$$

and $h_\alpha(s) = h_{\alpha,\alpha}(s)$ is the ordinary hook length of s in the partition λ_α.

Proof. Follows from taking $r = 1$ in the last proposition on each of the partitions $\lambda_1, \ldots, \lambda_r$ seperately. ☐

Note that $\mathcal{U}'_{\vec{\lambda}}$ picks out exactly the terms $\alpha = \beta$ from $\mathcal{U}_{\vec{\lambda}}$.

Corollary 2. *The weight space decomposition of the normal bundle $N_{\vec{\lambda}}$ to $\mathfrak{M}_{\vec{l}}(r,n)^{T'}$ in $\mathfrak{M}_{\vec{l}}(r,n)$ at $\vec{\lambda}$ is given by*

$$N_{\vec{\lambda}} = \sum_{\alpha\neq\beta} N_{\vec{\lambda}}^{\alpha,\beta}$$

Proof. This follows immediately from the previous two propositions. ☐

6.4 Weight Spaces at $\vec{\lambda} \in ({X_r}_{\vec{l}}^{[n]})^{T^\vee}$

Let $\vec{\lambda}$ be a $T^\vee$-fixed point in ${X_r}^{[n]}$. The tangent space $\mathcal{U}_{\vec{\lambda}}^\vee$ to ${X_r}^{[n]}$ at $\vec{\lambda}$ is a $T^\vee$ module, and the weight space decomposition is given by the following proposition.

Proposition 5. *The weight space decomposition of the $T^\vee$ module $\mathcal{U}_{\vec{\lambda}}^\vee$ is given by*

$$\mathcal{U}_{\vec{\lambda}}^\vee = \sum_{\alpha=1}^{r} (N^\vee)_{\vec{\lambda}}^{\alpha}$$

where

$$(N^\vee)_{\vec{\lambda}}^{\alpha} = \sum_{s\in\lambda_\alpha} t^{-rh_\alpha(s)} + \sum_{s\in\lambda_\alpha} t^{rh_\alpha(s)},$$

and $h_\alpha(s) = h_{\alpha,\alpha}(s)$ is the ordinary hook length of s in the partition λ_α.

Proof. We write the $T^\vee$ action at $x_\alpha \in {X_r}^{T^\vee}$ in terms of local coordinates- if u, v are local coordinates of the $\mathbb{C}^2$ chart who's origin is the fixed point $x_\alpha \in {X_r}^{T^\vee}$, then the action is

$$t(u, v) = (t^r u, t^{-r} v).$$

Thus, the weight space decomposition is just the same as in the bundle $\mathcal{U}'$ of the previous section, with t replaced by t^r. □

7 Equivariant Cohomology of Quiver Varieties

Let $\mathfrak{M}$ be a quiver variety of complex dimension $2m$, and suppose that $T = (\mathbb{C}^*)^k$ acts on $\mathfrak{M}$ with isolated fixed points. Let B_T be the classifying space of T, and let E_T be the universal bundle. T acts freely on the space E_T, and hence freely on the product $\mathfrak{M} \times E_T$. The equivariant cohomology of $\mathfrak{M}$ is defined to be the ordinary cohomology of the quotient space $\mathfrak{M} \times_T E_T$.

$$H_T^*(\mathfrak{M}) = H^*(\mathfrak{M} \times_T E_T)$$

We will always use complex coefficients for the equivariant cohomology of M. $H_T^*(\mathfrak{M})$ is a module over $R = H_T^*(pt)$. Let $\mathcal{R}$ denote the field of fractions of R, and let $\mathcal{H}_T^*(\mathfrak{M}) = H_T^*(\mathfrak{M}) \otimes_R \mathcal{R}$ be the localized equivariant cohomology of $\mathfrak{M}$.

All of the usual cohomological constructions carry over to the equivariant setting. In particular, if V is a T-equivariant vector bundle on $\mathfrak{M}$, we have equivariant Chern classes $c_k(V) \in H_T^{2k}(\mathfrak{M})$. If V is an n-dimensional vector bundle, the top equivariant Chern class $c_n(V)$ is called the equivariant Euler class of V, and is denoted by $e(V)$.

We endow $\mathcal{H}_T^*(\mathfrak{M})$ with an inner product given by

$$\langle , \rangle : \mathcal{H}_T^*(\mathfrak{M}) \times \mathcal{H}_T^*(\mathfrak{M}) \longrightarrow \mathcal{R}$$

$$\langle x, y \rangle = (-1)^m p_*(i_*)^{-1}(x \cup y)$$

where i is the inclusion

$$i : \mathfrak{M}^T \hookrightarrow \mathfrak{M}$$

and p is the unique map from $\mathfrak{M}^T$ to a point

$$p : \mathfrak{M}^T \longrightarrow \{pt\}$$

For a T-stable smooth subvariety $Y \subset \mathfrak{M}$, the normal bundle N_Y to Y in $\mathfrak{M}$ is a T-equivariant vector bundle. If a one-parameter subgroup $\mathbb{C}^* \hookrightarrow T$ acts trivially on Y, then this subgroup induces a splitting

$$N_Y = N_Y^+ \oplus N_Y^0 \oplus N_Y^-$$

where

$$N_Y^+ = \bigoplus_{n>0} N_Y(n)$$

is the positive weight space of $\mathbb{C}^*$,

$$N_Y^- = \bigoplus_{n<0} N_Y(n)$$

is the negative weight space of $\mathbb{C}^*$, and N_Y^0 is the zero weight space.

If $\tilde{T} \subset T$ is the subgroup of T which acts trivially on Y, then by choosing a generic one-parameter subgroup $\mathbb{C}^* \hookrightarrow \tilde{T}$ to define our splitting,we can guarantee that $N_Y^0 = 0$, so that the $\mathbb{C}^*$-fixed points of Y are isolated.

We can choose a splitting such that the equivariant Euler classes of the bundles N_Y^+, N_Y^- satisfy

$$e(N_Y^+) = (-1)^k e(N_Y^-),$$

where $k = \frac{1}{2} codim_{\mathbb{C}}(Y)$. We will describe this splitting for our quiver varieties $\mathfrak{M}_{\vec{l}}(r,n)$ and $X_{r_{\vec{l}}}^{[n]}$ in the next subsection.

7.1 Equivariant Euler classes for $\mathfrak{M}_{\vec{l}}(r,n)$ and $X_{r_{\vec{l}}}^{[n]}$

Example 1. *Let*

$$Y = \mathfrak{M}_{\vec{l}}(r,n)^{T'} = \coprod_{\sum n_i = n} \mathbb{C}^{2[n_1]} \times \ldots \times \mathbb{C}^{2[n_r]}$$

Then the Normal bundle $\mathcal{U}'$ to Y splits as a direct sum

$$\mathcal{U}' = \bigoplus_{\alpha \neq \beta} N_{\alpha,\beta}.$$

We choose the one-paramter subgroup given by $(1,1) \times (1,t,t^2,\ldots,t^{r-1}) \in T$ so that

$$\mathcal{U}'^+ = \bigoplus_{\alpha < \beta} N^{\alpha,\beta}$$

$$\mathcal{U}'^- = \bigoplus_{\alpha > \beta} N^{\alpha,\beta}$$

Then, looking at the character for the tangent bundle, we see that

$$e(\mathcal{U}'^+) = (-1)^{(r-1)n} e(\mathcal{U}'^-).$$

Example 2. *On the other hand, if we let*

$$Y = \mathfrak{M}_{\vec{l}}(r,n)^T,$$

then the fixed points are isolated, and the normal bundle at $\vec{\lambda}$ is the full tangent bundle $\mathcal{U}$. We choose the one-parameter subgroup $(t^r, t^{-r}) \times (1,t,t^2,\ldots,t^{r-1}) \in T$, which also has isolated fixed points. Then,

$$e(\mathcal{U}^+) = (-1)^{rn} e(\mathcal{U}^-).$$

Example 3. *Finally, let*

$$Y = (X_{r_{\vec{l}}}^{[n]})^{T^\vee}.$$

Since $T^\vee$ is already one-dimensional, there is no need to choose a one-parameter subgroup- the tangent bundle splits naturally, and

$$e(\mathcal{U}^{\vee +}) = (-1)^n e(\mathcal{U}^{\vee -}).$$

7.2 Localization and the Transport Map η

We return now to the general case of a smooth, T-stable subvariety $Y \subset \mathfrak{M}$. Let $i_Y : Y \longrightarrow \mathfrak{M}$ be the inclusion. For $x \in H_T^*(Y)$, define $\eta(x)$ by

$$\eta(x) = i_{Y*}(x) \cup e(N^-)^{-1} = \sum_j i_{Y_j *}(x) \cup e(N_j^-)^{-1}$$

where $\{Y_j\}$ are the connected components of Y.
A priori, the image of η lies in the localized equivariant cohomology of $\mathfrak{M}$, since we divide by the equivariant Euler class. However, the argument of [Nak], section 6 shows the following:

Lemma 6. *If $x \in H_T^*(Y)$ then $\eta(x) \in H_T^*(\mathfrak{M})$.*

Proof. Repeat the argument of [Nak] section 6. □

Thus, we have a well-defined map

$$\eta : H_T^*(Y) \longrightarrow H_T^*(\mathfrak{M}).$$

The image of η is a subspace of the equivariant cohomology of $\mathfrak{M}$ which is central to all of our constructions, so for the rest of the paper we will denote $\eta(H_T^0(\mathfrak{M}^T))$ by

$$H_T^{mid}(\mathfrak{M}) := \eta(H_T^0(\mathfrak{M}^T)).$$

Proposition 6. *The dimension of $H_T^{mid}(\mathfrak{M})$ is equal to the Euler characteristic $\chi(\mathfrak{M})$.*

Proof. The dimension of $H_T^{mid}(\mathfrak{M})$ is clearly equal to the number of fixed points $\#\{\mathfrak{M}^T\}$. This number is equal to the dimension of the total ordinary cohomology $H^*(\mathfrak{M})$, since $\mathfrak{M}$ has a Bialinicki-Birula decomposition with one complex cell for each fixed point. Thus $\mathfrak{M}$ has no odd dimensional homology, and

$$dim(H_T^{mid}(\mathfrak{M})) = dim(H^*(\mathfrak{M})) = \chi(\mathfrak{M}).$$

□

Lemma 7. *Let $Y \subset \mathfrak{M}$ be a T-stable smooth subvariety, and let*

$$\eta : H_T^*(\mathfrak{M}^T) \longrightarrow H_T^*(\mathfrak{M})$$

$$\eta' : H_T^*(Y^T) \longrightarrow H_T^*(Y)$$

$$\eta'' : H_T^*(Y) \longrightarrow H_T^*(\mathfrak{M})$$

be the corresponding transport maps. Then

$$\eta''(H_T^{mid}(Y)) \subset H_T^{mid}(\mathfrak{M})$$

Proof. Let $y \in Y^T$, and let N, N' be the tangent bundles to $\mathfrak{M}$ at y and to Y at y, respectively. Then $N = N' \oplus N''$, where N'' is the normal bundle to Y in $\mathfrak{M}$ at y. It follows that

$$\eta = \eta'' \eta'.$$

□

Proposition 7. *The restriction of* $\langle , \rangle$ *to* $H_T^{mid}(\mathfrak{M})$ *is non-degenerate and* $\mathbb{C}$ *valued.*

Proof. By the localization theorem, the classes $\eta(1_\lambda)$ for points $\lambda \in \mathfrak{M}^T$ form a basis of $H_T^{mid}(\mathfrak{M})$. So, we compute

$$\langle \eta(1_\lambda), \eta(1_\mu) \rangle = (-1)^m p_*(i_*)^{-1}(\eta(1_\lambda) \cup \eta(1_\mu)) =$$

$$= (-1)^m p_*(i_*)^{-1}(i_{\lambda *}(1_\lambda) \cup i_{\mu *}(1_\mu) \cup e(N_\lambda^-)^- \cup e(N_\mu^-)^-) = \delta_{\lambda,\mu}(-1)^m e(N_\lambda)^{-2} e(T_\lambda) = \delta_{\lambda,\mu}.$$

Thus, the classes $\eta(1_\lambda)$ form an orthonormal basis, and the bilinear form restricted to $H_T^{mid}(\mathfrak{M})$ is $\mathbb{C}$-valued and non-degenerate. ☐

We will denote the restriction of $\langle , \rangle$ to $H_T^{mid}(\mathfrak{M})$ by $\langle , \rangle$ as well.

Corollary 3.

$$\eta : H_T^{mid}(Y) \longrightarrow H_T^{mid}(\mathfrak{M})$$

is an isometry.

Proof. Let

$$\eta_1 : H_T^0(Y^T) \longrightarrow H_T^{mid}(Y)$$

$$\eta_2 : H_T^0(Y^T) \longrightarrow H_T^{mid}(\mathfrak{M})$$

Then, by the computation in the above lemma, η_1 and η_2 are isometries. But $\eta_2 = \eta\eta_1$, and η_1 is surjective. Thus η is an isometry. ☐

7.3 Localization of Correspondences

Let $\mathfrak{M}_1, \mathfrak{M}_2$ be quiver varieties with a T-action, and let $Y_1 \subset \mathfrak{M}_1$, $Y_2 \subset \mathfrak{M}_2$ be T-stable smooth subvarieties. Let $Z \subset Y_1 \times Y_2$ be a T-stable correspondence, so that the fundemental class $[Z]$ defines a linear map

$$[Z] : H_T^*(Y_1) \longrightarrow H_T^*(Y_2)$$

$$[Z](a) = q_{2*}(q_1^*(a) \cup [Z]).$$

Here q_1, q_2 are the projections from $Y_1 \times Y_2$ to Y_1, Y_2 respectively. We extend η to a map on correspondences by setting

$$\eta([Z]) = (i_1 \times i_2)_*([Z]) \cup (e(N_1^+)^{-1} \otimes e(N_2^-)^{-1})$$

so that $\eta([Z])$ defines a linear map

$$\eta([Z]) : H_T^*(\mathfrak{M}_1) \longrightarrow H_T^*(\mathfrak{M}_2)$$

$$\eta([Z])(b) = p_{2*}(p_1^*(b) \cup \eta([Z])).$$

Here p_1, p_2 are the projections from $\mathfrak{M}_1 \times \mathfrak{M}_2$ to $\mathfrak{M}_1, \mathfrak{M}_2$, respectively.

Theorem 1.

$$\eta([Z])(\eta(x)) = \eta([Z](x))$$

Proof. Let

$$p_j : \mathfrak{M}_1 \times \mathfrak{M}_2 \longrightarrow \mathfrak{M}_j \quad j = 1, 2$$

and

$$q_j : Y_1 \times Y_2 \longrightarrow Y_j \quad j = 1, 2$$

be the projection maps, and let

$$i_j : Y_j \longrightarrow \mathfrak{M}_j \quad j = 1, 2$$

be the inclusion. Then

$\eta([Z])(\eta(x)) =$
$= p_{2*}\big(p_1^*(\eta(x)) \cup \eta([Z])\big) =$
$= p_{2*}\big(p_1^*(i_{1*}(x) \cup e(N_1^-)^{-1}) \cup (i_1 \times i_2)_*([Z]) \cup e(N_1^+)^{-1} \cup e(N_2^-)^{-1}\big) =$
$= p_{2*}\big(p_1^*(i_{1*}(x)) \cup (i_1 \times i_2)_*([Z]) \cup e(N_1)^{-1} \cup e(N_2^-)^{-1}\big) =$
$= p_{2*}(i_1 \times i_2)_*\big((i_1 \times i_2)^* p_1^*(i_{1*}(x)) \cup [Z] \cup e(N_1)^{-1} \cup e(N_2^-)^{-1}\big) =$
$= i_{2*}q_{2*}\big(q_1^* i_1^*(i_{1*}(x)) \cup [Z] \cup e(N_1)^{-1} \cup e(N_2^-)^{-1}\big) =$
$= i_{2*}q_{2*}\big(q_1^*(x) \cup e(N_1) \cup [Z] \cup e(N_1)^{-1} \cup e(N_2^-)^{-1}\big) =$
$= i_{2*}q_{2*}\big(q_1^*(x) \cup [Z]\big) \cup e(N_2^-)^{-1} =$
$= \eta([Z](x))$

□

Corollary 4. *If $[Z] : H_T^{mid}(Y_1) \longrightarrow H_T^{mid}(Y_2)$ then*

$$\eta([Z]) : H_T^{mid}(\mathfrak{M}_1) \longrightarrow H_T^{mid}(\mathfrak{M}_2)$$

Proof. If $x \in H_T^{mid}(Y_1)$, then $\eta(x) \in H_T^{mid}(\mathfrak{M}_1)$. By assumption $[Z](x) \in H_T^{mid}(Y_2)$, so that $\eta([Z](x)) \in H_T^{mid}(\mathfrak{M}_2)$. The claim now follows from the above theorem. □

The above theorem and corollary will allow us to compute the (anti-)comutation relations of an operator $[X_r]$ and its adjoint $[X_r]^*$ on $\bigoplus_{n,\vec{l}} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$ by computing relations of the transported operators $\eta^{-1}([X_r]), \eta^{-1}([X_r]^*)$ on $\bigoplus_{n,\vec{l}} H_T^*(Y(n))$ for suitably chosen T-stable subvarieties $Y(n) \subset \mathfrak{M}_{\vec{l}}(r,n)$.

In particular, the inclusions

$$\mathfrak{M}_{\vec{l}}(r,n)^{T'} \hookrightarrow \mathfrak{M}_{\vec{l}}(r,n)$$
$$\mathfrak{M}_{\vec{l}}(r,n)^{T} \hookrightarrow \mathfrak{M}_{\vec{l}}(r,n)$$
$$(X_{r\vec{l}}^{[n]})^{T^\vee} \hookrightarrow X_{r\vec{l}}^{[n]}$$

give rise to three different maps

$$\eta' : H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)^{T'}) \longrightarrow H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$$

$$\eta : H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)^{T}) \longrightarrow H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$$

$$\eta^\vee : H_{T^\vee}^{mid}((X_{r\vec{l}}^{[n]})^{T^\vee}) \longrightarrow H_{T^\vee}^{mid}(X_{r\vec{l}}^{[n]}),$$

which will be used to check that the Heisenberg and Clifford operators defined in the subsequent chapters satisfy the defining relations of the Heisenberg and Clifford algebra.

8 Geometric Construction of the Clifford Operators

8.1 The spaces $\mathcal{B}^r$ and $\mathcal{B}^{r\vee}$

We begin by defining the fundamental spaces

$$\mathcal{B}^r = \bigoplus_{n,\vec{l}} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$$

and

$$\mathcal{B}^{r\vee} = \bigoplus_{n,\vec{l}} H_{T^\vee}^{mid}({X_{r}}_{\vec{l}}^{[n]}).$$

$\mathcal{B}^r$ and $\mathcal{B}^{r\vee}$ will be the underlying vector spaces of our representations of Heisenberg, Clifford, and affine Lie algebras.

8.2 Clifford Operators on $\mathcal{B}$ for $r = 1$

For simplicity, we begin with the case $r = 1$; note that in this case $\mathcal{B} = \mathcal{B}^\vee$.
For $x = (A, B, i) \in \mathbb{M}(1_l, \vec{v}), y = (A', B', i') \in \mathbb{M}(1_l, \vec{u})$, we write

$$x \twoheadrightarrow y$$

if there exists $S \subset V$ an A, B-stable subspace of dimension $\vec{v} - \vec{u}$ such that

$$(A_{V/S}, B_{V/S}, i_{V/S}) = (A', B', i').$$

Here $\mathbb{M}(1_l, \vec{v})$ is the affine space used to define the A_∞ quiver variety $M(1_l, \vec{v})$, and $A_{V/S}, B_{V/S}$ are the endomorphisms of the quotient space V/S induced from A and B.

For a $\mathbb{Z}$-graded vector space V of dimension $\vec{v}$ and homogeneous maps $A, B \in Hom(V, V)$, let $A(V)$ and $B(V)$ denote the images of the linear maps A and B. Let $a(\vec{v})$ and $b(\vec{v})$ denote the dimension vectors of the vector spaces $A(V), B(V)$ respectively.
If $x = (A, B, i) \in \mathbb{M}(1_l, \vec{v})$, note that $A(x) := (A_{A(V)}, B_{A(V)}, Ai)$ defines a point in $\mathbb{M}(1_{l+1}, a(\vec{v}))$.
Similarly, $B(x) := (A_{B(V)}, B_{B(V)}, Bi)$ defines a point in $\mathbb{M}(1_{l-1}, b(\vec{v}))$.
Given a pair of integers $k > l$ let $\vec{k}^l$ be the vector

$$k^l(i) = \begin{cases} 1 & l < i < k \\ 0 & \text{otherwise} \end{cases}$$

Define $\alpha(k)_{l,\vec{v}} \subset \mathbb{M}(1_l, \vec{v}) \times \mathbb{M}(1_{l+1}, \vec{v} + \vec{k}^l)$ as follows:

$$\alpha(k)_{l,\vec{v}} = \{(x, y) \mid y \twoheadrightarrow A(x)\}$$

Modding out by the $GL(V_k)$ actions in both factors, $\alpha(k)_{l,\vec{v}}$ defines a correspondence, which we denote by $\alpha(k)_{l,\vec{v}}$

$$\alpha(k)_{l,\vec{v}} \subset \mathfrak{M}(1_l, \vec{v}) \times \mathfrak{M}(1_{l+1}, \vec{v} + \vec{k}^l)$$

Let $\beta(k)_{l,\vec{v}}$ denote the adjoint correspondence obtained by swapping the factors $\mathfrak{M}(1_l,\vec{v})$ and $\mathfrak{M}(1_{l+1},\vec{v}+\vec{k}^l)$.
Similarly, if $k\leq l$, let $\vec{k}_l$ be the vector

$$k_l(i)=\left\{\begin{array}{ll}1 & k-1<i<l\\ 0 & \text{otherwise}\end{array}\right.$$

Define $\beta(k)_{l,\vec{v}}\subset\mathbb{M}(1_l,\vec{v})\times\mathbb{M}(1_{l-1},\vec{v}+\vec{k}_l)$ as follows:

$$\beta(k)_{l,\vec{v}}=\{(x,y)\mid y\twoheadrightarrow B(x)\}$$

Modding out by the $GL(V_k)$ actions in both factors, $\beta(k)_{l,\vec{v}}$ defines a correspondence

$$\beta(k)_{l,\vec{v}}\subset\mathfrak{M}(1_l,\vec{v})\times\mathfrak{M}(1_{l-1},\vec{v}+\vec{k}_l).$$

Let $\alpha(k)_{l,\vec{v}}$ denote the adjoint correspondence obtained by swapping the factors $\mathfrak{M}(1_l,\vec{v})$ and $\mathfrak{M}(1_{l-1},\vec{v}+\vec{k}_l)$.

For $k,l\in\mathbb{Z}$,$\vec{v}=(v_m)_{m\in\mathbb{Z}}$, let

$$n(k,l,\vec{v})=\left\{\begin{array}{ll}v_k & k>l\\ v_k+l-k & k\leq l\end{array}\right.$$

Define operators $\psi(k)$, $\psi^*(k)$,$k\in\mathbb{Z}$ by

$$\psi(k)=\bigoplus_{l\in\mathbb{Z},\vec{v}}(-1)^{n(k,l,\vec{v})}[\alpha(k)_{l,\vec{v}}]$$

$$\psi^*(k)=\bigoplus_{l\in\mathbb{Z},\vec{v}}(-1)^{n(k,l,\vec{v})}[\beta(k)_{l,\vec{v}}],$$

and these operators act on the cohomology of all of the A_∞ quiver varieties.

Since the fixed points $\coprod_{n,l}(\mathbb{C}^{2[n]}_l)^{\mathbb{C}^*}$ are canonically identified with the A_∞ quiver varieties, we may define operators

$$\eta(\psi_i(k)),\eta(\psi^*_i(k)):\mathcal{B}\longrightarrow\mathcal{B},$$

by using the map

$$\eta:\bigoplus_{n,l}H^{mid}_{\mathbb{C}^*}((\mathbb{C}^{2[n]}_l)^{\mathbb{C}^*})\longrightarrow\bigoplus_{n,l}H^{mid}_{\mathbb{C}^*}(\mathbb{C}^{2[n]}_l)=\mathcal{B},$$

extended naturally to a map on correspondences, as described in Section 7.3. Note that, by construction, the operators $\eta(\psi_i(k))$ and $\eta(\psi^*_i(k))$ are adjoint to one another with respect to the inner product on $\mathcal{B}$.

8.3 Clifford Operators on $\mathcal{B}^r$ for $r>1$

On r-component products of A_∞ quiver varieties we have r different correspondences

$$\alpha_i(k)_{l,\vec{v}},\beta_i(k)_{l,\vec{v}},\quad i=1,\dots r$$

modifying only the r different factors of the product. Denote by $\psi_i(k), \psi_i^*(k)$, $i = 1, \ldots, r$ the resulting operators, which act on the homology of the ith factor of the product. Since the fixed points $\coprod_{n,\vec{l}} \mathfrak{M}_{\vec{l}}(r,n)^T$ are naturally r-component products of A_∞ quiver varieties, we can define operators

$$\eta_r(\psi_i(k)), \eta_r(\psi_i^*(k)) : \mathcal{B}^r \longrightarrow \mathcal{B}^r,$$

where

$$\eta_r : \bigoplus_{n,\vec{l}} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)^T) \longrightarrow \bigoplus_{n,\vec{l}} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)) = \mathcal{B}^r$$

is extended to a map on correspondences as in Section 7.3. Note that, by construction, the operators $\eta_r(\psi_i(k))$ and $\eta_r(\psi_i^*(k))$ are adjoint to one another with respect to the geometric inner product on $\mathcal{B}^r$.

8.4 Clifford Operators on $\mathcal{B}^{r\vee}$ for $r > 1$

Since the $T^\vee$ fixed points of $X_{r}{}^{[n]}_{\vec{l}}$ are also naturally given by r-component products of A_∞ quiver varieties, we can also define operators

$$\eta_r^\vee(\psi_i(k)), \eta_r^\vee(\psi_i{}^*(k)) : \mathcal{B}^{r\vee} \longrightarrow \mathcal{B}^{r\vee}$$

using the map

$$\eta_r^\vee : \bigoplus_{n,\vec{l}} H_{T^\vee}^{mid}((X_{r}{}^{[n]}_{\vec{l}})^{T^\vee}) \longrightarrow H_{T^\vee}^{mid}(X_{r}{}^{[n]}_{\vec{l}}) = \mathcal{B}^{r\vee}.$$

Note that the operators $\eta_r^\vee(\psi_i(k))$ and $\eta_r^\vee(\psi_i{}^*(k))$ are adjoint with respect to the geometric inner product on $\mathcal{B}^{r\vee}$.

9 Geometric construction of Heisenberg operators

9.1 Heisenberg operators on $\mathcal{B}$ for $r = 1$

We begin first which the case $r = 1$; in this case the construction of Heisenberg operators is due independently to Nakajima [Nak99] and Grojnowski [Gro96]. Define

$$Z^o \subset \coprod_{n,k} \mathbb{C}^{2[n]} \times \mathbb{C}^{2[k]} \times \mathbb{C}^{2[n+k]}$$

to be the variety of triples (A, B, C) such that A and B have disjoint support, and there exists an exact sequence

$$0 \to A \to C \to B \to 0$$

Let $Z = \overline{Z^o}$. The fundemental classes $[Z]$ of the components of Z define a multiplication

$$[Z] : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B}$$

which makes $\mathcal{B}$ into a commutative algebra. Define a $\mathbb{C}^*$ action on $\mathbb{C}^2$ by

$$t \diamond (x, y) = (tx, y).$$

This induces a $\mathbb{C}^*$ action (also denoted by $\diamond$) on the Hilbert scheme ${\mathbb{C}^2}^{[n]}$, and this action commutes with symplectic action of $\mathbb{C}^*$ used when taking equivariant cohomology. The fixed point components with respect to the $\diamond$ action (which are not in general isolated) are naturally enumerated by paritions $\lambda \vdash n$ [Nak99], and we denote the fixed point component corresponding to λ by C_λ.
For $n > 0$, define classes $p(n), e(n), h(n) \in H^{mid}_{\mathbb{C}^*}({\mathbb{C}^2}^{[n]})$ by

$$p(n) = [\overline{\{z \in {\mathbb{C}^2}^{[n]} \mid \lim_{t\to 0} t \diamond z \in C_{(n)}\}}]$$

$$e(n) = [\overline{\{z \in {\mathbb{C}^2}^{[n]} \mid \lim_{t\to 0} t \diamond z \in C_{(1^n)}\}}]$$

$$h(n) = [\overline{\{z \in {\mathbb{C}^2}^{[n]} \mid \lim_{t\to 0} t \diamond z \quad exists\}}]$$

The class $p(n)$ has an alternative description as the fundamental class of the subvariety

$$P(n) \subset {\mathbb{C}^2}^{[n]}$$

of schemes supported at a single point somewhere on the x-axis of $\mathbb{C}^2$.

Multiplications by the above classes give operators

$$p(n), e(n), h(n) : \mathcal{B} \longrightarrow \mathcal{B}.$$

For $n < 0$, define $p(n), h(n), e(n)$ as the adjoints to $p(-n), h(-n), e(-n)$, respectively, with respect to the geometric inner product on $\mathcal{B}$. The operators $p(n)$ will be our Heisenberg operators, while the operators $e(n), h(n)$ will be important for our geometric interpretation of the Boson-Fermion correspondence.

9.2 Heisenberg Operators on $\mathcal{B}^r$ for $r > 1$

Recall the action of $T' = (\mathbb{C}^*)^r$ on $\mathfrak{M}_{\vec{l}}(r, n)$ defined in Chapter **??**. In particular, recall that the T'-fixed point components of $\mathfrak{M}_{\vec{l}}(r, n)$ are products of Hilbert schemes:

$$\mathfrak{M}_{\vec{l}}(r, n)^{T'} = \coprod_{\sum n_i = n} {\mathbb{C}^2}^{[n_1]} \times \ldots \times {\mathbb{C}^2}^{[n_r]}$$

We can define operators

$$\eta'_r(p(n)), \eta'_r(e(n)), \eta'_r(h(n)) : \mathcal{B}^r \longrightarrow \mathcal{B}^r$$

using the map

$$\eta'_r : \bigoplus_{n,\vec{l}} H^{mid}_T(\mathfrak{M}_{\vec{l}}(r, n)^{T'}) \longrightarrow \bigoplus_{n,\vec{l}} H^{mid}_T(\mathfrak{M}_{\vec{l}}(r, n)) = \mathcal{B}^r,$$

extended naturally to correspondences, as in Section 7.3.

9.3 Heisenberg Operators on $\mathcal{B}^{r\vee}$ for $r > 1$

Define

$$Z^o \subset \coprod_{n,k} {X_r}^{[n]} \times {X_r}^{[k]} \times {X_r}^{[n+k]}$$

to be the variety of triples (A, B, C) such that A and B have disjoint support, and there exists an exact sequence

$$0 \to A \to C \to B \to 0$$

Let $Z = \overline{Z^o}$. The fundemental classes $[Z]$ of the components of Z define a multiplication

$$[Z] : \mathcal{B}^{r\vee} \otimes \mathcal{B}^{r\vee} \longrightarrow \mathcal{B}^{r\vee}$$

whichmakes $\mathcal{B}^{r\vee}$ into a commutative algebra, just as in the case $r = 1$. For a closed, $T^\vee$-invariant curve $\Sigma \subset X_r$, we define the closed subvariety

$$P_\Sigma(n) \subset {X_r}^{[n]}$$

to be the set of all length n subschemes of X_r which are supported at a single point of Σ. Multiplication by the fundamental class $[P(\Sigma)] \in H^{mid}_{T^\vee}({X_r}^{[n]})$ gives an operator

$$p_\Sigma(n) : \mathcal{B}^{r\vee} \longrightarrow \mathcal{B}^{r\vee}$$

Since the fundamental classes of $T^\vee$-invariant curves Σ span the vector space $H^2_{T^\vee}(X_r)$, we may extend by linearity and define an operator $p_\alpha(n)$ for any $\alpha \in H^2_{T^\vee}(X_r)$. In particular, if η_r denotes the map

$$\eta_r^\vee : H^0_{T^\vee}({X_r}^{T^\vee}) \longrightarrow H^2_{T^\vee}(X_r),$$

then each fixed point x_i, $i = 1, \ldots, r$ gives rise to an operator $p(\eta_r^\vee(x_i))$. We will abuse notation slightly to preserve the symmetry with the construction of Section 9.2, and denote these operators by $\eta_r^\vee(p_i(n))$.

In order to define the operators $\eta_r^\vee(h_i(n))$ and $\eta_r^\vee(e_i(n))$, we will use the following lemma.

Lemma 8. *Let $r = 1$, so that $X_r = \mathbb{C}^2$, and suppose $n > 0$. Then the classes $h(n)$ and $e(n)$ are polynomials in the $\{p(k)\}_{k \in \mathbb{Z}}$.*

Proof. This statement is proven in the last chapter of [Nak99]. □

In other words, there are polynomials H_n, E_n such that

$$h(n) = H_n(\ldots, p(-2), p(-1), \ldots, p(k), \ldots)$$

$$e(n) = E_n(\ldots, p(-2), p(-1), \ldots, p(k), \ldots),$$

so, for $\alpha \in H^2_{T^\vee}(X_r)$, we define $h_\alpha(n)$ and $e_\alpha(n)$ by replacing $p(k)$ with $p_\alpha(k)$ in these expressions:

$$h_\alpha(n) = H_n(\ldots, p_\alpha(-2), p_\alpha(-1), \ldots, p_\alpha(k), \ldots)$$

$$e_\alpha(n) = E_n(\ldots, p_\alpha(-2), p_\alpha(-1), \ldots, p_\alpha(k), \ldots)$$

In particular, we have operators corresponding to the fixed point classes $\eta_r^\vee(x_i) \in H^2_{T^\vee}(X_r)$, and we denote these operators by $\eta_r^\vee(h_i(n))$ and $\eta_r^\vee(e_i(n))$. Finally, for $n < 0$, define

$$\eta_r^\vee(p_i(n)), \quad \eta_r^\vee(h_i(n)), \quad \eta_r^\vee(e_i(n))$$

as adjoints of

$$\eta_r^\vee(p_i(-n)), \quad \eta_r^\vee(h_i(-n)), \quad \eta_r^\vee(e_i(-n))$$

with respect to the inner product on $\mathcal{B}^{r\vee}$.

10 The Proof of the Relations

10.1 The Clifford Algebra

For simplicity in the statement of the main proposition, we will drop the $\eta, \eta^\vee$ notation of the previous sections in the statement of the main proposition. Thus, what follows, the operators $\psi_i(k), \psi_i^*(k)$ can be interpreted as either the operators $\eta_r(\psi_i(k)), \eta_r(\psi_i^*(k))$ of Section 8.3 or as the operators $\eta_r^\vee(\psi_i(k)), \eta_r^\vee(\psi_i^*(k))$ of Section 8.4.

Let $\nu_0 = 1 \in H^0_T(\mathfrak{M}(r,0)) = H^0_T(pt)$., respectively $\nu_0 = H^0_{T^\vee}(X_r^{[0]}) = H^0_{T^\vee}(pt)$.

Proposition 8. *The operators $\psi_i(k), \psi_i^*(k)$ satisfy the following anti-commutation relations:*

$$\psi_i(k)\nu_0 = 0 \quad \forall k \leq 0, i = 0, \ldots, r-1$$

$$\psi_i^*(k)\nu_0 = 0 \quad \forall k > 0, i = 0, \ldots, r-1$$

$$\{\psi_i(k), \psi_j(l)\} = \{\psi_i^*(k), \psi_j^*(l)\} = 0, \qquad \{\psi_i(k), \psi_j^*(l)\} = \delta_{ij}\delta_{kl}$$

Proof. We will prove the proposition for the operators of the first construction, Section 8.3. The proof for the operators of Section 8.4 is identical.

Let

$$\mathcal{F}^r = \bigoplus_{n,\vec{l}} H^0_T(\mathfrak{M}_{\vec{l}}(r,n)^T).$$

We will consider the operators

$$\eta^{-1}(\psi_i(k)), \eta^{-1}(\psi_i^*(k)) : \mathcal{F}^r \longrightarrow \mathcal{F}^r$$

and prove that they have the above commutation relations. For convenience, we will drop the notation η^{-1}, and denote our transported operators by $\psi_i(k), \psi_i^*(k)$ as well. In addition, since $\psi_i(k), \psi_i^*(k)$ and $\psi_j(k), \psi_j^*(k)$ for $i \neq j$ act on different coordinates in the product of A_∞ quiver varieties, the only interesting case is the case $i = j$; thus we may consider the case $r = 1$, and drop the subscripts i, j.

In order to prove this proposition for $r = 1$, it will be convenient to identify the vector space $\mathcal{F}$ with the semi-infinite wedge space. Given an A_∞ quiver variety $\mathfrak{M}(1_l, \vec{v})$, we define subsets $C^+_{\vec{v},l}, C^-_{\vec{v},l}, C_{\vec{v},l} \subset \mathbb{Z}$ by

$$C^+_{\vec{v},l} = \{k > l \mid v_k \neq v_{k-1}\}$$

$$C^-_{\vec{v},l} = \{k \leq l \mid v_k = v_{k-1}\}$$

$$C_{\vec{v},l} = C^+_{\vec{v},l} \cup C^-_{\vec{v},l}$$

Arranging the elements of $C_{\vec{v},l}$ in decending order,
$C_{\vec{v},l} = \{i_0, i_1, \ldots\}$
we get a semi-infinite wedge

$$C_{\vec{v},l} \mapsto i_0 \wedge i_1 \wedge i_2 \wedge \ldots$$

We define the charge of a semi infinite wedge $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$ to be the integer m such that $i_n = m - n$ for n sufficiently large; in this way the quiver variety $\mathfrak{M}(1_l, \vec{v})$ corresponds to a semi-infinite wedge of charge l. Let F_l denote the $\mathbb{C}$-span of the semi-infinite wedges of charge l. We define a vector space isomorphism

$$\mathcal{F}_l = \oplus_{\vec{v}} H^0_{\mathbb{C}^*}(\mathfrak{M}(1_l, \vec{v})) \longrightarrow F_l$$

by mapping $1 \in H^0_{\mathbb{C}^*}(\mathfrak{M}(1_l, \vec{v})$ to the semi-infinite wedge corresponding to $\mathfrak{M}(1_l, \vec{v})$. The following lemma, which is easy to check, relates to coordinate entries v_k of $\vec{v}$ to the integers appearing in the corresponding semi-infinite monomial.

Lemma 9. *If $\mathfrak{M}(1_l, \vec{v})$ corresponds to the wedge $i_0 \wedge i_1 \wedge i_2 \wedge \ldots$ then the number of elements in the set $\{i_0, i_1, i_2, \ldots\}$ which are greater than k is*

$$\begin{array}{ll} v_k & \text{if } k > l \\ v_k + l - k & \text{if } k \leq l. \end{array}$$

The anti-commutation relations of the operators $\psi(k), \psi^*(k)$ will follow immediately from the following proposition.

Proposition 9.

$$\psi(k)(i_0 \wedge i_1 \wedge \ldots) = \begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge k \wedge i_s \ldots & i_{s-1} > k > i_s \\ 0 & k = i_s \textit{ for some } s \end{cases}$$

$$\psi^*(k)(i_0 \wedge i_1 \wedge \ldots) = \begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge i_{s+1} \ldots & k = i_s \\ 0 & k \neq i_s \textit{ for all } s \end{cases}$$

Proof. Fix $l \in \mathbb{Z}$. It suffices to show that the proposition holds for the operators $\psi(k)$, $k > l$ and $\psi^*(k)$, $k \leq l$.
We begin with the operators $\psi(k)$, $k > l$.
Suppose $\mathfrak{M}(1_l, \vec{v})$ is nonempty and $\mathfrak{M}(1_l, \vec{v}) \leftrightarrow (i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots)$.
Then $\mathfrak{M}(1_{l+l}, \vec{v} + \vec{k}^l)$ is non-empty if and only if $k \notin \{i_0, i_1, i_2 \ldots\}$. If $\mathfrak{M}(1_{l+1}, \vec{v} + \vec{k}^l)$ is non-empty, then the projections

$$p_1 : \alpha(k) \longrightarrow \mathfrak{M}(1_l, \vec{v})$$

$$p_2 : \alpha(k) \longrightarrow \mathfrak{M}(1_{l+1}, \vec{v} + \vec{k}^l)$$

induce identity maps on cohomology, since the varieties involved are all points.
It follows that

$$[\alpha(k)](i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots) = i_0 \wedge i_1 \wedge \ldots \wedge i_s \wedge k \wedge i_{s+1} \wedge \ldots = (-1)^{s+1} k \wedge i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots$$

where $i_s > k \geq i_{s+1}$. Therefore

$$\psi(k)(i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots) = (-1)^{v_k} [\alpha(k)](i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots)$$

But by the previous lemma v_k is the number of elements in the set $\{i_0, i_1, \ldots\}$ which are greater than k, i.e. $v_k = s$, where $i_{s-1} > k > i_s$.

$$\Longrightarrow \psi(k)(i_0 \wedge i_1 \wedge \ldots) = \begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge k \wedge i_s \ldots & i_{s-1} > k > i_s \\ 0 & k = i_s \text{ for some } s \end{cases}$$

Now suppose that $k \leq l$. We will consider the operators $\psi^*(k)$. We have that $\mathfrak{M}(1_{l+1}, \vec{v} + \vec{k}_l)$ is non-empty if and only if $k \in \{i_0, i_1, i_2, \ldots\}$. As in the case $\psi(k)$,$k > l$, we have

$$\psi^*(k)(i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots) = (-1)^{v_k + l - k} [\beta(k)](i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots) = \sum_{j \in \mathbb{Z}} \delta_{i_j, k} (-1)^{v_k + l - k} i_0 \wedge i_1 \wedge \ldots \wedge \widehat{i_j} \wedge \ldots$$

But $v_k + l - k$ is the number of elements in $\{i_0, i_1, i_2, \ldots\}$ which are greater than k

$$\Longrightarrow \psi^*(k)(i_0 \wedge i_1 \wedge \ldots \wedge i_n \wedge \ldots) = \sum_{j \in \mathbb{Z}} \delta_{i_j,k}(-1)^j i_0 \wedge i_1 \wedge \ldots \wedge \widehat{i_j} \wedge \ldots =$$

$$\begin{cases} (-1)^s i_0 \wedge \ldots \wedge i_{s-1} \wedge i_{s+1} \ldots & k = i_s \\ 0 & k \neq i_s \text{ for all } s \end{cases}$$

□

This completes the proof of the proposition. □

As an immediate corollary, we get the following theorems.

Theorem 2. *The operators $\eta_r(\psi_i(k)), \eta_r(\psi_i^*(k))$ on the space*

$$\mathcal{B}^r = \bigoplus_{\vec{l},n} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$$

give a geometric realization of an irreducible module for the Clifford algebra Cl^r.

Theorem 3. *The operators $\eta_r^\vee(\psi_i(k)), \eta_r^\vee(\psi_i^*(k))$ on the space*

$$\mathcal{B}^{\vee r} = \bigoplus_{\vec{l},n} H_{T^\vee}^{mid}(X_{r\vec{l}}^{[n]})$$

give a geometric realization of an irreducible module for the Clifford algebra Cl^r.

10.2 The Heisenberg Algebra

Proposition 10. *The operators $\eta'(p_i(n))$, $n \in \mathbb{Z}$, $i = 0, \ldots r-1$ satisfy*

$$\eta'(p_i(n))\nu_0 = 0, \quad n < 0$$

$$[\eta'(p_i(n)), \eta'(p_j(m))] = n\delta_{i,j}\delta_{n+m,0} Id$$

Proof. Recall the isomorphism

$$\eta' : \bigoplus_n H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)^{T'}) \longrightarrow H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n))$$

We will consider the operators $p_i(n) := \eta'^{-1}(\eta(p_i(n))$, and show that they satisfy the same relations. Since $p_i(n)$ and $p_j(m)$ act on different factors of $\mathfrak{M}_{\vec{l}}(r,n)^{T'}$, the only interesting case is $i = j$; thus we may consider the case $r = 1$.
For a partition $\lambda \in (\mathbb{C}^{2[n]})^{\mathbb{C}^*}$, we have a class $[\lambda] = \eta(1_\lambda) \in H_{\mathbb{C}^*}^{mid}(\mathbb{C}^{2[n]})$. The classes $\{[\lambda]\}_{\lambda \vdash n}$ form an orthonormal basis of $H_{\mathbb{C}^*}^{mid}(\mathbb{C}^{2[n]})$, so we can construct an isometric vector space isomorphism

$$\phi : Sym \longrightarrow \bigoplus_n H_{\mathbb{C}^*}^{mid}(\mathbb{C}^{2[n]})$$

$$\phi(s_\lambda) = [\lambda]$$

by sending the Schur function s_λ to the class $[\lambda]$. Then we have the following two lemmas.

Lemma 10. *The map ϕ is an isomorphism of algebras.*

Proof. This is proved in [Vas01]. $\square$

Lemma 11. *The monomial symmetric functions are realized geometrically by the varieties L_λ:*

$$\phi(m_\lambda) = [L_\lambda].$$

Proof. This is proved in [Vas01]. $\square$

In particular, the operator $p(n)$, $n > 0$ is the image under ϕ of multiplication by the power-sum symmetric function p_n. Since ϕ is an isometry, the adjoint operator $p(-n)$ corresponds to the differential operator $\partial/\partial p_n$. The proposition then follows from

$$\partial/\partial p_n(1) = 0$$

and

$$[p_n, \partial/\partial p_m] = n\delta_{n,m} Id$$

$\square$

Thus we have the following theorem:

Theorem 4. *For any $\vec{l} \in \mathbb{Z}^r$, the operators $\eta'(p_i(n))$ on the space*

$$\mathcal{B}^r_{\vec{l}} = \bigoplus_n H^{mid}_T(\mathfrak{M}_{\vec{l}}(r,n))$$

give a geometric realization of an irreducible module for the Heisenberg algebra $\mathcal{H}^r$.

Moreover, the same proof works on the space $\mathcal{B}^{\vee r}$, (see also [Nak99] and [QW]) giving us the companion theorem:

Theorem 5. *For any $\vec{l} \in \mathbb{Z}^r$, the operators $\eta^\vee(p_i(n))$ on the space*

$$\mathcal{B}^{\vee r}_{\vec{l}} = \bigoplus_n H^{mid}_{T^\vee}({X_r}^{[n]}_{\vec{l}})$$

give a geometric realization of an irreducible module for the Heisenberg algebra $\mathcal{H}^r$.

Of course, the construction of Heisenberg algebra actions on ordinary cohomology analog of the space $\mathcal{B}^{\vee r}$ is due to Nakajima [Nak99] and Grojnowski [Gro96]. The modification of this Heisenberg action to equivariant cohomology in the case of the Hilbert scheme $\mathbb{C}^{2[n]}$ ($r = 1$) is due to Vasserot [Vas01], while the straightforward modification to equivariant cohomology on ${X_r}^{[n]}$ also appears in Qin-Wang [QW]. The main new point for us is that these same representations can be realized using moduli spaces $\mathfrak{M}(r,n)$ of higher rank torsion-free sheaves.

10.3 Level One Representations of $\widehat{gl(r)}$

Passing from representations of the Heisenberg algebra $\mathcal{H}^r$ or Clifford Algebra Cl^r to a representation of the affine Lie algebra $\widehat{gl(r)}$, we can use our geometric constructions identify the basic representation of $\widehat{gl(r)}$. Let $Q \simeq \mathbb{Z}^{r-1} \subset \mathbb{Z}^r$ be the balanced sublattice whose entries sum to 0. Then our construction of Heisenberg and Clifford modules immediately implies the following theorem.

Theorem 6. *The basic reprsesentation of $\widehat{gl(r)}$ can be realized geometrically on the vector space*

$$\bigoplus_{\vec{l} \in Q, n} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)),$$

as well as on the vector space

$$\bigoplus_{\vec{l} \in Q, n} H_{T^\vee}^{mid}({X_r}_{\vec{l}}^{[n]}).$$

Proof. This construction can be accomplished using the Heisenberg operators $\eta'_r(p_i(n))$ and $\eta_r^\vee(p_i(n))$ as in the Frenkel-Kac construction [FK81], or using the Clifford operators $\eta_r(\psi_i(k)), \eta_r(\psi_i^*(k))$ and $\eta_r^\vee(\psi_i(k)), \eta_r^\vee(\psi_i^*(k))$, as in [Fre80] and [KP81]. $\square$

In [Nak99], [Nak98], [Gro96], the basic representation of the affine Lie algbera $\widehat{sl(r)}$ is constructed on the ordinary cohomology of moduli spaces of rank one torsion-free sheaves on X_r. Our construction on $\mathcal{B}^{r\vee}$ extends this construction to an action of $\widehat{gl(r)}$ on equivariant cohomology. The $\widehat{gl(r)}$ action on $\mathcal{B}^r$, however, is quite different from the action on $\mathcal{B}^{r\vee}$, in that the appearance of the rank r Lie algebra $gl(r)$ is related to the rank of the sheaves in $\mathfrak{M}(r,n)$, whereas in [Nak99], [Nak98],[Gro96], the Lie algebra is related to the geometry of the underlying surface X_r.

11 Geometric Interpretation of the Boson-Fermion Correspondence

11.1 The Boson-Fermion Correspondence

We have constructed fermionic operators $\eta(\psi_i(k)), \eta(\psi_i^*(k))$ and bosonic operators $\eta(p_i(n))$ on the space

$$\mathcal{B}^r = \bigoplus_{n,\vec{l}} H_T^{mid}(\mathfrak{M}_{\vec{l}}(r,n)),$$

as well as fermionic operators $\eta^\vee(\psi_i(k)), \eta^\vee(\psi_i^*(k))$ and bosonic operators $\eta^\vee(p_i(n))$ on the common space

$$\mathcal{B}^{r\vee} = \bigoplus_{n,\vec{l}} H_{T^\vee}^{mid}({X_r}_{\vec{l}}^{[n]}).$$

In both cases, we can relate the bosonic operators to the fermionic operators, giving a geometric realization of the Boson-Fermion correspondence. The result in this section should be compared to the main result of [Sav], which considers the case $r = 1$ and uses localization to relate the action of the bosonic operators $p(n)$ on $\mathcal{B}$ with an action of $sl(\infty)$ on $\mathcal{B}$. We remark that the $sl(\infty)$ action considered in [Sav] can be constructed from the Clifford algebra action we constructed in the previous chapter. In order to state the result concisely, we drop the η_r and $\eta_r^\vee$ from the notation, so that $p_i(n)$ denotes either $\eta_r(p_i(n))$ or $\eta_r^\vee(p_i(n))$.

In order to state our geometric Boson-Fermion correpondence, we need to define r other operators q_i, $i = 1, \dots r$ which acts as our translation operator in the lattice $\mathbb{Z}^r$. Recall that

$$H^2_{\mathbb{C}^*}(\mathfrak{M}(r,1)^{T'}, \mathbb{Z}) \simeq \mathbb{Z}^r$$

$$H^2_{\mathbb{C}^*}(X_r, \mathbb{Z}) \simeq \mathbb{Z}^r,$$

and that for all $\vec{l} \in \mathbb{Z}^r$ there are line bundles $L_{\vec{l}}, L^\vee_{\vec{l}}$ on $\mathfrak{M}(r,1)^{T'}, X_r$ respectively with equivariant Chern class equal to $\vec{l}$. In particular, for coordinate vectors $1_i = (0, \dots, 1, \dots, 0)$ we have line bundles $L_i, L_i^\vee$. Define operators, both denoted by Q_i, by tensoring with these line bundles:

$$Q_i = \otimes L_i^\vee : \mathcal{B}^{r\vee} \longrightarrow \mathcal{B}^{r\vee},$$

$$Q_i = \eta'(\otimes L_i) : \mathcal{B}^r \longrightarrow \mathcal{B}^r$$

These operators are geometric versions of the shift operators q_i needed in the Boson-Fermion correspondence.

Theorem 7. *a) As operators on $\mathcal{B}^r$ or $\mathcal{B}^{r\vee}$, the bosons can be written in terms of the fermions:*

$$p_i(n) = \sum_{k\in\mathbb{Z}} \psi_i(k)\psi_i^*(k+n)$$

if $n \neq 0$, and

$$p_i(0) = \sum_{j>0} \psi_i(k)\psi_i^*(k) - \sum_{j\leq 0} \psi_i^*(k)\psi_i(k)$$

b) For fixed $\vec{m} \in \mathbb{Z}^r$, as operators $\mathcal{B}^r(\vec{m}) \longrightarrow \mathcal{B}^r(\vec{m} \pm 1_i)$ or $\mathcal{B}^{r\vee}(\vec{m}) \longrightarrow \mathcal{B}^{r\vee}(\vec{m} \pm 1_i)$, the fermions can be written in terms of the bosons and the shift operators:

$$\psi_i(k) = \sum_{n\in\mathbb{Z}} Q_i h_i(n) e_i(n - m_i + k)$$

$$\psi_i^*(k) = \sum_{n\in\mathbb{Z}} Q_i^{-1} e_i(n) h_i(n + m_i + k)$$

Proof. Recall the algebraic version of the r-colored bosonic Fock space

$$\mathcal{B}^r_{alg} = \mathcal{B}_{alg} \otimes \ldots \otimes \mathcal{B}_{alg},$$

where

$$\mathcal{B}_{alg} = \mathbb{C}[q, q^{-1}, p_1, p_2, \ldots].$$

We denote the vector $q_0^{l_0} s_{\lambda_0} \otimes \ldots \otimes q^{l_{r-1}} s_{\lambda_{r-1}} \in \mathcal{B}^r_{alg}$ by $q^{\vec{l}} s_{\vec{\lambda}}$. The vectors $\{q^{\vec{l}} s_{\vec{\lambda}}\}_{\lambda,\vec{l}}$ form an orthonormal basis of $\mathcal{B}^r_{alg}$.
The maps

$$\phi : \mathcal{B}^r_{alg} \longrightarrow \mathcal{B}^r$$

and

$$\phi^\vee : \mathcal{B}^r_{alg} \longrightarrow \mathcal{B}^{r\vee}$$

given by

$$\phi(q^{\vec{l}} s_{\vec{\lambda}}) = \eta([\vec{\lambda}]_{\vec{l}}) \in H_T^{mid}(M_{\vec{l}}(r,n))$$

and

$$\phi^\vee(q^{\vec{l}} s_{\vec{\lambda}}) = \eta^\vee([\vec{\lambda}]_{\vec{l}}) \in H_{T^\vee}^{mid}(X_{r_{\vec{l}}}^{[n]})$$

are isometric algebra isomorphisms. This implies that the operators $h_i(n), e_i(n)$ correspond under ϕ to multiplication by the homogeneous symmetric functions $(h_i)_n$ and the elementary symmetric functions $(e_i)_n$ for $n > 0$ and to their adjoints for $n < 0$. Of course, since ϕ is an isometry and an algebra isomorphism, it is also an isomorphism of Heisenberg modules.

Similarly, our Clifford algebra action was constructed so that ϕ is also an isomorphism of Clifford modules. The theorem now follows from the algebraic formulation of the boson-fermion correspondence. □

11.2 Geometric Realization of Level k Representations

The decomposition of $\coprod_{n,\vec{l}} M_{\vec{l}}(r,n)$ into connected components induces a natural grading on the vector space

$$\mathcal{B}^r = \bigoplus_{n,\vec{l}} H_T^{mid}(M_{\vec{l}}(r,n)).$$

For any matrix unit $e_{i,j} \in gl(r)$, the operator $e_{i,j} \otimes t^m \in \widehat{gl(r)}$ is homogeneous with respect to this grading in the sense that if $x \in \mathcal{B}^r$ is supported in one summand, then
$e_{i,j} \otimes t^m(x)$ will be a class supported in one summand of $\mathcal{B}^r$.

The inclusion $\mathbb{Z}_k \hookrightarrow \mathbb{C}^* \times 1 \subset T$ induces an action of $\mathbb{Z}_k$ on the spaces $M_{\vec{l}}(r,n)$ such that the connected components of the fixed point set $\coprod_{n,\vec{l}} M_{\vec{l}}(r,n)^{\mathbb{Z}_k}$ are $\widehat{A_{k-1}}$ quiver varieties.

Since the T- fixed points of $\coprod_{n,\vec{l}} M_{\vec{l}}(r,n)$ are the same as the T-fixed points of $\coprod_{n,\vec{l}} M_{\vec{l}}(r,n)^{\mathbb{Z}_k}$, there is a natural vector space isomorphism

$$\mathcal{B}^r = \bigoplus_{n,\vec{l}} H_T^{mid}(M_{\vec{l}}(r,n)) \simeq \bigoplus_{n,\vec{l}} H_T^{mid}(M_{\vec{l}}(r,n)^{\mathbb{Z}_k}),$$

and the decomposition of $\coprod_{n,\vec{l}} M_{\vec{l}}(r,n)^{\mathbb{Z}_k}$ into connected components induces another (more refined) grading of $\mathcal{B}^r$. In general, elements of the form $e_{i,j} \otimes t^m$ are not homogeneous with respect to this new grading, but elements of the form $e_{i,j} \otimes t^{km}$ are homogeneous with respect to it. Thus, the operators in the subalgebra $\widehat{gl(r)}_k = gl(r) \otimes \mathbb{C}[t^k, t^{-k}] \oplus \mathbb{C}c$ can be constructed naturally as equivariant cohomology classes inside the products of $\widehat{A_{k-1}}$ quiver varieties. We hope to study this geometric realization of both the subalgebra $\widehat{gl(r)}_k$ and of Nakajima's commuting level r action of $\widehat{sl(k)}$ in future work. We expect these commuting operators to give a geometric realization of the level-rank duality first discovered in [Fr 82].

12 Comparing the Geometry of $\mathfrak{M}(r,n)$ and $X_r^{[n]}$

Since the same representation can be geometrically constructed using two different geometries, we may ask for geometric relationships between the underlying moduli spaces. In the present case, we seek a relationship between

$$U(r) \ \textit{instantons on} \ \widetilde{\mathbb{C}^2/\mathbb{Z}_k}$$

and

$$U(k) \ \textit{instantons on} \ \widetilde{\mathbb{C}^2/\mathbb{Z}_r}$$

In the rest of this section we address this question for $k = 1$ by considering the associated semismall resolutions.

12.1 The Resolution $\pi^\vee : {X_r}^{[n]} \longrightarrow S^n(\mathbb{C}^2/\mathbb{Z}_r)$

Let $S^n(\mathbb{C}^2/\mathbb{Z}_r)$ denote the nth symmetric product of the singular variety $\mathbb{C}^2/\mathbb{Z}_r$. $S^n(\mathbb{C}^2/\mathbb{Z}_r)$ is naturally stratified

$$S^n(\mathbb{C}^2/\mathbb{Z}_r) = \coprod_{0 \le k \le n \ \lambda \vdash n-k} S^{n-k}_\lambda(\mathbb{C}^2/\mathbb{Z}_r),$$

where $S^{n-k}_\lambda(\mathbb{C}^2/\mathbb{Z}_r)$ is the subspace of the symmetric product where the singular point (the origin) occurs with multiplicity k, and the rest of the configuration is of partition type λ. The Hilbert-Chow morphism

$${X_r}^{[n]} \longrightarrow S^n(X_r)$$

and the resolution

$$X_r \longrightarrow \mathbb{C}^2/\mathbb{Z}_r$$

together give a resolution

$$\pi^\vee : {X_r}^{[n]} \longrightarrow S^n(\mathbb{C}^2/\mathbb{Z}_r)$$

which is semismall with respect to the above stratification, [Na-ICM]. For a point $y \in S^{n-k}_\lambda(\mathbb{C}^2/\mathbb{Z}_r)$, denote the fiber $(\pi^\vee)^{-1}(y)$ by ${F^\vee}^\lambda_k$. Then the decomposition theorem gives a graded vector space isomorphism

$$H^*({X_r}^{[n]}) = \bigoplus_{k,\lambda} IH^*(\overline{S^{n-k}_\lambda(\mathbb{C}^2/\mathbb{Z}_r)}) \otimes H^{top}({F^\vee}^\lambda_k)$$

12.2 The Resolution $\pi : \mathfrak{M}(r,n) \longrightarrow \mathfrak{M}_0(r,n)$

Let $\mathfrak{M}_0(r,n)$ denote the Uhlenbeck compactification of the moduli space of framed *locally-free* sheaves $\mathfrak{M}^{reg}(r,n)$ ([Nak99] chapter 3). $\mathfrak{M}_0(r,n)$ has a stratification

$$\mathfrak{M}_0(r,n) = \coprod_{0 \le k \le n \ \lambda \vdash n-k} \mathfrak{M}^k_\lambda$$

where

$$\mathfrak{M}^k_\lambda = \mathfrak{M}^{reg}(r,k) \times S^{n-k}_\lambda(\mathbb{C}^2),$$

and $\mathfrak{M}^{reg}(r,k)$ is the moduli space of framed locally-free sheaves on $\mathbb{P}^2$ with second Chern class $c_2 = k$. By a result of Baranovsky [Bar00], the resolution

$$\pi : \mathfrak{M}(r,n) \longrightarrow \mathfrak{M}_0(r,n)$$

$$\mathcal{E} \mapsto (\mathcal{E}^{\vee\vee}, supp(\mathcal{E}^{\vee\vee}/\mathcal{E}))$$

is semismall with respect to the above stratification. For $x \in \mathfrak{M}^k_\lambda$, let $\pi^{-1}(x) = F^\lambda_{n-k}$ denote the fiber of π over x. The decomposition theorem gives a graded vector space isomorphism

$$H^*(\mathfrak{M}(r,n)) = \bigoplus_{k,\lambda} IH^*(\overline{\mathfrak{M}^k_\lambda}) \otimes H^{top}(F^\lambda_{n-k}).$$

12.3 Numerical Symplectic Duality for $\mathfrak{M}(r, n)$ and ${X_r}^{[n]}$

We may now state the main result of this section. Recall that F^{λ}_{n-k} denotes the fiber of π over a point $x \in \mathfrak{M}^k_{\lambda}$, while ${F^{\vee}}^{\lambda}_{k}$ denotes the fiber of $\pi^{\vee}$ over a point $y \in S^{n-k}_{\lambda}(\mathbb{C}^2/\mathbb{Z}_r)$.

Theorem 8. *For all k, $0 \leq k \leq n$, and any partition $\lambda \vdash n-k$, we have*

$$dim\big(IH^*(\overline{S^{n-k}_{\lambda}(\mathbb{C}^2/\mathbb{Z}_r)})\big) = dim\big(H^{top}(F^{\lambda}_{n-k})\big)$$

and

$$dim\big(IH^*(\overline{\mathfrak{M}^k_{\lambda}})\big) = dim\big(H^{top}(F_k^{\lambda\vee})\big).$$

Thus, in the decomposition of the vector spaces $H^*(\mathfrak{M}(r, n))$ and $H^*({X_r}^{[n]})$ provided by the resolutions π and $\pi^{\vee}$, the dimensions of top fiber cohomology and total intersection cohomology are exchanged. In the rest of this section we will prove theorem 8.

Proposition 11. *There is an equality of generating functions*

$$\sum_{n=0}^{\infty} dim\big(H^*({X_r}^{[n]})\big)q^n = \sum_{n=0}^{\infty} dim\big(H^*(\mathfrak{M}(r, n))\big)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^r}$$

Proof. A Bialinicki-Birula cell decomposition of ${X_r}^{[n]}$ and $\mathfrak{M}(r, n)$ gives

$$dim\big(H^*({X_r}^{[n]})\big) = \#({X_r}^{[n]})^{T^{\vee}}$$

and

$$dim\big(H^*(\mathfrak{M}(r, n)) = \#(\mathfrak{M}(r, n))^{T}.$$

Since both fixed point sets are naturally identified with the set of r-tuples of partitions of total size n, the proposition follows from the standard generating function for partitions.

□

Proposition 12. *The generating function for the Uhlenbeck compactification is given by*

$$\sum_{n=0}^{\infty} dim\big(IH^*(\mathfrak{M}_0(r, n))\big)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{r-1}}$$

Proof. In [Bar00], Baranovsky computes the ratio of generating functions

$$\frac{\sum_{n=0}^{\infty} dim\big(H^*(\mathfrak{M}(r, n))\big)q^n}{\sum_{n=0}^{\infty} dim\big(IH^*(\mathfrak{M}_0(r, n))\big)q^n} = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)},$$

which, along with the previous proposition, gives the result. □

Proposition 13. *The generating function for the middle oridinary cohomology of the Hilbert scheme is given by*

$$\sum_{n=0}^{\infty} dim\big(H^{mid}({X_r}^{[n]})\big)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{r-1}}$$

Proof. This follows directly from a theorem of Göttsche, see [Nak99], which computes the generating function of graded Poincare polynomials for the Hilbert scheme. □

Let $F^\vee(n) = \pi^{\vee -1}(0)$ denote the central fiber of the resolution

$$\pi^\vee : {X_r}^{[n]} \longrightarrow S^n(\mathbb{C}^2/\mathbb{Z}_r).$$

Corollary 5. *For all $n \geq 0$,*

$$dim\big(H^{top}(F^\vee(n))\big) = dim\big(IH^*(\mathfrak{M}_0(r,n))\big)$$

Proof. It follows from Proposition 11 and Proposition 12 that

$$\sum_{n=0}^{\infty} dim\big(IH^*(\mathfrak{M}_0(r,n))\big)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{r-1}}$$

which, by the Proposition 13, implies that

$$\sum_{n=0}^{\infty} dim\big(H^{mid}({X_r}^{[n]})\big)q^n = \sum_{n=0}^{\infty} dim\big(IH^*(\mathfrak{M}_0(r,n))\big)q^n,$$

i.e.

$$dim\big(H^{mid}({X_r}^{[n]})\big) = dim\big(IH^*(\mathfrak{M}_0(r,n))\big).$$

Since $F^\vee(n)$ is a deformation retract of ${X_r}^{[n]}$ and

$$dim(F^\vee(n)) = \frac{1}{2} dim({X_r}^{[n]}),$$

it follows that

$$H^{top}(F^\vee(n)) \simeq H^{mid}({X_r}^{[n]}),$$

which proves the corollary. □

We can now prove Theorem 8.

Proof. We first show that

$$dim\big(IH^*(\overline{S_\lambda^{n-k}(\mathbb{C}^2/\mathbb{Z}_r)})\big) = dim\big(H^{top}(F_{n-k}^\lambda)\big) \simeq \mathbb{C}.$$

Let $S^\lambda(\mathbb{C}^2/\mathbb{Z}_r) = S^{\nu_1}(\mathbb{C}^2/\mathbb{Z}_r) \times \ldots \times S^{\nu_n}(\mathbb{C}^2/\mathbb{Z}_r)$, where $\lambda = (1^{\nu_1}2^{\nu_2}\ldots n^{\nu_n}) \vdash k$. The map

$$\kappa : S^\lambda(\mathbb{C}^2/\mathbb{Z}_r) \longrightarrow \overline{S_\lambda^{n-k}(\mathbb{C}^2/\mathbb{Z}_r)}$$

$$\kappa(C_1, \ldots, C_n) = \sum_j jC_j$$

is finite and birational [Nak99], and respects the natural stratifications on both varieties. Thus,

$$IH^*(\overline{S_\lambda^{n-k}(\mathbb{C}^2/\mathbb{Z}_r)} \simeq IH^*(S^\lambda(\mathbb{C}^2/\mathbb{Z}_r) \simeq \mathbb{C}.$$

On the other hand, F_{n-k}^λ is irreducible, [Bar00], so

$$H^{top}(F_{n-k}^\lambda) \simeq \mathbb{C}$$

as well, proving the first half of Theorem 8.

Now we will show that
$$dim\big(IH^*(\overline{\mathfrak{M}^k_\lambda})\big) = dim\big(H^{top}(F^{\lambda\vee}_k)\big).$$
Once again, there is a finite birational morphism respecting the induced stratifications [Bar00]
$$\kappa : \mathfrak{M}_0(r,k) \times S^\lambda(\mathbb{C}^2) \longrightarrow \overline{\mathfrak{M}^k_\lambda}.$$
so that
$$dim\big(IH^*(\overline{\mathfrak{M}^k_\lambda})\big) = dim\big(IH^*(\mathfrak{M}_0(r,k))\big).$$

On the other hand, ${F^\vee}^\lambda_k$, the fiber over $x \in S^{n-k}_\lambda(\mathbb{C}^2/\mathbb{Z}_r)$, is isomorphic to $F^\vee(k) \times Q$, where Q is irreducible, and $F^\vee(k)$ is the central fiber of the resolution
$$\pi^\vee : {X_r}^{[k]} \longrightarrow S^k(\mathbb{C}^2/\mathbb{Z}_r).$$
Since Q is irreducible,
$$dim\big(H^{top}({F^\vee}^\lambda_k)\big) = dim\big(H^{top}(F^\vee(k))\big).$$
Thus, the second half of Theorem 8 follows from the equality
$$dim\big(IH^*(\mathfrak{M}_0(r,k))\big) = dim\big(H^{top}(F^\vee(k))\big)$$
of Corollary 5.

□

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