

MATH 53 MIDTERM EXAM 2

Name:

Student ID Number:

TA Section Number:

Signature:

1. INSTRUCTIONS

- Print your name, student ID number, TA name and section number.
- Sign the signature line to indicate that you accept the Honor Code.
- There are 13 problems on this exam, and most of them have multiple parts- so it's a long exam. You are not expected to solve all of the problems.
- Please check that the your exam has all of the pages, and is stapled correctly.
- This is a 2 hour, closed book, closed notes exam. No calculators or electronic computational devices are allowed.
- If you need extra space to write your solutions you may write on the back of the pages. If you use extra paper, make sure to write your name on the extra pages. **Unless explicitly directed otherwise, you should justify all of your answers.**

1	2	3	4	5	6	7

8	9	10	11	12	13

(1) (a) (5 points) Let

$$\mathbf{x}_1 = \begin{pmatrix} 5 \cos\left(\frac{3t}{2}\right) \\ 2 \cos\left(\frac{3t}{2}\right) + 6 \sin\left(\frac{3t}{2}\right) \end{pmatrix}, \text{ and } \mathbf{x}_2 = \begin{pmatrix} 5 \sin\left(\frac{3t}{2}\right) \\ 2 \sin\left(\frac{3t}{2}\right) - 6 \cos\left(\frac{3t}{2}\right) \end{pmatrix}.$$

Are \mathbf{x}_1 and \mathbf{x}_2 a set of fundamental solutions to the differential equation

$$\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{pmatrix} \mathbf{x}?$$

Solution. First, we compute the Wronskian:

$$\det \begin{pmatrix} 5 \cos\left(\frac{3t}{2}\right) & 5 \sin\left(\frac{3t}{2}\right) \\ 2 \cos\left(\frac{3t}{2}\right) + 6 \sin\left(\frac{3t}{2}\right) & 2 \sin\left(\frac{3t}{2}\right) - 6 \cos\left(\frac{3t}{2}\right) \end{pmatrix} = -30 \neq 0.$$

Next, we check that x_1 and x_2 are solutions of the given system. It holds:

$$\begin{aligned} x_1'(t) &= \begin{pmatrix} 5 \cos\left(\frac{3t}{2}\right) \\ 2 \cos\left(\frac{3t}{2}\right) + 6 \sin\left(\frac{3t}{2}\right) \end{pmatrix}' = \begin{pmatrix} -\frac{15}{2} \sin\left(\frac{3t}{2}\right) \\ -3 \sin\left(\frac{3t}{2}\right) + 9 \cos\left(\frac{3t}{2}\right) \end{pmatrix}, \\ x_2'(t) &= \begin{pmatrix} 5 \sin\left(\frac{3t}{2}\right) \\ 2 \sin\left(\frac{3t}{2}\right) - 6 \cos\left(\frac{3t}{2}\right) \end{pmatrix}' = \begin{pmatrix} \frac{15}{2} \cos\left(\frac{3t}{2}\right) \\ 3 \cos\left(\frac{3t}{2}\right) + 9 \sin\left(\frac{3t}{2}\right) \end{pmatrix}. \end{aligned}$$

In addition, we have:

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{pmatrix} x_1(t) &= \begin{pmatrix} -\frac{15}{2} \sin\left(\frac{3t}{2}\right) \\ -3 \sin\left(\frac{3t}{2}\right) + 9 \cos\left(\frac{3t}{2}\right) \end{pmatrix}, \\ \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{pmatrix} x_2(t) &= \begin{pmatrix} \frac{15}{2} \cos\left(\frac{3t}{2}\right) \\ 3 \cos\left(\frac{3t}{2}\right) + 9 \sin\left(\frac{3t}{2}\right) \end{pmatrix}. \end{aligned}$$

This shows that x_1 and x_2 solve the system and the fact that the Wronskian is non-zero shows that they form a fundamental set of solutions.

(b) (5 points) Compute the matrix exponential e^{At} for

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{pmatrix}.$$

Solution. From (a) we have

$$\begin{aligned}
 e^{At} &= (x_1(t) \ x_2(t))(x_1(0) \ x_2(0))^{-1} \\
 &= \begin{pmatrix} 5 \cos\left(\frac{3t}{2}\right) & 5 \sin\left(\frac{3t}{2}\right) \\ 2 \cos\left(\frac{3t}{2}\right) + 6 \sin\left(\frac{3t}{2}\right) & 2 \sin\left(\frac{3t}{2}\right) - 6 \cos\left(\frac{3t}{2}\right) \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 2 & -6 \end{pmatrix}^{-1} \\
 &= \begin{pmatrix} 5 \cos\left(\frac{3t}{2}\right) & 5 \sin\left(\frac{3t}{2}\right) \\ 2 \cos\left(\frac{3t}{2}\right) + 6 \sin\left(\frac{3t}{2}\right) & 2 \sin\left(\frac{3t}{2}\right) - 6 \cos\left(\frac{3t}{2}\right) \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{1}{15} & -\frac{1}{6} \end{pmatrix} \\
 &= \begin{pmatrix} \cos\left(\frac{3t}{2}\right) + \frac{1}{3} \sin\left(\frac{3t}{2}\right) & -\frac{5}{6} \sin\left(\frac{3t}{2}\right) \\ \frac{4}{3} \sin\left(\frac{3t}{2}\right) & -\frac{1}{3} \sin\left(\frac{3t}{2}\right) + \cos\left(\frac{3t}{2}\right) \end{pmatrix}.
 \end{aligned}$$

- (2) (a) (5 points) Write down a system of two linear differential equations in two unknown functions and find the general solution.

Solution: Here, you had total freedom of what to write down. Hence, writing down a simple system of differential equations was a good idea. For example, you could choose

$$\begin{aligned}x' &= x \\y' &= 2y\end{aligned}$$

which can be solved in two ways. You can recognize that the first equation only involves x , and the second equation only involves y , and that the solution to $x' = x$ is $x(t) = c_1 e^t$, and the solution to $y' = y$ is $y(t) = c_2 e^{2t}$. Therefore, the solution to this is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$$

You can also write this down as the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$$

and as usual, find the eigenvalues and eigenvectors of the matrix. Since

$$\det \begin{pmatrix} 1 - \lambda & 0 \\ 0 & 2 - \lambda \end{pmatrix} = (1 - \lambda)(2 - \lambda)$$

we see that the eigenvalues are 1 and 2. Therefore, solving for the eigenvectors, we get that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

are eigenvectors with eigenvalues 1 and 2, respectively. Therefore, the general solution to our system is

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$$

Note that both methods of solving yielded the same solution (as they very well should have!)

The moral of the story: When you are asked to write down a system of linear equations, or anything else: make life easy for yourself! Write down something that's not difficult to solve.

- (b) (5 points) Write down a second order linear differential equation and find the general solution.

Solution: Again, we want to pick a second order differential equation which is easy to solve. Therefore, instead of picking coefficients for the

differential equation at the beginning, we will let them be variables until we can decide which coefficients will be simplest to deal with. Accordingly, consider the linear differential equation

$$y'' + ay' + by = 0$$

where a and b are constants. Using our usual procedure for converting linear differential equations into systems of DEs, we let

$$x_1 = y, x_2 = y'$$

and therefore get the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= y'' = -ax_2 - bx_1 \end{aligned}$$

using the fact that $y'' + ay' + by = 0$, and therefore, $y'' = -ay' - by = -ax_2 - bx_1$. This system of differential equations is precisely

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \mathbf{x}$$

The characteristic polynomial of this matrix is

$$\det \begin{pmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{pmatrix} = \lambda^2 + a\lambda + b$$

Since we will need to use the eigenvalues of this differential equations, let us pick a and b so the above characteristic polynomial has simple roots. For our example, we will pick a and b so that our characteristic polynomial has roots 1 and 2. Since

$$(\lambda - 1)(\lambda - 2) = \lambda^2 - 3\lambda + 2$$

we will pick $a = -3$, and $b = 2$. Thus, we will be solving the differential equation

$$y'' - 3y' + 2y = 0$$

whose corresponding matrix is $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

Now, we already know that the eigenvalues of this matrix are 1 and 2. Therefore, let us find the eigenvectors for these eigenvalues.

$\lambda = 1$: We solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Thus, we have

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

and row-reducing, we get

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

from which we get $v_1 - v_2 = 0$, so $v_1 = v_2$ and therefore

$$\mathbf{v} = \begin{pmatrix} v_2 \\ v_2 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Therefore, from $\lambda = 1$ we get the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = 2$: Again, we solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Thus, we get

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

and row-reducing, we have

$$\begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

from which we get $v_1 - \frac{v_2}{2} = 0$, so $v_2 = 2v_1$ and therefore

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Therefore, from $\lambda = 2$ we get the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus, let us combine this information. From $\lambda = 1$, and its corresponding eigenvector, we get the solution

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and from $\lambda = 2$, we get the solution.

$$\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We know from class that this is a fundamental set of solutions. Therefore, the general solution to our system of DEs is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{2t} \\ e^t + 2c_2 e^{2t} \end{pmatrix}$$

Note, however, that we aren't done! This is the general solution to the system of DEs

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} \mathbf{x}$$

not to our original differential equation $y'' - 3y' + 2y = 0$. To find this, we need to do one more thing: recall that we defined $x_1 = y$ when converting our second order differential equation into a system of linear DEs.

Therefore, to find y , we just need to take the first coordinate of \mathbf{x} . Thus, the general solution to our original differential equation is

$$y(t) = c_1 e^t + c_2 e^{2t}$$

Note: A lot of people lost marks by not doing the last step. Therefore, you should always make sure that your answer to the problem answer the actual question being asked, and doesn't just solve some intermediate (if very important!) step.

- (3) In each of the parts of the following problem, you should
- construct a *real* 2 by 2 matrix A with the required properties, and then
 - write down the general *real* solution to the associated differential equation

$$\mathbf{x}' = A\mathbf{x}$$

using the matrix A you construct.

(Recall that a matrix A is *real* when the entries of A are real numbers, and a solution \mathbf{x} is *real* if all the component function of $\mathbf{x}(t)$ are real-valued.)

- (a) (10 points) All entries of A should be non-zero, and the eigenvalues of A should be equal to 1 and 2.

Solution. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}.$$

This matrix satisfies the required properties. By construction, it has eigenvalues 1, 2 with respective eigenspaces

$$E_1 = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_2 = \text{span} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Hence, the general solution to the associated differential equation is

$$c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- (b) (10 points) All entries of A should be non-zero, and $1 + i$ should be one of the eigenvalues of A .

Solution. Since A is a real, the characteristic polynomial of A is real. Hence, the roots come in conjugate pairs and $1 - i$ is also necessarily a root of the characteristic polynomial. Hence, A has characteristic polynomial

$$(1 + i - \lambda)(1 - i - \lambda) = \lambda^2 + 2\lambda + 2.$$

Thus, we need to write down a matrix A with this characteristic polynomial (i.e A should have trace equal to 2 and determinant equal to -2). In particular,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is an example of a matrix with the desired properties.

By a simple row reduction, we compute that $E_{1+i} = \text{span} \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Hence, a general solution to the associated differential equation is given by

$$c_1 \text{Re}\{e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}\} + c_2 \text{Im}\{e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}\} = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

- (c) (10 points) A should *not* be diagonalizable, and 3 should be an eigenvalue of A .

Solution. $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

is an example of such a matrix. $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A , and

$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a generalized eigenvector satisfying

$$(A - 3I)\mathbf{w} = \mathbf{v}.$$

Thus, the general solution of the associated differential equation $\mathbf{x}' = A\mathbf{x}$ is given by

$$c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

(4) (a) (10 points) Find the general solution to the differential equation

$$\mathbf{x}' = \begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix} \mathbf{x}.$$

Solution: As usual, let us first find the eigenvalues. We have

$$\begin{aligned} \det \begin{pmatrix} -3 - \lambda & -1 \\ 1 & -5 - \lambda \end{pmatrix} &= (-3 - \lambda)(-5 - \lambda) + 1 \\ &= \lambda^2 + 8\lambda + 16 \\ &= (\lambda + 4)^2 \end{aligned}$$

Thus, the only eigenvalue is $\lambda = -4$. Let us find the eigenvectors for this eigenvalue. We solve $(A + 4I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

and row-reducing, we get

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

Thus, we get $v_1 - v_2 = 0$, and therefore $v_1 = v_2$. Therefore,

$$\mathbf{v} = \begin{pmatrix} v_2 \\ v_2 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Therefore, let our eigenvector \mathbf{v} be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Since we only found one eigenvector for our eigenvalue, we need to find the generalized eigenvector \mathbf{w} . We solve $(A + 4I)\mathbf{w} = \mathbf{v}$:

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, we get the equations

$$w_1 - w_2 = 1$$

$$w_1 - w_2 = 1$$

which clearly reduce to $w_1 = w_2 + 1$. Letting $w_2 = 0$, since we are allowed to choose a particular \mathbf{w} , we get

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now, from lecture we know that

$$\mathbf{x}_1(t) = e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{x}_2(t) = te^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a set of fundamental solutions to our system of differential equations. Thus, the general solution to our system is

$$\begin{aligned}\mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \\ &= c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)\end{aligned}$$

(b) (5 points) How do the solutions behave as $t \rightarrow \infty$?

Solution: We know that $\lim_{t \rightarrow \infty} e^{-4t} = 0$. Furthermore, since

$$\lim_{t \rightarrow \infty} t = \infty, \quad \lim_{t \rightarrow \infty} e^{4t} = \infty$$

we can use L'Hopital's rule to get that

$$\lim_{t \rightarrow \infty} t e^{-4t} = \lim_{t \rightarrow \infty} \frac{t}{e^{4t}} = \lim_{t \rightarrow \infty} \frac{1}{4e^{4t}} = 0$$

Thus, since

$$\lim_{t \rightarrow \infty} e^{-4t} = \lim_{t \rightarrow \infty} t e^{-4t} = 0$$

and those are the terms that appear in our general solution above, all the solutions approach the origin.

Note: In general, a negative exponential always shrinks faster than a polynomial grows (you can use this information without proving it on exams, if you like.) Using the same reasoning, we could show that

$$\lim_{t \rightarrow \infty} t^{20} e^{-0.005t} = 0$$

even though certainly for some values of t , the quantity $t^{20} e^{-0.005t}$ can be quite large.

(c) (5 points) Find the solution to part (a) that satisfies the initial value

$$\mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Solution: We solve for c_1 and c_2 . If

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(t e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

then $\mathbf{x}(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 \end{pmatrix}$. Thus, we get that

$$\begin{pmatrix} c_1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

and so

$$\begin{aligned}c_1 + c_2 &= 3 \\ c_1 &= 4\end{aligned}$$

Thus, solving, we get $c_1 = 4$, $c_2 = -1$, so

$$\begin{aligned}\mathbf{x}(t) &= 4e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left(te^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3e^{-4t} - te^{-4t} \\ 4e^{-4t} - te^{-4t} \end{pmatrix}\end{aligned}$$

Note: You might have gotten a different eigenvector \mathbf{v} or a different generalized eigenvector \mathbf{w} in part (a), but your solution to (c) after simplification should be the same as the one above.

- (5) (a) (5 points) Let A be an n by n matrix. Give the definition of exponential e^{At} .

Solution: The definition is

$$e^{At} = I_n + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Notes: There were a number of common mistakes in giving the definition. First of all, while it is true that $e^{At} = X(t)X^{-1}(0)$, where $X(t)$ is a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$, that is not the *definition* of e^{At} . This happens to be a fact we derived from the definition given above.

Secondly, make sure that your definition involves an infinite sum! This is an infinite series, and does not 'stop' at any n . Therefore, writing that

$$e^{At} = I_n + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^nt^n}{n!}$$

or that

$$e^{At} = \sum_{k=0}^n \frac{A^k t^k}{k!}$$

is incorrect.

- (b) (5 points) Using the definition, compute e^{At} when

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution: We have that

$$e^{At} = I_n + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Thus, to calculate e^{At} from the definition, we need to know what A^k is for every k . As shown many times in class, we have

$$A^k = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}$$

Therefore,

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{2^k t^k}{k!} & 0 \\ 0 & \frac{3^k t^k}{k!} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{2^k t^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{3^k t^k}{k!} \end{pmatrix} \end{aligned}$$

Now, recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus,

$$\sum_{k=0}^{\infty} \frac{2^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} = e^{2t}$$

and similarly, $\sum_{k=0}^{\infty} \frac{3^k t^k}{k!} = e^{3t}$. Thus, we see that

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

Note: If sigma notation confuses you, I was perfectly happy to accept answers that just used sums and '...', starting with

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} + \frac{t^3}{3!} \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix} + \dots$$

and proceeding similarly.

(c) (5 points) Using the definition, compute e^{At} when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Solution: Again, to apply the definition, we need to figure out A^k for every k . Since this has not been used extensively in class, we will have to prove that our answer is correct. Note that

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

so we guess that

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Let us now prove this by induction. From above, we see that the formula holds for $k = 0$. Now, assume the formula holds for k . Then,

$$A^{k+1} = A^k A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$$

we see that the formula holds for $k + 1$, so we're done. Thus,

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{t^k}{k!} & \frac{kt^k}{k!} \\ 0 & \frac{t^k}{k!} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{kt^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{pmatrix} \end{aligned}$$

Now,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots = e^t$$

Furthermore, for $k \neq 0$, $k \cdot (k-1)! = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1 = k!$, and so $\frac{kt^k}{k!} = \frac{t^k}{(k-1)!}$. For $k = 0$, we have $0! = 1$, and therefore:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{kt^k}{k!} &= 0 + t + \frac{2t^2}{2!} + \frac{3t^3}{3!} + \frac{4t^4}{4!} + \dots \\ &= t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \dots \\ &= t \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \\ &= te^t \end{aligned}$$

Thus, we have

$$e^{At} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

(d) (5 points) Compute e^{At} (using the definition or otherwise) when

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Solution: Since we're not told to do this by definition, there are two approaches we can take. We can either find a formula for A^k for a general k , or we can use the fundamental matrix approach.

Method 1: Let us first figure out e^{At} from definition. This method is definitely trickier than Method 2 below: in general, I would recommend that unless you are told to find e^{At} from the definition, you should use other methods. Nevertheless, some people tried doing it this way, so I'll present it; however, if you just want to see a solution to the question, skip to Method 2.

That being said, let us figure out what A^k is for any k . Let's try the first few k .

$$\begin{aligned} A^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ A^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ A^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A^3 &= A^2 A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ A^4 &= A^3 A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, we see that $A^4 = I_2$, the 2×2 identity matrix. Therefore, we see that if $k = 4j + x$, then

$$A^k = A^{4j+x} = (A^4)^j \cdot A^x = I_2^j \cdot A^x = A^x$$

Thus, we see that all that matters for A^k is the remainder of k when divided by 4, so we conclude:

$$A^k = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & k = 4j \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & k = 4j + 1 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & k = 4j + 2 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & k = 4j + 3 \end{cases}$$

Thus,

$$\begin{aligned} e^{At} &= I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \frac{A^5 t^5}{5!} + \frac{A^6 t^6}{6!} + \dots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 1 \end{pmatrix} + \begin{pmatrix} \frac{-t^2}{2!} & 0 \\ 0 & \frac{-t^2}{2!} \end{pmatrix} + \begin{pmatrix} 0 & \frac{-t^3}{3!} \\ \frac{t^3}{3!} & 0 \end{pmatrix} + \begin{pmatrix} \frac{t^4}{4!} & 0 \\ 0 & \frac{t^4}{4!} \end{pmatrix} + \dots \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \end{pmatrix} \end{aligned}$$

Now, note that

$$1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}$$

is the Taylor series for $\cos t$. Similarly,

$$t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$$

is the Taylor series for $\sin t$. Thus, looking at the entries we got for e^{At} , we see that

$$e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Method 2: The less tricky (and actually more straightforward) way of doing this question uses fundamental matrices. Recall that if $X(t)$ is a fundamental matrix for the differential equation $\mathbf{x}' = A\mathbf{x}$, then $e^{At} = X(t)X^{-1}(0)$. Therefore, let us find a fundamental matrix for the differential equation

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

As usual, let us find the eigenvalues for this matrix. We have

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

Thus, if $\lambda^2 + 1 = 0$,

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i$$

Therefore, the eigenvalues are $\pm i$. Since the eigenvalues are complex, it suffices to find an eigenvector \mathbf{v} for i , and then split $e^{it}\mathbf{v}$ into real and complex parts. Let us find an eigenvector for i . We solve $(A - iI)\mathbf{v} = \mathbf{0}$. Thus, we get

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \mathbf{v} = \mathbf{0}$$

and row-reducing, noting that i times the first row is $(1 \ i)$, we get the following:

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

Therefore, we have

$$v_1 + iv_2 = 0 \Rightarrow v_1 = -iv_2$$

and so we get that

$$\mathbf{v} = \begin{pmatrix} -iv_2 \\ v_2 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

Thus, we take the eigenvector \mathbf{v} to be $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. Now, using that

$$e^{it} = \cos t + i \sin t$$

we get

$$\begin{aligned} e^{it}\mathbf{v} &= (\cos t + i \sin t) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} \sin t - i \cos t \\ \cos t + i \sin t \end{pmatrix} \\ &= \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ i \sin t \end{pmatrix} \end{aligned}$$

Therefore, splitting this up into real and imaginary parts, we get that

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \mathbf{x}_2(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is a set of fundamental solutions for $\mathbf{x}' = A\mathbf{x}$, using what we learned in lecture. Thus,

$$X(t) = \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix}$$

is a fundamental matrix for our system of equations. Thus,

$$\begin{aligned} e^{At} &= X(t)X^{-1}(0) = \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \end{aligned}$$

(e) (5 points) Suppose that $A = TBT^{-1}$. Show that

$$e^{At} = Te^{Bt}T^{-1}.$$

Solution: Note that

$$A^k = (TBT^{-1})(TBT^{-1}) \cdots (TBT^{-1}) = TB^kT^{-1}$$

since all the $T^{-1}T$ in the middle cancel out. Thus,

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots \\ &= TT^{-1} + TBT^{-1}t + \frac{TB^2T^{-1}t^2}{2!} + \frac{TB^3T^{-1}t^3}{3!} + \frac{TB^4T^{-1}t^4}{4!} + \dots \\ &= TT^{-1} + TBT^{-1} + T\frac{B^2t^2}{2!}T^{-1} + T\frac{B^3t^3}{3!}T^{-1} + T\frac{B^4t^4}{4!}T^{-1} + \dots \\ &= T(I + Bt + \frac{B^2t^2}{2!} + \frac{B^3t^3}{3!} + \frac{B^4t^4}{4!} + \dots)T^{-1} \\ &= Te^{Bt}T^{-1} \end{aligned}$$

(f) (5 points) Compute e^{At} when

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}^{-1}.$$

Solution. Given the form we were given for A , we should use part (e). We have that

$$e^{At} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} e^{Bt} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}^{-1}$$

where $B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. Since B is diagonal, we have

$$\begin{aligned} e^{At} &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 4/9 & 1/9 \\ -1/9 & 2/9 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{3t} & -e^{4t} \\ e^{3t} & 4e^{4t} \end{pmatrix} \begin{pmatrix} 4/9 & 1/9 \\ -1/9 & 2/9 \end{pmatrix} \\ &= \begin{pmatrix} \frac{8}{9}e^{3t} + \frac{1}{9}e^{4t} & \frac{2}{9}e^{3t} - \frac{2}{9}e^{4t} \\ \frac{4}{9}e^{3t} - \frac{4}{9}e^{4t} & \frac{1}{9}e^{3t} + \frac{8}{9}e^{4t} \end{pmatrix} \end{aligned}$$

Note: Remember that for invertible 2×2 matrices,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

A lot of people messed up the calculation in (f), using the algorithm for finding A^{-1} by row-reducing next to the identity matrix. This formula saves a LOT of work!

- (6) (a) (5 points) Let A be an n by n matrix. Let \mathbf{v}_1 be a non-zero eigenvector of A with eigenvalue λ_1 , and let \mathbf{v}_2 be a non-zero eigenvector of A with eigenvalue λ_2 . Suppose that $\lambda_1 \neq \lambda_2$. Show that \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.

Solution. Suppose

$$(1) \quad c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0}.$$

We must show that $c_1 = c_2 = 0$. Note that

$$(A - \lambda_1 I) \mathbf{v}_1 = \mathbf{0} \text{ and } (A - \lambda_1 I) \mathbf{v}_2 = (\lambda_2 - \lambda_1) \mathbf{v}_2$$

since \mathbf{v}_1 is an eigenvector of A with eigenvalue λ_1 , and \mathbf{v}_2 is an eigenvector of A with eigenvalue λ_2 . Thus, applying $(A - \lambda_1 I)$ on the left to both sides of equation 1, we obtain

$$(A - \lambda_1 I)(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_2(\lambda_2 - \lambda_1) \mathbf{v}_2 = \mathbf{0}.$$

Now \mathbf{v}_2 is a non-zero vector, and by assumption $\lambda_2 - \lambda_1$ is a non-zero scalar. Thus $c_2 = 0$, and equation 1 reduces to

$$c_1 \mathbf{v}_1 = \mathbf{0}.$$

Thus $c_1 = 0$, too, as desired. \square

- (b) (5 points) Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly independent *generalized* eigenvectors of A corresponding to a multiplicity k eigenvalue λ_1 . (Recall that, by definition, this means that

$$(A - \lambda_1 I)^k \mathbf{v}_l = \mathbf{0}$$

for $l = 1, \dots, k$.) Let \mathbf{w} be a non-zero eigenvector of A with eigenvalue λ_2 , and suppose that $\lambda_1 \neq \lambda_2$. Show that the $k + 1$ vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}$ are linearly independent.

Solution. Suppose

$$(2) \quad c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k + c_{k+1} \mathbf{w} = \mathbf{0}.$$

We must show that $c_1 = c_2 = \dots = c_k = c_{k+1} = 0$. Note that the vectors

$$(A - \lambda_1 I)^k \mathbf{v}_1, \dots, (A - \lambda_1 I)^k \mathbf{v}_k$$

are all $\mathbf{0}$, while

$$(A - \lambda_1 I)^k \mathbf{w} = (\lambda_2 - \lambda_1)^k \mathbf{w},$$

(check to see that you understand this last line- many of you missed it on the exam). Thus, applying $(A - \lambda_1 I)^k$ on the left of both sides of equation 2, we are left with

$$c_{k+1}(\lambda_2 - \lambda_1)^k \mathbf{w} = \mathbf{0}.$$

Since \mathbf{w} is a non-zero vector and $\lambda_2 - \lambda_1$ is a non-zero scalar, we conclude that $(\lambda_2 - \lambda_1)^k \mathbf{w}$ is a non-zero vector, and thus that $c_{k+1} = 0$. Now equation 2 becomes

$$c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}.$$

But, by assumption, the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent. Thus $c_1 = c_2 = \dots = c_k = 0$ as well. \square

(7) (a) (5 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be the following vectors:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Write down a system of differential equations $\mathbf{x}' = A\mathbf{x}$ whose general solution is

$$c_1 e^t \mathbf{u} + c_2 e^{2t} \mathbf{v} + c_3 e^{-3t} \mathbf{w}.$$

Solution. The matrix A has to satisfy the equations $A\mathbf{u} = \mathbf{u}$, $A\mathbf{v} = 2\mathbf{v}$, $A\mathbf{w} = -3\mathbf{w}$. The first two equations mean that the first column of A is \mathbf{u} and the second column of A is $2\mathbf{v}$. The third column of our matrix is

given by $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

we have

$$A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ -3 \end{pmatrix}.$$

Thus we have

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 2 & -5 \\ 0 & 0 & -3 \end{pmatrix}.$$

(b) (5 points) Solve the initial value problem

$$(\mathbf{x})' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\mathbf{x}' = A\mathbf{x}$ is the system of differential equations found in part (a). (Note: you don't necessarily have to have found the matrix A in order to solve this part.)

Solution. We need to solve $\mathbf{x}(0) = c_1 \mathbf{u} + c_2 \mathbf{v} + c_3 \mathbf{w}$ for the constants c_1, c_2, c_3 . We get the equations $c_1 + c_3 = 0$, $c_2 + c_3 = 0$, $c_3 = 1$. Hence, $c_1 = c_2 = -1$, $c_3 = 1$, so the solution is $-\mathbf{u} - \mathbf{v} + \mathbf{w}$.

(c) (5 points) Compute the matrix e^{At} for the matrix A which occurs in the solution of part (a) (Note: you don't necessarily have to have found the matrix A in order to solve this part.)

Solution. We have

$$\begin{aligned} e^{At} &= X(t)X(0)^{-1} = (e^t \mathbf{u} \ e^{2t} \mathbf{v} \ e^{-3t} \mathbf{w})(\mathbf{u} \ \mathbf{v} \ \mathbf{w})^{-1} \\ &= (e^t \mathbf{u} \ e^{2t} \mathbf{v} \ e^{-3t} \mathbf{w}) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} e^t & 0 & -e^t + e^{-3t} \\ 0 & e^{2t} & -e^{2t} + e^{-3t} \\ 0 & 0 & e^{-3t} \end{pmatrix} \end{aligned}$$

Note: In the above expressions we have denoted several 3 by 3 matrices in shorthand by writing them as a vector of columns. Thus $(\mathbf{u} \ \mathbf{v} \ \mathbf{w})$ denotes the 3 by 3 matrix whose first column is \mathbf{u} , second column is \mathbf{v} and third column is \mathbf{w} .

(more space)

(8) (10 points) Is it possible that

$$e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, te^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are all solutions to the equation $\mathbf{x}' = A\mathbf{x}$ for some 3 by 3 real matrix A ? (Hint: Think about the content of problem 6.)

Solution. Let's assume that

$$x_1 = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = e^t \left(t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right), x_3 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are all solutions of the differential equation $x' = Ax$ for some 3×3 matrix A . We will derive a contradiction.

Plugging x_1 into the differential equation it satisfies, we see that

$$x_1' = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Ax_1 = A \left(e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = e^t A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Dividing both sides by e^{2t} (which is non-zero for all t), we get that

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

i.e. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1.

A completely analogous argument proves that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of

A with eigenvalue 2.

Now we plug x_2 into the differential equation and see that

$$e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + te^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x_2' = Ax_2 = te^t A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\text{Since } A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

we may simplify the above equation to give

$$e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^t A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Dividing through by the non-zero quantity e^t , we see that

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (A - I) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,

$$(A - I)^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (A - I) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

and so $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

is a generalized eigenvector of A for the eigenvalue 1.

We know, however, (see problem 6) that the generalized 1-eigenspace must be linearly independent of the 2-eigenspace. But the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are not linearly independent:

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0.$$

This contradicts the linear independence of the generalized 1 and 2 eigenspaces! Hence, our original assumption was incorrect, and x_1, x_2, x_3 cannot all be solutions to a differential equation of the form $x' = Ax$.

(9) Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

when

(a) (7 points)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Solution.

$$x(t) = e^{At}x(0) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) (7 points)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Solution.

$$x(t) = e^{At} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e^{At} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{At} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{At} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

both lie in the 2-eigenspace of A . Hence,

$$e^{At} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$e^{At} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Also, since $(A - 2I)^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (A - 2I) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$, we can compute the third summand easily just using the definition of e^{At} :

$$e^{At} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t} e^{t(A-2I)} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t} \left[I \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t(A-2I) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = e^{2t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

Hence,

$$x(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e^{2t} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

(c) (7 points)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution. We can write $A = \begin{pmatrix} M & 0 \\ 0 & 2 \end{pmatrix}$

where M denotes the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus,

$$e^{At} = \begin{pmatrix} e^{Mt} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

But we already computed (problem 5 (d)):

$$e^{Mt} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence,

$$x(t) = e^{At} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ -\sin t + \cos t \\ e^{2t} \end{pmatrix}.$$

- (10) (a) (5 points) Consider the system of linear differential equations given by

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t)$$

Prove that if $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ are solutions to the above system of differential equations, then

$$\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_n\mathbf{x}_n(t)$$

is also a solution.

Solution. We plug \mathbf{y} into the differential equation and get:

$$\begin{aligned} \mathbf{y}'(t) &= (c_1\mathbf{x}_1(t) + \dots + c_n\mathbf{x}_n(t))' = c_1\mathbf{x}'_1(t) + \dots + c_n\mathbf{x}'_n(t) \\ &= c_1P(t)\mathbf{x}_1(t) + \dots + c_nP(t)\mathbf{x}_n(t) = P(t)c_1\mathbf{x}_1(t) + \dots + P(t)c_n\mathbf{x}_n(t) \\ &= P(t)\mathbf{y}(t). \end{aligned}$$

This shows that \mathbf{y} is a solution of the differential equation.

- (b) (5 points) Let \mathbf{x}_0 be a solution to

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t).$$

Assuming the result in part (a), show that the set of all solutions to

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t)$$

is the set

$$\{\mathbf{x} + \mathbf{x}_0 \mid \mathbf{x} \text{ is a solution to } \mathbf{x}'(t) = P(t)\mathbf{x}(t)\}.$$

Solution. First, we show that each $\mathbf{x} + \mathbf{x}_0$ is a solution to the inhomogeneous system:

$$\begin{aligned} (\mathbf{x}(t) + \mathbf{x}_0(t))' &= \mathbf{x}'(t) + \mathbf{x}'_0(t) = P(t)\mathbf{x}(t) + P(t)\mathbf{x}_0(t) + \mathbf{g}(t) \\ &= P(t)(\mathbf{x}(t) + \mathbf{x}_0(t)) + \mathbf{g}(t). \end{aligned}$$

Finally, we show that each solution of the inhomogeneous system is of the form $\mathbf{x} + \mathbf{x}_0$. To this end, let \mathbf{y} be a solution of the inhomogeneous system and let $\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{x}_0(t)$. Then

$$\begin{aligned} \mathbf{x}'(t) &= (\mathbf{y}(t) - \mathbf{x}_0(t))' = P(t)\mathbf{y}(t) + \mathbf{g}(t) - P(t)\mathbf{x}_0(t) - \mathbf{g}(t) \\ &= P(t)(\mathbf{y}(t) - \mathbf{x}_0(t)) = P(t)\mathbf{x}(t), \end{aligned}$$

so \mathbf{x} solves the homogeneous system, q.e.d.

- (11) (a) (5 points) Rewrite the following third order linear equation as a system of linear equations:

$$y'' + 5y' + 6y = 0.$$

Solution. Setting $x_1 = y, x_2 = y'$, we get the 2 by 2 system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{x}.$$

□

- (b) (5 points) Find the general solution to the system of linear equations you wrote down in part (a).

Solution. The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$$

is

$$\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3).$$

Thus the eigenvalues of A are $\lambda = -2$ and $\lambda = -3$. An eigenvector with

eigenvalue -2 is $\mathbf{v}_{-2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ while an eigenvector with eigenvalue -3

is $\mathbf{v}_{-3} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \mathbf{v}_{-2} + c_2 e^{-3t} \mathbf{v}_{-3} = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ -2c_1 e^{-2t} - 3c_2 e^{-3t} \end{pmatrix}.$$

□

- (c) (5 points) Use part (b) to solve the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

Solution. In terms of our 2 by 2 system, this initial value problem becomes $x_1(0) = 2$ and $x_2(0) = 3$. Thus we need to find c_1, c_2 such that

$$c_1 + c_2 = 2, \quad \text{and}$$

$$-2c_1 - 3c_2 = 3.$$

Solving this system gives

$$c_1 = 9, \quad c_2 = -7.$$

Now we have to translate this back to y variables. Since $y = x_1$, we see that the solution to our initial value problem is

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

It's a good idea to check that this solution satisfies the original differential equation and initial values. \square

- (12) (a) (5 points) Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are both solutions to the differential equation

$$\mathbf{x}' = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \mathbf{x}$$

on an interval $I \subset \mathbb{R}$; here $p_{11}(t), p_{12}(t), p_{21}(t)$ and $p_{22}(t)$ are continuous functions. Let $\mathbf{X}(t)$ be the associated fundamental matrix (so that the columns of $\mathbf{X}(t)$ are $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.) The Wronskian of $W(t)$ of $\mathbf{X}(t)$ satisfies the differential equation

$$W'(t) = (p_{11}(t) + p_{22}(t))W(t).$$

Explain why this implies that $W(t)$ is either identically 0 on I or else never equal to 0 on I .

Solution. This problem- especially part b)- was much easier than it may have appeared. Before giving the solution, let's give some context. We are given a system of linear differential equations

$$\mathbf{x}' = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \mathbf{x}.$$

In class we noted that anytime you have a fundamental matrix $\mathbf{X}(t)$ of solutions to this system, the Wronskian $W(t)$ will either be identically 0 or never equal to 0. This problem asks you to explain why this is true, starting from the fact that the Wronskian $W(t)$ itself is a solution to a differential equation,

$$(3) \quad W'(t) = (p_{11}(t) + p_{22}(t))W(t).$$

So, starting from this differential equation, we need to say why $W(t)$ is identically 0 or never 0. The shortest way to answer is to note that equation 3 is a first order linear differential equation, and therefore initial value problems have unique solutions. Since the function $W(t) = 0$ is clearly a solution to equation 3, it follows that every other solution must never be equal to 0. Several of you referenced existence and uniqueness in your answer, but usually it was in reference to the original system of differential equations, rather than the differential equation 3.

Alternatively (and perhaps even better), you could just solve the first order linear differential equation 3 (by using an integrating factor, or just by remembering the solution). The few of you who solved this problem did this. The general solution to 3 is

$$W(t) = c \exp \left(\int (p_{11}(t) + p_{22}(t)) dt \right).$$

If $c = 0$, $W(t)$ is the zero function, while if $c \neq 0$, $W(t)$ is also never zero, because the exponential function never vanishes.

(b) (5 points) Consider the linear second order differential equation

$$y^{(2)} + a(t)y' + b(t)y = 0,$$

and let $y_1(t)$ and $y_2(t)$ be two solutions. Let $W(t)$ be the Wronskian of the associated fundamental matrix. Write down a differential equation satisfied by $W(t)$. (Hint: Use the first part.)

Solution. We have a second order linear differential equation

$$y^{(2)} + a(t)y' + b(t)y = 0.$$

Transforming this equation into a system of two first order linear differential equations in the standard way, we get

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix} \mathbf{x}.$$

In the notation of part (a), we have $p_{11}(t) = 0$ and $p_{22}(t) = -a(t)$, so that $(p_{11}(t) + p_{22}(t)) = -a(t)$. Thus, by part (a), the differential equation satisfied by the Wronskian $W(t)$ is

$$W'(t) = -a(t)W(t).$$

□

- (13) (a) (5 points) Let A be a real symmetric n by n matrix. Show that the eigenvalues of A are real numbers. (Hint: let \mathbf{v} be an eigenvector of A with eigenvalue λ and consider the dot product $A\mathbf{v} \cdot \bar{\mathbf{v}}$. Now use the fact that, for all vectors \mathbf{v} and \mathbf{w} , $A\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot A^T\mathbf{w}$.)

Solution. Before giving the proof, here's a brief reminder about complex numbers. If $z = a + bi$ is a complex number, then to say z is real is to say that $b = 0$. Since $\bar{z} = a - bi$, we see that z is real if and only if $z = \bar{z}$. (The real numbers are precisely those complex numbers which are equal to their own complex conjugates.) We need to show that an eigenvalue λ is real, so we should show that $\lambda = \bar{\lambda}$.

Now for the proof. We're encouraged in the hint to compute the dot product

$$(A\mathbf{v}) \cdot \bar{\mathbf{v}},$$

so let's do that. On the one hand, $A\mathbf{v} = \lambda\mathbf{v}$, so

$$(A\mathbf{v}) \cdot \bar{\mathbf{v}} = \lambda(\mathbf{v} \cdot \bar{\mathbf{v}}).$$

On the other hand,

$$(A\mathbf{v}) \cdot \bar{\mathbf{v}} = \mathbf{v} \cdot (A^T\bar{\mathbf{v}}) = \mathbf{v} \cdot (A\bar{\mathbf{v}}),$$

where the first equality used the fact given in the hint in the last equality we used the fact that $A = A^T$ (A is symmetric). Now, A is a real matrix- so if \mathbf{v} is an eigenvector of A with eigenvalue λ , $\bar{\mathbf{v}}$ is also an eigenvector of A , with complex conjugate eigenvalue $\bar{\lambda}$. Thus

$$\mathbf{v} \cdot (A\bar{\mathbf{v}}) = \bar{\lambda}(\mathbf{v} \cdot \bar{\mathbf{v}}),$$

and we conclude that

$$\lambda(\mathbf{v} \cdot \bar{\mathbf{v}}) = \bar{\lambda}(\mathbf{v} \cdot \bar{\mathbf{v}}).$$

Since \mathbf{v} is a non-zero vector, $(\mathbf{v} \cdot \bar{\mathbf{v}})$ is a non-zero scalar. Hence $\lambda = \bar{\lambda}$, and therefore λ is a real number. \square

- (b) (5 points) A real matrix X is said to be *skew-symmetric* if $X^T = -X$. What kind of statement can be made about the eigenvalues of X when X is skew-symmetric? Prove whatever statement you come up with.

Solution. Let's proceed just as in the first part of the problem, and see what kind of statement we can come up with. On the one hand, $X\mathbf{v} = \lambda\mathbf{v}$, so

$$(X\mathbf{v}) \cdot \bar{\mathbf{v}} = \lambda(\mathbf{v} \cdot \bar{\mathbf{v}}).$$

On the other hand,

$$(X\mathbf{v}) \cdot \bar{\mathbf{v}} = \mathbf{v} \cdot (X^T\bar{\mathbf{v}}) = \mathbf{v} \cdot (-X\bar{\mathbf{v}}),$$

where in the last equality we used the fact that $X^T = -X$ (X is skew-symmetric). Now, X is a real matrix- so if \mathbf{v} is an eigenvector of X with eigenvalue λ then

$\bar{\mathbf{v}}$ is an eigenvector of $-X$, with eigenvalue $-\bar{\lambda}$. (Do you understand why this is true?) Thus

$$\mathbf{v} \cdot (X\bar{\mathbf{v}}) = -\bar{\lambda}(\mathbf{v} \cdot \bar{\mathbf{v}}),$$

and we conclude that

$$\lambda(\mathbf{v} \cdot \bar{\mathbf{v}}) = -\bar{\lambda}(\mathbf{v} \cdot \bar{\mathbf{v}}).$$

Since \mathbf{v} is a non-zero vector, $(\mathbf{v} \cdot \bar{\mathbf{v}})$ is a non-zero scalar. Hence $\lambda = -\bar{\lambda}$. Thus the eigenvalues of a real skew-symmetric matrix satisfy $\lambda = -\bar{\lambda}$. To see what this means geometrically, write $\lambda = a + bi$. Then $-\bar{\lambda} = -a + bi$, so $\lambda = -\bar{\lambda}$ if and only if $a = 0$. Thus the eigenvalues of a real skew-symmetric matrix lie on the *imaginary* axis. \square