1. Problems

Problem 1.1. In the following five examples, you should

- give the order of the differential equation,
- state (without proof) whether or not the differential equation is linear, and
- if the differential equation is linear, solve it (i.e. give the general solution.)

(1) (2 points)
$$y = 2.$$
(2) (2 points)
$$y' = y.$$
(3) (2 points)
$$y'' + 3ty' + 5t^2 \sin(ty) = e^{3t}.$$
(4) (2 points)
$$yy'' + y''' = 2.$$
(5) (2 points)
$$y' - \sin^2(y'') + 2y = \cos^2(y'') + 3e^{-t}.$$

Problem 1.2. Suppose you know that a 2 by 2 matrix A satisfies
$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \text{ and } A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(1) (5 points) Find A.

(2) (5 points) Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(3) (5 points) Compute e^{At}.

Note: If you can't compute A in part (1), then it might not be clear how to do parts (2) and (3). If you choose, you may leave (1) blank, and get credit on parts (2) and (3) by taking A to be the matrix
$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 3 & 0 \end{pmatrix}.$$

Problem 1.3. (5 points) Construct a first order differential equation $y' + a(t)y = g(t)$ such that the graphs of all solutions approach the line $y = 6 + t$ as $t \to \infty$.

Problem 1.4. (10 points) Find a 2 by 2 matrix A such that the differential equation
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$
has the following properties:

- Lines in the xy-plane through the origin of slope 2 and slope -1 are characteristic curves of solutions,
- All characteristic curves approach the origin as $t \to \infty$, and

- with the exception of the line of slope -1, other characteristic curves approach the origin as $t \to \infty$ with slope 2.

Problem 1.5. For each of the following matrices A you should write down the general *real* solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

(1) (5 points) $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

(2) (5 points) $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.

(3) (5 points) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Problem 1.6. (1) (10 points) Solve the initial value problem

$$y'' - 2y' + -3y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

using the Laplace transform. (Feel free to use a table of Laplace transforms.)

(2) (10 points) Solve the same initial value problem *without* using the Laplace transform.

Problem 1.7. (1) (5 points) Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a function. Define the Laplace transform of f.

(2) (5 points) Show that the Laplace transform of $\sin(t)$ is $\frac{1}{1+s^2}$. (You may use the fact that the Laplace transform of $e^{at} = \frac{1}{s-a}$ or the fact that the Laplace transform of t^k is $\frac{k!}{s^{k+1}}$ if you so desire.)

Problem 1.8. (10 points) Sove the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 1.9. (10 points) Recall that a real n by n matrix A is *orthogonal* if $A^T = A^{-1}$, and called *skew-symmetric* if $A^T = -A$. Show that if X is skew-symmetric then e^X is orthogonal.

Problem 1.10. (5 points) Find the general solution to the differential equation

$$\frac{dy}{dx} = xy^2.$$

Problem 1.11. (10 points) Mike the salt mixer is doing what he does best- mixing salt into a tank filled with water. His tank initially contains 2 gallons of pure water. Water containing $\frac{1}{3}$ lb of salt per gallon is poured into the tank at a rate of 5 gallons per minute, and the mixture is allowed to leave at the same rate. After 5 minutes, Mike has to stop the process so he can go to a number theory seminar. After the seminar, he returns to the tank and pours fresh water in at a rate of 3 gallons per minute for 10 minutes, with mixture again allowed to leave at a rate of 3 gallons per minute. Find the amount of salt in the tank.

Problem 1.12. (5 points) Suppose that x is a continuous solution to the initial value problem

$$x' = (\cos^2 t)x, \quad x(0) = 1,$$

and that x is defined on all of $\mathbb{R}$. Show that $x(t) > 0$ for all t.

Problem 1.13. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are given by

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(1) (5 points) Write down a differential equation $\mathbf{x}' = A\mathbf{x}$ whose general solution is

$$c_1 e^{2t}\mathbf{u} + c_2 e^{2t}\mathbf{v} + c_2 t e^{2t}\mathbf{u}.$$

(2) (5 points) Compute e^{tA} for the matrix A which you obtained in the first part.

Problem 1.14. (1) (5 points) Consider the system of linear differential equations given by

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t)$$

Prove that if $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots \mathbf{x}_n(t)$ are solutions to the above system of differential equations, then

$$\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$$

is also a solution.

(2) (5 points) Let $\mathbf{x}_0$ be a solution to

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t).$$

Assuming the result in part (a), show that the set of all solutions to

$$\mathbf{x}'(t) = P(t)\mathbf{x}(t) + \mathbf{g}(t)$$

is the set

$$\{\mathbf{x} + \mathbf{x}_0 \mid \mathbf{x} \text{ is a solution to } \mathbf{x}'(t) = P(t)\mathbf{x}(t)\}.$$

Problem 1.15. (1) (5 points) Write down four linearly independent vectors in $\mathbb{R}^4$. Write down four distinct real numbers. Now write down a matrix A such that your four distinct real numbers are eigenvalues and your four linearly independent vectors are eigenvectors. Write down the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

(2) (5 points) Write down four linearly independent vectors in $\mathbb{R}^4$, a non-zero real number, and a *non-diagonalizable* matrix A such that your vectors are all *generalized* eigenvectors (some of your vectors can be genuine eigenvectors if you wish). Write down the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ for your matrix A.

2. Comments

The material on this practice exam is skewed a bit in the direction of material we've done since the second exam. In particular, problems 1.4, 1.6, 1.7 are all from material covered since the second exam. In fact, 1.4 was really only discussed in class on Monday of this last week of class. The actual final exam won't necessarily be skewed in this way (although I can promise a problem like 1.6). I want to give you a sense of the full spectrum of problems you could reasonably expect on the exam:**You should really think of the first exam, second exam, the practice second exam, and this practice exam together as a collection of practice exams, since any of the kinds of problems on these three exams are fair game**.

The practice exam is also skewed a little bit in favor of reverse engineering questions-I give you some properties of a matrix, and ask you to write down the matrix and solve a differential equation involving it. The actual exam will have problems like this, but it will also have some more standard questions where I give you the matrix and ask you to solve something. In this sense the practice exam might be a bit harder than the actual final, depending on your personal problem preferences.

Here are some hints/comments about individual questions.

- Problem 1: be careful on part 5.
- Problem 2: remember that in order to find A it suffices to find the columns of A. The first column of A is $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while the second column of A is $A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So if you can figure out what $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are you're in business. It's not yet clear how lucrative this business is; ask
- Problem 3: this problem appeared on the first exam.
- Problem 4: we talked about this material in class on Monday. That the lines of slope 2 and -1 are characteristic curves should tell you something about the eigenvectors of A. The fact that all characteristic curves approach the origin tells you something about the eigenvalues of A. The fact that all but one characteristic curve approaches the origin with slope 2 tells you something more about the eigenvalues of A. Find an A whose eigenvalues and eigenvectors have these required properties.
- Problem 5: this is a bread and butter differential equations problem. You will definitely have to do something like this on the final, and I expect all of you will get it correct.
- Problem 6: the first part is new material since the last homework-you will definitely have to do something like the first part on the exam.
- Problem 7: part 2 was a quiz problem (though perhaps I asked it for $\cos(t)$ on the quiz. You should try to remember a couple of ways to do it.
- Problem 8: should be straightforward.

- Problem 9: You need to show that e^X is orthogonal if X is skew-symmetric. To do this, write down what you are assuming and what you need to show. You are assuming that X is skew-symmetric, i.e. that $X^T = -X$. You need to show that e^X is orthogonal, that is, $(e^X)^T = (e^X)^{-1}$.
- Problem 10: separable.
- Problem 11: this is more or less from the first exam.
- Problem 12: for any real numbers a, b, we know that the initial value problem

  $$y' = (\cos^2 t)y, \quad y(a) = b$$

  has a unique solution. If our solution $x(t)$ is ever non-positive, then by the intermediate value theorem there is a real number t_0 with $x(t_0) = 0$. Show that this implies that a certain initial value problem would have 2 solutions, violating uniqueness.
- Problem 13: the form of the general solution tells you that $\mathbf{u}$ is an eigenvector with eigenvalue 2, and $\mathbf{v}$ satisfies $(A - 2I)\mathbf{v} = \mathbf{u}$. Now find A.
- Problem 14: you may have seen this before...
- Problem 15: Asking this in $\mathbb{R}^4$ is perhaps a bit computationally intensive. On the exam this question would be more likely to be asked in $\mathbb{R}^2$ or $\mathbb{R}^3$. But if you can do it in $\mathbb{R}^4$ then you'll be fine if something like this is asked on the exam. For part 1, once you've written down 4 linearly independent vectors and declared them to be eigenvectors with certain eigenvalues, you've completely determined your matrix A. When thinking about part (2), you can write down a matrix with 1,2, or 3 linearly independent eigenvectors. You can't write down one with 4 linearly independent eigenvectors, for then A would be diagonalizable.