1 Hamiltonians

Recall. Last time we considered the classical physical system on \((\mathbb{R}^n \times \mathbb{R}^n, \omega_0 = \sum_{j=1}^n dx_j \wedge dy_j)\) governed by the energy \(H(x,y) := \frac{1}{2}|y|^2 + V(x)\).

In that case \(dH = \frac{1}{2} \sum_{j=1}^n \left(y_j dy_j + \frac{\partial V}{\partial x_j} dx_j\right)\), so using \(i_X \omega = dH\) we got that
\[
X = \frac{1}{2} \sum_{j=1}^n \left(y_j \frac{\partial}{\partial x_j} - \frac{\partial V}{\partial x_j} \frac{\partial}{\partial y_j}\right).
\]
The flow of \(X\) defines Hamilton’s equations:
\[
\begin{cases}
\frac{\partial x_j}{\partial t} &= y_j = \frac{\partial H}{\partial y_j}, \\
\frac{\partial y_j}{\partial t} &= -\frac{\partial V}{\partial x_j} = -\frac{\partial H}{\partial x_j}.
\end{cases}
\]

Recall. In general, given a function \(H : M \rightarrow \mathbb{R}\) called the Hamiltonian, one gets a vector field \(X_H\) via \(i_{X_H} \omega = dH\) which defines a flow \(\phi_t\) on \(M\) (the flow is defined globally assuming \(M\) is compact).

**Theorem 1** (Conservation of the Hamiltonian). The Hamiltonian \(H\) is a conserved quantity under the flow \((\phi_t)\) or equivalently

- \(\frac{d}{dt} \phi_t^* H = 0\), which is the same as
- \(\phi_t\) preserves the level sets of \(H\), which in turn is the same as
- the trajectory of every point lies on a level set of \(H\).

**Proof.**
\[
\frac{d}{dt} \phi_t^* H = \phi_t^* \mathcal{L}_{X_H} \omega = \phi_t^* i_{X_H} \omega = \phi_t^* i_{X_H} i_{X_H} \omega = 0.
\]

We can generalize the discussion about Hamiltonian flows to the case of a time-dependent Hamiltonian, i.e. a smooth function \(H : \mathbb{R} \times M \rightarrow \mathbb{R}\) viewed as a 1-parameter family \((H_t)_{t \in \mathbb{R}}\) of Hamiltonians given by \(H_t(x) := H(t,x)\).

Given such a time dependent Hamiltonian \(H_t\), we define its (time-dependent) Hamiltonian vector field \(X_t := X_{H_t}\) as before by \(i_{X_t} \omega = dH_t\) which in turn defines a flow \((\phi_t)\) via the ODE
\[
\frac{d}{dt} \phi_t = X_t \circ \phi_t \tag{1}
\]
with the initial condition $\phi_0 = \text{id}_M$.

**Definition 1.** A symplectomorphism $\phi : (M, \omega) \to (M, \omega)$ which is a time $t_0$ map ($\phi = \phi_{t_0}$) for some $t_0 \in \mathbb{R}$ of some Hamiltonian flow $(\phi_t)$ is called a Hamiltonian symplectomorphism.

**Note.** After scaling time and $H$ we can assume that any Hamiltonian symplectomorphism is a time 1 map $\phi_1$ of some (time-dependent) Hamiltonian.

**Proposition 2.** The Hamiltonian symplectomorphisms of $(M, \omega)$ form path-connected normal subgroup Ham$(M, \omega)$ of Sympl$(M, \omega)$.

**Proof.** The space of Hamiltonian symplectomorphisms is obviously path connected: the flow $(\phi_t)_{t \in [0,1]}$ defines a path of Hamiltonian symplectomorphisms from $\text{id}_M$ to $\phi_1$.

Note that if $\phi \in \text{Diff}(M)$ and $(\phi_t)$ is a path in Diff$(M)$, then the vector field $X_t$ associated to $\phi_t$ (via the ODE (1)) is the same as the vector field associated to $\phi_t \circ \phi = \phi^*(\phi_t)$ via the same ODE.

Assume that the flows $\phi_t$ and $\psi_t$ are generated by the Hamiltonians $H_t$ and $G_t$, respectively. Then $\phi_t \circ \psi_t$ is generated by $H_t + G_t \circ \phi_t^{-1}$ and $\phi_t^{-1}$ is generated by $-H_t \circ \phi_t$ (check it!). Hence, Ham$(M, \omega)$ is, indeed, a subgroup of Sympl$(M, \omega)$.

Finally, we show that Ham$(M, \omega)$ is normal in Sympl$(M, \omega)$. (Note: it not necessarily normal in Diff$(M)$!) Indeed, given $\phi \in \text{Sympl}(M, \omega)$, $\phi \circ \phi_t \circ \phi^{-1}$ is generated by $H_t \circ \phi^{-1}$ (check it!), hence $\phi \circ \phi_t \circ \phi^{-1}$ is an element of Ham$(M, \omega)$.

**Recall.** For a symplectic vector field $X$, $i_X \omega$ is closed and for a Hamiltonian vector field $X$, $i_X \omega$ is exact. Thus, the obstruction to $X$ being Hamiltonian is $[i_X \omega] \in H^1_{dR}(M)$.

Similarly, the obstruction to a path $(\phi_t)_{t \in [0,1]}$ of symplectomorphisms to be symplectically deformed to a path of Hamiltonian symplectomorphisms is the flux

$$\text{Flux}(\phi_t) := \int_0^1 [i_{X_t} \omega] dt \in H^1_{dR}(M).$$

**Exercise 1.** The flux is invariant under symplectic deformations of $(\phi_t)$ relative the end points.

## 2 Arnold conjectures

**Conjecture 3** (Arnold conjecture, version 1). Assume $(M, \omega)$ is a compact symplectic manifold and $\phi \in \text{Ham}(M, \omega)$ has nondegenerate fixed points, then the number of fixed points of $\phi$ is at least the sum of Betti numbers of $M$, i.e. $\sum_{i=0}^{2n} \text{rk} H^i(M)$.

**Note.** Proven for coefficients in $\mathbb{R}$.

**Conjecture 4** (Arnold conjecture, version 2). Under the same assumptions the number of fixed points of $\phi$ is at least the minimal number of critical points of a Morse function on $M$.

The motivation of the first version is question of existence of closed orbits of a time-dependent Hamiltonian (or 1-periodic orbits of a 1-periodic Hamiltonian).

Note that the time $t$ periodic orbits of a flow $(\phi_t)$ are in a bijective correspondence with the initial conditions $x_0 \in M$ such that $\phi_t(x_0) = x_0$, i.e. the fixed points of $\phi_t$ which can be viewed as the intersection points of the graph $\Gamma_{\phi_t}$ of $\phi_t$ and the diagonal $\Delta$ inside $M \times M$. 

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**Definition 2.** A fixed point $x_0$ of $\phi \in \text{Diff}(M)$ is called *nondegenerate* if the graph $\Gamma_\phi$ of $\phi$ intersects the diagonal $\Delta$ transversely at $x_0$. This is equivalent to $(d\phi(x_0) - \text{id}_{T_x M})$ being invertible, i.e. 1 not being an eigenvalue of $d\phi(x_0)$.

**Note.** The *Lefschetz fixed point theorem* says that if $\phi \in \text{Diff}(M)$ has nondegenerate fixed points then the number of fixed points of $\phi$ is at least

$$\left| \sum_{i=0}^{2n} (-1)^i \text{rk} H^i(M) \right| = |\chi(M)|.$$ 

In general, *any* smooth function $\phi : M \to M$ with nondegenerate fixed points has at least

$$\left| \sum_{i=0}^{2n} (-1)^i \text{Tr}(\phi^*|_{H^i_{dR}(M)}) \right|$$

fixed points.

**Note.** Arnold conjecture fails for symplectomorphisms $\phi : M \to M$ which do *not* come from Hamiltonian functions. For example consider the 2-torus $M = (T^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega_0)$ with the standard symplectic form and let $\phi$ be the time 1/2 map of the symplectic flow generated by $\partial / \partial x + \alpha \partial / \partial y$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$ does not have closed orbits of *any* period (check it!).

**Proposition 5.** Assume $(M, \omega)$ compact and $H^1_{dR}(M) = 0$. Then any symplectomorphism $\phi$ of $M$ that is sufficiently $C^1$-close to the identity $\text{id}_M$ has at least two fixed points.

**Proposition 6.** Assume $L$ is a compact Lagrangian in $(M, \omega)$ with $H^1_{dR}(L) = 0$. Then any sufficiently $C^1$-small Lagrangian deformation of $L$ intersects $L$ in at least two points.

**Note.** Lagrangian deformation = deformation through Lagrangians.

**Note.** Proposition 5 implies Proposition 6 by taking $L$ to be the diagonal $\Delta$ inside $(M \times M, -\omega \oplus \omega)$. Then the graph $\Gamma_\phi$ of a $C^1$-small deformation $\phi$ of $\text{id}_M$ is $C^1$-small Lagrangian deformation of $L$. 

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