1 Consequences of Moser’s trick

**Theorem 1** (Moser’s stability theorem). Assume $\omega_t$ is a smooth path (1-parameter family) of symplectic forms on a closed manifold $M$ such that $[\omega_t] \in H_{dR}^2(M)$ is constant in $t$. Then there exists a smooth family $\phi_t$ of diffeomorphisms of $M$ such that $\phi_t^*\omega_t = \text{const}$.

**Proof.** The proof is “the same is before”: use the flow of some vector field $X_t$ to find $\phi_t$ such that
\[
\frac{d}{dt}(\phi_t^*\omega_t) = 0.
\]
By applying the chain rule to the equation above we get
\[
\frac{d}{dt}\omega_t + L_{X_t}\omega_t = 0
\]
which after using Cartan’s formula becomes
\[
\frac{d}{dt}\omega_t + d\omega_t + i_{X_t}\omega_t + i_{X_t}d\omega_t = 0.
\]
Note that $[\frac{d\omega_t}{dt}] = \frac{d}{dt}[\omega_t] = 0$ in $H_{dR}^2(M)$. Hence, for each $t$ there exists 1-form $\beta_t$ such that $\frac{d}{dt}\omega_t = d\beta_t$.

Hence, equation (1) holds if and only if
\[
d\beta_t + i_{X_t}\omega_t = 0.
\]
Since $\omega_t$ is nondegenerate we can find a vector field $X_t$ such that
\[
i_{X_t}\omega_t = -\beta_t
\]
and so (1) is satisfied.

The last thing need to check is that we can choose $\beta_t$ so that it depends smoothly on $t$. There are two ways to go about it.

The first way is to cover the manifold with “well-behaved” charts (i.e. intersections of charts are contractible). Then use Poincaré lemma and Mayer-Vietoris.

The second way is to use the Hodge decomposition: any $k$-form has a unique decomposition as a harmonic form plus an exact form plus a co-exact form:
\[
k\text{-forms} = \text{harmonic} \oplus \text{im } d \oplus \text{im } d^*
\]
as follows

\[ \alpha = h([\alpha]) + dd^* G\alpha + d^* dG\alpha \]

where \( G \) is Green’s operator \((G := (dd^* + d^* d)^{-1} \) is the inverse of the Laplacian away from harmonic forms). In particular, the decomposition above defines an isomorphism between \( k^{th} \) de Rham cohomology group and the space of harmonic \( k \)-forms:

\[ H^k_{dR}(M) \equiv \text{Harm}^k(M). \]

In our case take \( \alpha_t := \frac{d}{dt} \omega_t \). Hodge decomposition tells us that

\[ \alpha_t = dd^* G\alpha_t \]

(we used that \( d\alpha_t = 0 \) and \([\alpha_t] = 0\), hence \( h(\alpha_t) = 0\)).

We can let \( \beta_t := d^* G\alpha_t \).

2 Symplectic tubular neighborhoods

**Proposition 2.** Assume \( W \) is a compact submanifold (without boundary) of a symplectic manifold \( M \). Consider two symplectic forms \( \omega_0 \) and \( \omega_1 \) on \( M \) which agree along \( W: \omega_0|_W = \omega_1|_W \) on \( TM|_W \) (the pullback of \( TM \) to \( M \)). Then there exist neighborhoods \( U_0 \) and \( U_1 \) of \( W \) in \( M \) and a diffeomorphism \( \phi: U_0 \to U_1 \) such that

1. \( \phi^* \omega_1 = \omega_0; \)
2. \( \phi|_W = \text{id}_W. \)

**Corollary 3** (Darboux’s Theorem). Any symplectic manifold \((M, \omega)\) is locally symplectomorphic to \((\mathbb{R}^{2n}, \omega_0)\).

**Proof of the Corollary.** Darboux’s theorem follows by applying the proposition to \( W := p \), an arbitrary point of \( M \).

More explicitly, choose an neighborhood \( U \) of \( p \) in \( M \) that is diffeomorphic to an open subset \( V \) of \( \mathbb{R}^{2n} \) via \( \psi: U \to V \). By a linear change of coordinates on \( \mathbb{R}^{2n} \) we can assume that \((\psi^* \omega_0)_p = (\omega_1)_p\) (i.e. the standard symplectic form \( \omega_0 \) on \( \mathbb{R}^{2n} \) pulls back to \( \omega_1 \) at the point \( p \)). By abuse of notation, let \( \omega_0 \) denote the pullback \( \psi^* \omega_0 \) of the standard symplectic form \( \omega_0 \) on \( \mathbb{R}^{2n} \) to \( U \).

Apply the proposition to \( W := \text{point } p \) viewed as a compact submanifold of the symplectic manifold \( U \) with two symplectic forms \( \omega_0 \) (constructed above) and \( \omega_1 \) (coming from \( M \)) that agree at the point \( p \). We get that there exists open neighborhoods \( U_0 \) and \( U_1 \) of \( p \) in \( U \) and a diffeomorphism \( \phi: U_0 \to U_1 \) such that \( \phi^* \omega_1 = \omega_0 \) and \( \phi(p) = p \). Then \( \psi \circ \phi^{-1}: U_1 \to \mathbb{R}^{2n} \) defines a symplectomorphism from \((U_1, \omega_1)\) onto an open subset of \( \mathbb{R}^{2n} \) with the standard symplectic form \( \omega_0 \).

**Proof of the Proposition.** As before, consider the linear interpolation between the two symplectic forms as a path of closed forms:

\[ \omega_t := t\omega_0 + (1-t)\omega_1. \]

We know that \( \omega_0|_W = \omega_1|_W \), so \( \omega_t|_W = \omega_0|_W \) for every \( t \). In particular, \( \omega_t \) is nondegenerate at all points of \( W \). Nondegeneracy is an open condition, i.e. the subset of nondegenerate 2-forms is open in the set of all 2-forms (viewed as a topological vector space). Hence, \( \omega_t \) is nondegenerate in a neighborhood of \( U \) of \( W \).
Since \( W \) is compact we can assume that \( U \) is a tubular neighborhood (i.e. diffeomorphic to a disk bundle over \( W \)).

We have that \( \omega_t \) is a path of symplectic forms on \( U \) such that are constant in \( t \) on all points of \( W \). We want to produce an open set neighborhood \( U_0 \) of \( W \) in \( U \) and a path of maps \( \phi_t : U_0 \to M \) that are diffeomorphisms onto their images such that

\[
\begin{aligned}
\phi_t^* \omega_t &= \omega_0; \\
\phi_t | W &= \text{id}_W; \\
\phi_0 &= \text{id}_{U_0}.
\end{aligned}
\]  

We look for a path of vector fields \( X_t \) on \( U \) such that its flow \( \phi_t \) is defined at least for time \( t \in [0,1] \) (on some smaller open set \( U_0 \subset U \)) and \( \phi_t \) satisfies \( (2) \).

We need \( \frac{d}{dt}(\phi_t^* \omega_t) = 0 \). As before, this is equivalent to

\[
\frac{d}{dt} \omega_t + d i_{X_t} \omega_t = 0.
\]

By definition, \( \frac{d}{dt} \omega_t = \omega_0 - \omega_1 \), let’s call this difference \( \alpha \). We know that \( \alpha \) is a closed form on \( U \) and \( \alpha | W = 0 \) by our assumptions.

We will show that there exists a 1-form \( \beta \) on \( U \) such that

\[
\begin{aligned}
\alpha &= d \beta \\
\beta | W &= 0
\end{aligned}
\]

using relative de Rham theory as follows.

Consider the following long exact sequence associated with the embedding \( W \hookrightarrow U \):

\[
\cdots \to H^1_{dR}(U) \to H^1_{dR}(W) \to H^2_{dR}(U,W) \to H^2_{dR}(U) \to H^2_{dR}(W) \to \cdots
\]

By assumption, \( U \) is a tubular neighborhood of \( W \), so, in particular, \( W \) is a deformation retract of \( U \). Hence, the inclusion \( W \hookrightarrow U \) induces an isomorphism between the de Rham cohomologies of \( W \) and \( U \). In particular, the maps \( H^1_{dR}(U) \to H^1_{dR}(W) \) and \( H^2_{dR}(U) \to H^2_{dR}(W) \) above are isomorphisms. Hence, \( H^2_{dR}(U,W) = 0 \).

Let’s recall what \( H^2_{dR}(U,W) \) means explicitly: it is the quotient of all closed 2-forms on \( U \) that vanish on \( W \) by the differentials of 1-forms on \( U \) that vanish on \( W \). Hence, \( H^2_{dR}(U,W) \) being zero means that \( \alpha = d \beta \) for some 1-form \( \beta \) vanishing on \( W \).

Recall, that we are trying to solve \( \frac{d}{dt} \omega_t + d i_{X_t} \omega_t = 0 \) for a vector field \( X_t \) on \( U \) that is zero on \( W \). By definition of \( \beta \) this equation is the same as

\[
d \beta + d i_{X_t} \omega_t = 0.
\]

As before, we can solve \( i_{X_t} \omega_t = -\beta \) for \( X_t \) in \( U \) since \( \omega_t \) is nondegenerate.

A little note on smoothness that did not come up in class: view each \( \omega_t \) as a isomorphisms (of vector bundles on \( U \)) \( \tilde{\omega}_t : T U \to T^* U \) via \( \tilde{\omega}_t(Y) = \omega_t(Y, \bullet) \) for a vector field \( Y \) and note that it depends smoothly on \( t \). By considering its inverse (which also depends smoothly on \( t \)) we can let \( X_t := \tilde{\omega}_t^{-1}(\beta) \). Hence, \( X_t \) depends smoothly on \( t \).

Also, since \( \beta \) vanishes on \( W \), so does \( X_t \).
Since $W$ is compact and $X_t$ is zero on $W$, we can choose a neighborhood $U_0$ of $W$ in $U$ such that $X_t$ stays “sufficiently small” in $U_0$, namely so that the flow $\phi_t$ is defined on $U_0$ for all $t \in [0, 1]$ (this uses some standard but slightly messay-to-prove facts about ODEs).

Then $\phi_t|_W = \text{id}_W$ (because $X_t|_W = 0$).

In particular, $\phi_1^* \omega_1 = \omega_0$ and $\phi_1|_W = \text{id}_W$, so $\phi_1$ is the desired diffeomorphism.

**Corollary 4** (Lagrangian Tubular Neighborhood Theorem, due to Weinstein). Assume $W$ is a compact Lagrangian submanifold of a symplectic manifold $(M, \omega)$. Then there exists a neighborhood $U_0$ of the zero section in $T^*W$, a neighborhood $U_1$ of $W$ in $M$ and a diffeomorphism $\phi : U_0 \rightarrow U_1$ such that $\phi^* \omega = \omega_{\text{can}}$ and $\phi|_W = \text{id}|_W$.

**Proof.** (Only the basic idea, more details in the next lecture)

The previous result implies that if $U$ is a sufficiently small neighborhood of any submanifold of $(M, \omega)$ then $\omega$ on $U$ is determined (up to a change of choordinates) by $\omega|_W$ on $TM|_W$.

Hence, we need to check that for a Lagrangian submanifold $W$, $(T_pM, \omega_p)$ is naturally isomorphic to $(T_pW \oplus T^*_pW, \omega_{\text{can}})$ as a symplectic vector space for every point $p \in W$. 

\[\square\]