1 Overview

Loose plan:

- Linear algebra preliminaries;
- Symplectic manifolds: definition, some examples: surfaces, $\mathbb{C}P^n$ and cotangent bundles;
- Symplectic and Lagrangian submanifolds, symplectic vector bundles, tubular neighborhood theorems;
- Hamiltonian flows and their closed orbits, Arnold conjecture (multiple versions);
- Almost complex structures, the first Chern class $c_1$ and the Maslov index;
- Symplectic construction toolkit: symplectic fibrations, Lefschetz fibrations, symplectic sum (the analog of the connect sum in the symplectic category), symplectic blow-up;
- Gromov non-squeezing theorem (statement only?);
- Basic notions of contact geometry: contact structures, Legendrian submanifolds, Reeb orbits and the Weinstein conjecture;
- Symplectic and Hamiltonian group actions and moment maps (time permitting).

The main two references are Lectures on Symplectic Geometry by Anna Cannas da Silva (freely available online at http://www.math.ist.utl.pt/~acannas/Books/lsg.pdf) and Introduction to Symplectic Topology by Dusa McDuff and Dietmar Salamon.

2 Linear Algebra

2.1 Basic Definitions

Assume $V$ is an $2n$-dimensional (real) vector space.

Note. Any bilinear form $\omega : V \times V \to \mathbb{R}$ induces a linear map $\tilde{\omega} : V \to V^*$ via

$$\tilde{\omega}(v) := \omega(v, \cdot).$$

Definition 1. A bilinear form $\omega$ is called nondegenerate if the map $\tilde{\omega}$ is an isomorphism or, equivalently, if $\ker(\tilde{\omega}) = 0$ (since $V$ and $V^*$ have the same dimension).
Definition 2. A 2-form $\omega$ on $V$ is an element of $\Lambda^2(V) := V^* \wedge V^*$, i.e. a bilinear skew-symmetric map $\omega : V \times V \to \mathbb{R}$.

Definition 3. A symplectic form $\omega$ on $V$ is a nondegenerate 2-form.

Note. If $\omega$ is non-degenerate then the dimension of $V$ is even.

Definition 4. A symplectic vector space is a vector space $V$ together with a symplectic form $\omega$.

Example 1. Define the standard symplectic form on $V = \mathbb{R}^{2n}$ as follows. Identify $\mathbb{R}^{2n}$ with $\mathbb{R}^n \times \mathbb{R}^n$, label the standard basis by $x_1, \ldots, x_n, y_1, \ldots, y_n$. Let $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^*$ be the dual basis of $V^*$. Then let

$$\omega_0 := \sum_{j=1}^n x_j^* \wedge y_j^*.$$ 

The associated matrix of $\omega_0$ is

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

Example 2. Let $E$ be a real or complex vector space. Then $E \oplus E^*$ has a canonical symplectic form $\omega_{\text{can}}$ given by

$$\omega_{\text{can}}((v,\alpha), (u,\beta)) := \beta(u) - \alpha(v)$$

for $\alpha, \beta \in E^*$, $u, v \in E$.

Exercise 1. Check that the form above is nondegenerate!

Note. If we identify $\mathbb{R}^{2n}$ with $\mathbb{R}^n \times (\mathbb{R}^n)^*$ then the previous example gives us the standard symplectic form on $\mathbb{R}^{2n}$.

Exercise 2. If $e_1, \ldots, e_n$ is a basis of $E$ and $e_1^*, \ldots, e_n^*$ is the dual basis of $E^*$ then

$$\omega_{\text{can}} = \sum_{j=1}^n e_j^* \wedge e_j.$$ 

Definition 5. A linear map $\phi : (V_1, \omega_1) \to (V_2, \omega_2)$ between two symplectic vector spaces is called a linear symplectomorphism if

$$\phi^* \omega_2 = \omega_1,$$

i.e.

$$\omega_1(u, v) = \omega_2(\phi(u), \phi(v)), \quad \forall u, v \in V.$$

Definition 6. The linear symplectomorphisms form a group $\text{Sympl}(V, \omega)$. The symplectic group $\text{Sp}(2n)$ is the group of linear symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$.

Exercise 3. Explicitly,

$$\text{Sp}(2n) = \left\{ A \in M_{2n}(\mathbb{R}) \left| A^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right. \right\}.$$ 

Exercise 4. $\dim \text{Sp}(2n) = n(2n + 1)$.

Exercise 5. If $A$ is an element of $\text{Sp}(2n)$ then $\det A = 1$ (obviously, $\det A$ has to be plus or minus one, less obvious that minus one cannot be attained).

Proposition 1. Any symplectic 2n-dimensional vector space $(V, \omega)$ is linearly isomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Proof. See Practice Problem 1.

Proposition 2. A 2-form $\omega$ on $V^{2n}$ is nondegenerate iff its $n^{th}$ exterior power $\Lambda^n\omega$ is nonzero.

Proof. Follows from Practice Problem 1.
2.2 Linear subspaces

Let \((V^{2n}, \omega)\) be a symplectic vector space.

**Definition 7.** Assume \(W\) is a linear subspace of \(V\). Its *symplectic complement* is defined by

\[ W^\omega := \{ v \in V | \omega(v, w) = 0, \ \forall w \in W \}. \]

**Warning!** Unlike the case of symmetric forms, usually, \(W^\omega \cap W \neq \{0\} \).

**Proposition 3.**

1. \(\dim W + \dim W^\omega = \dim V\).
2. \((W^\omega)^\omega = W\).
3. \((W_1 \cap W_2)^\omega = W_1^\omega + W_2^\omega\).
4. If \(W_1 \subseteq W_2\) then \(W_1^\omega \supseteq W_2^\omega\).

**Definition 8.** A linear subspace \(W\) of \((V, \omega)\) is called

1. *symplectic* if \(\omega|_W\) is nondegenerate, i.e. if \((W, \omega|_W)\) is a symlectic vector space;
2. *isotropic* if \(\omega|_W = 0\), i.e. \(W \subseteq W^\omega\);
3. *coisotropic* if \(W \supseteq W^\omega\);
4. *Lagrangian* if \(W = W^\omega\).

*Note.* The dimension of any isotropic subspace is at most half of the dimension of \(V\); the dimension of any coisotropic subspace is at least half.

*Note.* A subspace \(W\) is Lagrangian iff it is isotropic and \(\dim W = \frac{1}{2} \dim V\) (or, equivalently, if it is coisotropic and \(\dim W = \frac{1}{2} \dim V\)).

**Example 3.** In \((\mathbb{R}^{2n}, \omega_0)\):

1. Symplectic: \(\mathbb{R}^k \times \mathbb{R}^k = \{(x_1, \ldots, x_k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0)\}\).
2. Isotropic: \(\mathbb{R}^k \times \{0\} = \{(x_1, \ldots, x_k, 0, \ldots, 0)\}\).
3. Coisotropic: \(\mathbb{R}^n \times \mathbb{R}^k = \{(x_1, \ldots, x_n, y_1, \ldots, y_k, 0, \ldots, 0)\}\).
4. Lagrangian: \(\mathbb{R}^n \times \{0\} = \{(x_1, \ldots, x_n, 0, \ldots, 0)\}\) (can also be viewed as \(\mathbb{R}^n \subset \mathbb{C}^n\)).

*Note.* Given any symplectic/isotropic/coisotropic/Lagrangian subspace \(W\) of a symplectic vector space \((V^{2n}, \omega)\) one can find a basis \(x_1, \ldots, x_n, y_1, \ldots, y_n\) of \(V\), such that with respect to that basis \(\omega = \omega_0\) and \(W\) is one of the examples above.

2.3 Natural examples of Lagrangian subspaces

**Example 4.** Assume \((V, \omega)\) is a symplectic vector space and \(\phi : V \rightarrow V\) is a linear transformation. Then \(\phi\) is a linear symplectomorphism iff its graph \(\Gamma_\phi\) is Lagrangian in \((V \oplus V, -\omega \oplus \omega)\) where \(-\omega \oplus \omega\) denotes the difference \(-\pi_1^*\omega + \pi_2^*\omega\) of the pullbacks of \(\omega\) along the two canonical projections \(\pi_1, \pi_2 : V \oplus V \rightarrow V\) and

\[ \Gamma_\phi = \{(v, \phi(v)) | v \in V\}. \]

**Example 5.** Assume \(\phi : V \rightarrow V^*\) is a linear transformation. Then \(\Gamma_\phi\) is Lagrangian in \((V \oplus V^*, \omega_{can})\) iff \(\phi\) is symmetric:

\[ \phi(v)(w) = \phi(w)(v), \ \forall v, w \in V. \]
3 Symplectic Manifolds

Let $M^{2n}$ be a smooth manifold (not necessarily closed).

**Definition 9.** A symplectic form $\omega$ on $M$ is a closed ($d\omega = 0$) nondegenerate (pointwise) 2-form. The pair $(M, \omega)$ is called a symplectic manifold.

**Note.** The nondegeneracy condition means that for every $p \in M$, $\omega_p : T_p M \times T_p M \to \mathbb{R}$ is a symplectic form. It also means that the $n^{th}$ exterior power $\omega^n$ of $\omega$ is a volume form on $M$.

**Example 6.** $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$ with

$$\omega_0 := \sum_{j=1}^{n} dx_j \wedge dy_j.$$  

We can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z_j = x_j + iy_j$, in which case

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

**Example 7.** Consider a surface $\Sigma^2$. A symplectic form on a surface is exactly an area form ($d\omega = 0$ for dimensional reasons and $\omega^n = \omega$ has to be a volume form on $\Sigma$). Hence $\Sigma$ admits a symplectic structure iff it is orientable.

**Example 8.** $S^2$ := the unit sphere in $\mathbb{R}^3$ equipped with a symplectic form $\omega$ given at $p \in S^2 \subset \mathbb{R}^3$ by

$$\omega_p(u, v) := \langle p, u \times v \rangle$$

for $u, v \in T_p S^2 \subset T_p \mathbb{R}^3$ (where we canonically identify $T_p \mathbb{R}^3$ with $\mathbb{R}^3$ in order to get the vector product $u \times v$). In cylindrical coordinates ($z =$ height and $\theta =$ polar angle in $(x, y)$-plane) we get

$$\omega_p = d\theta \wedge dz$$

(away from $z$-axis). The above formula shows that $\omega_p$ is nondegenerate and rotation invariant.

**Example 9.** Kähler manifolds. *(to be continued...)*