Problem Set 3 Solutions

Math 120

2.4.2 If A is a subset of B, then any subgroup of G which contains B also contains A, so then by definition

$$\langle A\rangle = \bigcap_{A\subseteq H} H \leq \bigcap_{B\subseteq H} H = \langle B\rangle.$$

For an example of a proper subset $A \subset B$ which generates the same subgroup of G as B, consider $A = \{r, s\}$, $B = \{r, r^3, s\} \subset D_8$. Clearly we have $\langle A\rangle = \langle B\rangle = D_8$.

2.4.7 Let $G = \langle(1\ 2),(1\ 3)(2\ 4)\rangle$, and note first that $[(1\ 3)(2\ 4)](1\ 2) = (1\ 4\ 2\ 3)$, an element of order 4 in S_4. Note too that $(1\ 4\ 2\ 3)(1\ 2) = (1\ 3)(2\ 4) = (1\ 2)(1\ 4\ 2\ 3)^{-1}$. We may thus define a homomorphism $\varphi : D_8 \to G$ by $\varphi(r) = (1\ 4\ 2\ 3)$ and $\varphi(s) =(1\ 2)$. As G is generated by $(1\ 2)$ and $(1\ 4\ 2\ 3)$, and these generators have the same orders and satisfy the same relations as their counterparts in D_8, the map must be surjective. Hence G has order at most 8. But one can easily write down eight distinct elements of G, so we conclude that $|G| = 8$ and φ must therefore be injective and hence an isomorphism.

2.4.8 Here is one approach (though certainly not the only one): First note that

$$(1\ 2\ 3\ 4)(1\ 2\ 4\ 3)(1\ 2\ 3\ 4)^{-1} = (1\ 4\ 2\ 3) \text{ and } (1\ 2\ 4\ 3)(\ 1\ 2\ 3\ 4)^{-1}(1\ 2\ 4\ 3) = (1\ 2)$$

so by the previous problem, $G = \langle(1\ 2\ 3\ 4),(1\ 2\ 4\ 3)\rangle$ contains a subgroup of order 8. On the other hand, $(1\ 2\ 3\ 4)(1\ 2\ 4\ 3) = (1\ 3\ 2)$, so G contains an element of order three and thus a cyclic subgroup of order three generated by that element. Therefore, by Lagrange's theorem, both 8 and 3 divide the order of G, so $|G|$ must be at least 24, and thus all of S_4.

2.5.2 (a) Reading from the lattice of subgroups for D_{16}, we see that the only subgroups contained in $\langle sr^2, r^4\rangle$ are 1, $\langle sr^6\rangle$, $\langle sr^2\rangle$, $\langle r^4\rangle$, and $\langle sr^2, r^4\rangle$.

(b) Notice that $sr^7r^4 = sr^3 = r^4sr^7$, and it is easy to check that $\langle sr^7, r^4\rangle = \langle sr^3, r^4\rangle$. We see from the lattice for D_{16} then that the only subgroups of $\langle sr^7, r^4\rangle$ are 1, $\langle r^4\rangle$, $\langle sr^3\rangle$, $\langle sr^7\rangle$, $\langle sr^7, r^4\rangle$.

(c) From the lattice for D_{16}, we see that the only subgroups of D_{16} that contain $\langle r^4\rangle$ are $\langle r^4\rangle$, $\langle sr^2, r^4\rangle$, $\langle s, r^4\rangle$, $\langle r^2\rangle$, $\langle sr^3, r^4\rangle$, $\langle sr^5, r^4\rangle$, $\langle s, r^2\rangle$, $\langle r\rangle$, $\langle sr, r^2\rangle$, and D_{16}.

(d) From the lattice for D_{16}, we see that the only subgroups of D_{16} that contain $\langle s\rangle$ are $\langle s\rangle$, $\langle s, r^4\rangle$, $\langle s, r^2\rangle$, and D_{16}.

2.5.4 Note that, for example, the group $\langle s, rs\rangle$ is a subgroup of D_8 which contains both s and rs, but we see from the given lattice for D_8 that the smallest such subgroup is all of D_8. More generally, by similar reasoning we see from the lattice that given any element which does not lie in the subgroup $\langle s, r^2\rangle$, the subgroup generated by that element and s must be all of D_8. The same holds true for r^2s. This gives us eight generating pairs: (s, rs), (s, r^3s), (s, r), (s, r^3), (r^2s, rs), (r^2s, r^3s), (r^2s, r), and (r^2s, r^3). On the other hand, given any element which does not lie in the subgroup $\langle rs, r^2\rangle$, the subgroup generated by that element and rs or that element and r^3s is all of D_8. This observation gives us the last four pairs: (rs, r^3), (rs, r), (r^3s, r), (r^3s, r^3).

3.1.1 $\varphi^{-1}(E)$ is nonempty since it contains the identity element. Now if $x, y \in \varphi^{-1}(E)$, then $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in E$ since E is a subgroup of H, so $xy^{-1} \in \varphi^{-1}(E)$ and so $\varphi^{-1}(E)$ is a subgroup. Now if $E \trianglelefteq H$, and $x \in \varphi^{-1}(E)$, then for any $g \in G$ we have $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} \in E$, and so $gxg^{-1} \in \varphi^{-1}(E)$, proving that $\varphi^{-1}(E) \trianglelefteq G$. Since $\{1\} \leq H$ is trivially a normal subroup, we deduce that $\ker\varphi = \varphi^{-1}(\{1\})$ is a normal subgroup of G.

3.1.3 Any subgroup of an abelian group is normal, so by Theorem 6 in section 3.1 of Dummit and Foote, the set of cosets A/B has a natural group structure. Given any $x, y \in A$, we have $xyx^{-1}y^{-1} = 1 \in B$ and therefore $xyx^{-1}y^{-1}B = B \Rightarrow xyB = yxB \Rightarrow (xB)(yB) = (yB)(xB)$, proving that A/B is abelian. Next consider the nonabelian group D_8 and its normal subgroup $\langle r^2\rangle$. Then the quotient group $D_8/\langle r^2\rangle$ has order 4 and is easily checked to be abelian.

3.1.14 **(a)** Given rational numbers $q_1, q_2 \in [0, 1)$, if $q_1 + \mathbb{Z} = q_2 + \mathbb{Z}$ then we necessarily have $q_1 - q_2 \in \mathbb{Z}$, which obviously implies $q_1 = q_2$ since $|q_1 - q_2| < 1$.

(b) Any rational number can be written in the form p/q where p and q are relatively prime integers. Since $q \cdot p/q = p/q + p/q + \cdots + p/q = p \in \mathbb{Z}$, the coset represented by this rational number has order q which is finite. Given any $N \in \mathbb{Z}$, the element $\frac{1}{N}\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ has order N, proving the second assertion.

(c) We have already shown in part (b) that every element of $\mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ is torsion, but an irrational number multiplied by an integer is never an integer, and so no other element of $\mathbb{R}/\mathbb{Z}$ has finite order. Hence $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.

(d) Every root of unity in $\mathbb{C}^\times$ has the form $e^{\frac{2\pi ip}{q}}$ for relatively prime integers p and q. So we define a map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ by $p/q\mathbb{Z} \mapsto e^{\frac{2\pi ip}{q}}$. One easily checks that this map is a well-defined surjective homomorphism. The kernel is also easily seen to be the integers, which is just the identity element $\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, so this homomorphism has trivial kernel so is thus injective, proving that it is an isomorphism.

3.1.32 The subgroups 1, $\langle -1\rangle$, and Q_8 are all easily seen to be normal. To show that the subgroup $\langle i\rangle$ is normal, we need only show that it is preserved under conjugation by $i^{\pm 1}$ and $j^{\pm 1}$ since i and j generate Q_8. Since $\langle i\rangle$ is cyclic, this amounts to showing that when the element i is conjugated by one of these four elements, the result is still in $\langle i\rangle$. We have

$$\begin{gathered}
iii^{-1} = i \in \langle i\rangle \\
jij^{-1} = ji(-j) = -jk = -i \in \langle i\rangle \\
i^{-1}ii = i \in \langle i\rangle \\
j^{-1}ij = -jij = -jk = -i \in \langle i\rangle
\end{gathered}$$

proving that $\langle i\rangle$ is a normal subgroup. The proof that $\langle j\rangle$ and $\langle k\rangle$ are normal is almost identical, and from the lattice of subgroups for Q_8 we see that this accounts for all subgroups. For the quotient groups, obviously $Q_8/1 \cong Q_8$ and $Q_8/Q_8 \cong 1$. Also, the quotient groups $Q_8/\langle i\rangle$, $Q_8/\langle j\rangle$, and $Q_8/\langle k\rangle$ each have order two (for example, the nontrivial coset in the first group is $j\langle i\rangle$), so they are each isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$. Finally, the quotient group $Q_8/\langle -1\rangle$ consists of four cosets $\{\langle -1\rangle, i\langle -1\rangle, j\langle -1\rangle, k\langle -1\rangle\}$. Since each of these elements has order two in the quotient group, we conclude that this group is isomorphic to the Klein 4-group V_4.

3.1.34 (a) Since $\langle r^k\rangle$ is cyclic, we need only show that the element r^k is sent by conjugation with any element in D_{2n} to another element of $\langle r^k\rangle$. Clearly $r^i r^k r^{-i} = r^k$ for any i. Moreover, for any i we have $sr^ir^k(sr^i)^{-1} = sr^ir^k(sr^i) = sr^{i+k}sr^i = s^2r^{-i-k}r^i = r^{-k} \in \langle r^k\rangle$, proving that $\langle r^k\rangle$ is normal.

(b) We define a map $\pi : D_{2k} \to D_{2n}/\langle r^k\rangle$ by $R \mapsto r\langle r^k\rangle$ and $S \mapsto s\langle r^k\rangle$, where $D_{2k} = \langle R, S \mid S^2 = R^k = 1, RS = SR^{-1}\rangle$. Since $r\langle r^k\rangle$ has order k, $s\langle r^k\rangle$ has order 2, and $(r\langle r^k\rangle)(s\langle r^k\rangle) = (s\langle r^k\rangle)(r\langle r^k\rangle)^{-1}$, this map is a surjective homomorphism. However, $D_{2n}/\langle r^k\rangle$ has order $2k$, so the map must be injective as well and thus an isomorphism.

3.1.42 Since K is normal, $x^{-1}y^{-1}x \in K$, and so multiplying on the right by $y \in K$ gives $x^{-1}y^{-1}xy \in K$. Similarly, since H is normal, $y^{-1}xy \in H$, and so multiplying on the left by $x^{-1} \in H$ gives $x^{-1}y^{-1}xy \in H$. We thus conclude that $x^{-1}y^{-1}xy \in H \cap K \Rightarrow x^{-1}y^{-1}xy = 1 \Rightarrow xy = yx$.