Problem Set 3 Solutions
Math 120

2.4.2 If $A$ is a subset of $B$, then any subgroup of $G$ which contains $B$ also contains $A$, so then by definition

$$\langle A \rangle = \bigcap_{A \subseteq H} H \leq \bigcap_{B \subseteq H} H = \langle B \rangle.$$ 

For an example of a proper subset $A \subset B$ which generates the same subgroup of $G$ as $B$, consider $A = \{r, s\}$, $B = \{r, r^3, s\} \subset D_8$. Clearly we have $\langle A \rangle = \langle B \rangle = D_8$.

2.4.7 Let $G = \langle (1 \ 2), (1 \ 3)(2 \ 4) \rangle$, and note first that $[(1 \ 3)(2 \ 4)](1 \ 2) = (1 \ 4 \ 2 \ 3)$, an element of order 4 in $S_4$. Note too that $(1 \ 4 \ 2 \ 3)(1 \ 2) = (1 \ 3)(2 \ 4) = (1 \ 2)(1 \ 4 \ 2 \ 3)^{-1}$. We may thus define a homomorphism $\varphi : D_8 \to G$ by $\varphi(r) = (1 \ 4 \ 2 \ 3)$ and $\varphi(s) = (1 \ 2)$. As $G$ is generated by $(1 \ 2)$ and $(1 \ 4 \ 2 \ 3)$, and these generators have the same orders and satisfy the same relations as their counterparts in $D_8$, the map must be surjective. Hence $G$ has order at most 8. But one can easily write down eight distinct elements of $G$, so we conclude that $|G| = 8$ and $\varphi$ must therefore be injective and hence an isomorphism.

2.4.8 Here is one approach (though certainly not the only one): First note that

$$(1 \ 2 \ 3 \ 4)(1 \ 2 \ 4 \ 3)(1 \ 2 \ 3 \ 4)^{-1} = (1 \ 4 \ 2 \ 3) \text{ and } (1 \ 2 \ 4 \ 3)(1 \ 2 \ 3 \ 4)^{-1}(1 \ 2 \ 4 \ 3) = (1 \ 2)$$

so by the previous problem, $G = \langle (1 \ 2 \ 3 \ 4), (1 \ 2 \ 4 \ 3) \rangle$ contains a subgroup of order 8. On the other hand, $(1 \ 2 \ 3 \ 4)(1 \ 2 \ 4 \ 3) = (1 \ 3 \ 2)$, so $G$ contains an element of order three and thus a cyclic subgroup of order three generated by that element. Therefore, by Lagrange’s theorem, both 8 and 3 divide the order of $G$, so $|G|$ must be at least 24, and thus all of $S_4$.

2.5.2 (a) Reading from the lattice of subgroups for $D_{16}$, we see that the only subgroups contained in $\langle sr^2, r^4 \rangle$ are $\langle 1 \rangle$, $\langle sr^6 \rangle$, $\langle sr^2 \rangle$, $\langle r^4 \rangle$, and $\langle sr^2, r^4 \rangle$.

(b) Notice that $sr^7 r^4 = sr^3 = r^4 sr^7$, and it is easy to check that $\langle sr^7, r^4 \rangle = \langle sr^3, r^4 \rangle$. We see from the lattice for $D_{16}$ then that the only subgroups of $\langle sr^7, r^4 \rangle$ are $\langle r^4 \rangle$, $\langle sr^3 \rangle$, $\langle sr^7 \rangle$, $\langle sr^7, r^4 \rangle$.

(c) From the lattice for $D_{16}$, we see that the only subgroups of $D_{16}$ that contain $\langle r^4 \rangle$ are $\langle r^4 \rangle$, $\langle sr^2, r^4 \rangle$, $\langle s, r^4 \rangle$, $\langle r^2 \rangle$, $\langle sr^3, r^4 \rangle$, $\langle s, r^2 \rangle$, $\langle r \rangle$, $\langle sr, r^2 \rangle$, and $D_{16}$.

(d) From the lattice for $D_{16}$, we see that the only subgroups of $D_{16}$ that contain $\langle s \rangle$ are $\langle s \rangle$, $\langle s, r^4 \rangle$, $\langle s, r^2 \rangle$, and $D_{16}$.
2.5.4 Note that, for example, the group $\langle s, rs \rangle$ is a subgroup of $D_8$ which contains both $s$ and $rs$, but we see from the given lattice for $D_8$ that the smallest such subgroup is all of $D_8$. More generally, by similar reasoning we see from the lattice that given any element which does not lie in the subgroup $\langle s, r^2 \rangle$, the subgroup generated by that element and $s$ must be all of $D_8$. The same holds true for $r^2s$. This gives us eight generating pairs: $(s, rs), (s, r^3s), (s, r), (r^2, rs), (r^2, r^3), (r^5, r), (r^2, r^5)$, and $(r^5, r^3)$. On the other hand, given any element which does not lie in the subgroup $\langle rs, r^2 \rangle$, the subgroup generated by that element and $rs$ or that element and $r^3s$ is all of $D_8$. This observation gives us the last four pairs: $(rs, r^3), (rs, r), (r^3s, r), (r^3s, r^3)$.

3.1.32 The subgroups $1$, $\langle \pm 1 \rangle$, and $Q_8$ are all easily seen to be normal. To show that the subgroup $\langle i \rangle$ is normal, we need only show that it is preserved under conjugation by $i^\pm 1$ and $j^\pm 1$ since $i$ and $j$ generate $Q_8$. Since $\langle i \rangle$ is cyclic, this amounts to showing that when the element $i$ is conjugated
by one of these four elements, the result is still in \( \langle i \rangle \). We have

\[
\begin{align*}
iii^{-1} &= i \in \langle i \rangle \\
ji^{-1} &= ji(-j) = -jk = -i \in \langle i \rangle \\
i^{-1}ii &= i \in \langle i \rangle \\
j^{-1}ij &= -ji = -jk = -i \in \langle i \rangle
\end{align*}
\]

proving that \( \langle i \rangle \) is a normal subgroup. The proof that \( \langle j \rangle \) and \( \langle k \rangle \) are normal is almost identical, and from the lattice of subgroups for \( Q_8 \) we see that this accounts for all subgroups. For the quotient groups, obviously \( Q_8/1 \cong Q_8 \) and \( Q_8/Q_8 \cong 1 \). Also, the quotient groups \( Q_8/\langle i \rangle \), \( Q_8/\langle j \rangle \), and \( Q_8/\langle k \rangle \) each have order two (for example, the nontrivial coset in the first group is \( j \langle i \rangle \)), so they are each isomorphic to the group \( \mathbb{Z}/2\mathbb{Z} \). Finally, the quotient group \( Q_8/\langle -1 \rangle \) consists of four cosets \( \{\langle -1 \rangle, i\langle -1 \rangle, j\langle -1 \rangle, k\langle -1 \rangle\} \). Since each of these elements has order two in the quotient group, we conclude that this group is isomorphic to the Klein 4-group \( V_4 \).

3.1.34 (a) Since \( \langle r^k \rangle \) is cyclic, we need only show that the element \( r^k \) is sent by conjugation with any element in \( D_{2n} \) to another element of \( \langle r^k \rangle \). Clearly \( r^i r^k r^{-i} = r^k \) for any \( i \). Moreover, for any \( i \) we have \( sr^i r^k (sr^i)^{-1} = sr^i r^k (sr^i) = sr^i+k sr^i = s^2 r^{-i-k} r^i = r^{-k} \in \langle r^k \rangle \), proving that \( \langle r^k \rangle \) is normal.

(b) We define a map \( \pi : D_{2k} \to D_{2n}/\langle r^k \rangle \) by \( R \mapsto r \langle r^k \rangle \) and \( S \mapsto s \langle r^k \rangle \), where \( D_{2k} = \langle R, S \mid S^2 = R^k = 1, RS = SR^{-1} \rangle \). Since \( r \langle r^k \rangle \) has order \( k \), \( s \langle r^k \rangle \) has order 2, and \( (r \langle r^k \rangle)(s \langle r^k \rangle) = (s \langle r^k \rangle)(r \langle r^k \rangle)^{-1} \), this map is a surjective homomorphism. However, \( D_{2n}/\langle r^k \rangle \) has order \( 2k \), so the map must be injective as well and thus an isomorphism.

3.1.42 Since \( K \) is normal, \( x^{-1} y^{-1} x \in K \), and so multiplying on the right by \( y \in K \) gives \( x^{-1} y^{-1} xy \in K \). Similarly, since \( H \) is normal, \( y^{-1} xy \in H \), and so multiplying on the left by \( x^{-1} \in H \) gives \( x^{-1} y^{-1} xy \in H \). We thus conclude that \( x^{-1} y^{-1} xy \in H \cap K \Rightarrow x^{-1} y^{-1} xy = 1 \Rightarrow xy = yx \).