Limiting Forms of the Trace Formula

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Abstract

We carry out the first nontrivial cases of the limiting process proposed by Langlands in his manuscript *Beyond Endoscopy*, with technical variations that enable us to treat the limit unconditionally. This gives an elementary proof, on GL(2), of the classification of forms such that the symmetric square L-function has a pole (including, implicitly, the construction of these forms). The result of this may be seen as one of the simplest cases of the "pipe-dream" Langlands proposes. We also apply similar methods to derive a converse theorem, and to produce a result that generalizes Duke's estimate on the dimension of weight 1 forms to arbitrary number fields – but is sharper, even over $\mathbb{Q}$, than Duke's original estimate.

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I would also like to express my gratitude to Professor Robert Langlands, whose paper *Beyond Endoscopy* was the inspiration for this work, and whose interest, support and suggestions were most helpful.

I would like to thank all my dear friends at Princeton; I would rather not list their names here, but merely trust that they know who I am referring to should they ever read this.

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Contents

Chapter 1

Introduction

1.1 Limits and the Trace Formula

In the most approximate terms, this Thesis is devoted to the technique of *taking a limit in a trace-type formula* to isolate a spectrally "small" set of forms. In Chapters 2 and 4 the "small" set is that of dihedral forms, those corresponding to an orthogonal 2-dimensional Galois representation; in Chapter 3 we isolate a *single* form, and in Chapter 5 we isolate forms of "Galois type."

The precise theorems that can be proved with this technique vary. In Chapters 2 and 4, we give an alternate proof of the classification of dihedral forms due to Langlands-Labesse, [12]. In Chapter 3, we show how one may derive versions of the converse theorem from the trace formula. In Chapter 5 new estimates are obtained for the number of automorphic forms of "Galois type." The method of proof also makes clear the connection of this with the *amplification* method now standard in the analytic theory of automorphic forms.

The motivation for this work was the development of ideas in [13]. We now turn to a more detailed explanation of Langlands' idea, at least in the context we will be using it.

1.2 Langlands' Idea

We begin by discussing Langlands' idea in relative generality; we will eventually be somewhat more specialized in our approach.

Let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. Let π range over all automorphic, cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. If ρ is a representation of the dual group $\mathrm{GL}(2, \mathbb{C})$, we denote by $m(\pi, \rho)$ the order of the pole at $s = 1$ of $L(s, \pi, \rho)$, when defined. Let f be a nice function on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$; we shall denote by $\mathrm{tr}(\pi)(f)$ the trace of the operator defined by f on the representation π.

In *Beyond Endoscopy* ([13]), Langlands suggests the development of a formula of the form:

$$\sum_{\pi} m(\pi, \rho)\mathrm{tr}(\pi)(f) = \sum \ldots \tag{1.1}$$

where the right hand sum ranges over a "geometric" contribution (that is, something resembling a sum over conjugacy classes.) This would therefore "isolate" the π for which $m(\pi, \rho) > 0$. Such π are expected to be functorial transfers from other groups; therefore, one might hope to be able to match the resulting formula with the trace formulae for these groups, and thereby prove these functorial lifts. (One hopes, of course, to do this for more general groups than GL(2).)

In concrete terms, the idea of Langlands can be expressed as follows. From the trace formula we will be able to evaluate:

$$\sum_{\pi} \lambda(n, \pi, \rho)\mathrm{tr}(\pi)(f) = \ldots$$

where $\lambda(n, \pi, \rho)$ is the coefficient of n^{-s} in the Dirichlet series $L(s, \pi, \rho)$. (If ρ is the standard representation, for example, $\lambda(n, \pi, \rho)$ is just the nth Hecke eigenvalue.) At least, this is true for n coprime with those primes where f ramifies – we can then express the summand as the trace of a new function related to f. This is quite enough for our purposes.

The right hand side is a geometric contribution, a sum over conjugacy classes. In concrete terms, the most involved part (the "elliptic" term) is a sum over all quadratic orders and involves their class numbers.

We take $n = p$ to be prime, and sum over all $p < X$, weighted by $\log(p)$. For each π, the quantity $\lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \lambda(p,\pi,\rho)\log(p)$ is equal to $m(\pi,\rho)$. We therefore instead evaluate:

$$\sum_{\pi} \frac{1}{X} \mathrm{tr}(\pi)(f) \sum_{p<X} \log(p)\lambda(p,\pi,\rho)$$

by means of the trace formula. We will then obtain a sum over primes and conjugacy classes on the right hand side, and one can hope to evaluate the resulting limit as $X \to \infty$ by techniques of analytic number theory; we will then obtain a sum just over those forms for which $L(s,\pi,\rho)$ has a pole, and we will obtain an expression for Equation 1.1, as desired.

Implicit in this is the hope of being able to identify the multiplicity of the pole of the L-function purely from the trace formula, without recourse to integral representations; this itself is of interest. Unfortunately the technical details in inverting the spectral sum and limit are rather formidable.

1.3 Discussion of Contents

This thesis carries through, and attempts to understand the ramifications of Langlands' idea in the first few nontrivial cases, and in a setting where the limit on the right hand side can be evaluated.

1.3.1 Dihedral Forms

We continue to use the notation of the previous section. In Chapter 2, we will take ρ to be the symmetric square representation of the dual group and use the Petersson (or Petersson-Kuznetsov) formula instead of the trace formula and sum over *all integers* rather than primes. These introduce important technical simplifications, and make the details manageable. In some sense, it seems that for the application to dihedral representations the Petersson formula is more natural than the trace formula. (The method also works for ρ the standard representation, although it is not written here, and we shall get the expected answer of 0!)

The Petersson-Kuznetsov formula, which is a central tool of the analytic theory of GL(2), may be regarded as a "trace formula with weights": a cuspidal form π appears with a weight given by $1/L(1, \pi, \mathrm{Ad})$, where Ad is the adjoint representation (that is, symmetric square twisted by inverse determinant.) It also includes a contribution of the continuous spectrum. In some sense, this weight is *precisely* what we need for our purposes.

For, in summing over primes, the limit $\lim_{X\to\infty} \frac{1}{X}\sum_{p<X}\log(p)\lambda(p,\pi,\rho)$ is a relatively harmless constant: the multiplicity $m(\pi, \rho)$. In summing over integers, we will be considering a sum of the form $\lim_{X\to\infty} \frac{1}{X}\sum_{n<X}\lambda(n,\pi,\rho)$, which essentially evaluates the residue of $L(s, \pi, \rho)$ at $s = 1$. This is a much less manageable weight, because it varies in some relatively incomprehensible way as π varies. However, for $\rho = \mathrm{Sym}^2$ this residue is perfectly canceled by the weights of the Petersson-Kuznetsov formula! (Or, to be exact, for π such that $L(s, \pi, \mathrm{Sym}^2)$ has a pole, the residue of this L-function divided by $L(1, \pi, \mathrm{Ad})$ is just $1/L(1, \omega_\pi)$ where ω_π is the central character. Since we can sum over π with a prescribed central character, this is effectively a constant.) We should stress, however, that this "miracle" is convenient but not at all essential.

Here is a loose description of the method. We will not be dealing with the function

f from above any more and f will now always denote a modular form with respect to a group $\Gamma_0(N)$. For such a form, let $a_n(f)$ be the nth Fourier coefficient.

The Kuznetsov formula evaluates:

$$\sum_f a_n(f)\overline{a_m(f)}h(t_f)$$

the sum ranging over an L^2 basis of, say, Maass forms f of prescribed level and Nebentypus, where $1/4+t_f^2$ is the Laplacian eigenvalue of f. h is a test function with various good properties, and it will show up (in a transformed fashion) on the other side of the formula. Actually, there is a contribution from holomorphic forms and Eisenstein series, but for ease of explanation we shall ignore these for now.

The "weight" referred to earlier enters through the difference between $a_n(f)$, the nth Fourier coefficient, and λ_n, the nth Hecke eigenvalue (one can regard this as the difference between the normalization to have L^2 norm one and the normalization to have first Fourier coefficient 1; this is expressed as a value at 1 of an L-function by Rankin-Selberg.)

We will then sum over all $n < X$ and divide by X. (For technical reasons, we sum with a "smooth weight function," but this is peripheral to the concept.) We will then obtain an evaluation of

$$\sum_f \left(\frac{1}{X}\sum_{n<X} a_n(f)\right)\overline{a_m(f)}h(t_f)$$

Now, as X approaches infinity, the bracketed term is merely the residue at $s = 1$ of the L-function of f at zero. Of course, the standard L-function is holomorphic (for cusp forms) and so this residue is zero. We therefore expect that the limit on the right hand side equals 0; this is not difficult to do, and we carry it out in Chapter 2.

More interesting is the cases where one proceeds as above, but with the limit $\frac{1}{X}\sum_{n<X} a_{n^2}(f)$. In that case, we are (more or less) evaluating the residue at $s = 1$ of

the symmetric square L-function, and in that case this will be nonzero precisely for the forms originally constructed by Maass: those associated to Grössencharacters of quadratic fields! One expects, in carrying out the limiting process on the right hand side, both to be able to construct these forms and to show that they exhaust all forms f for which $L(s, \mathrm{Sym}^2 f)$ has a pole at $s = 1$.

To be precise, carrying this through gives amounts (more or less, ignoring complications of continuous spectrum) to evaluating:

$$\sum_{f\colon m(f,\mathrm{Sym}^2)=1} \lambda_m(f) h(t_f)$$

where $\lambda_m(f)$ is the mth Hecke eigenvalue of f. To give the flavor of the answer, a special case (with $m = 1$, and other simplifications) of what we show is the following Theorem. (We introduce somewhat *ad hoc* notation to avoid having to define all the notation of Chapter 2.)

Theorem 1. *(Special case of Chapters 2 and 4) Let $\mathfrak{o}$ be an order of the real quadratic field $\mathbb{Q}(\sqrt{D})$, and let N be the discriminant of $\mathfrak{o}$. Fixing an embedding of $\mathbb{Q}(\sqrt{D})$ into $\mathbb{R}$, let ϵ_0 be a positive fundamental unit for $\mathfrak{o}^\times$; set $\delta = 1$ or 2 according to whether, respectively, $\mathrm{Norm}(\epsilon_0) = 1$ or $\mathrm{Norm}(\epsilon_0) = -1$, and let $h(\mathfrak{o})$ be the class number of $\mathfrak{o}$. Let h be any test function on $\mathbb{R}$. Let f vary over cuspidal Maass newforms of level dividing N and Nebentypus χ_D, the quadratic character associated with $\mathbb{Q}(\sqrt{D})$; for such f, let t_f be defined so that f has eigenvalue $1/4 + t_f^2$. Then*

$$\sum_{\substack{f:m(f,\mathrm{Sym}^2)=1 \\ t_f \neq 0}} h(t_f) = \frac{h(\mathfrak{o})}{\delta} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} h\Big(\frac{k\pi}{\delta \log(\epsilon_0)}\Big) \tag{1.2}$$

Although this is not precisely in the form of Equation 1.1, it is nevertheless precisely a formula of the type desired: if it is known for all m and a sufficiently large class of test functions h, it amounts to a construction and classification of all forms with $m(f, \mathrm{Sym}^2) = 1$.

These results are known, of course: the construction of the forms is due to Hecke [9] (in the holomorphic case) and Maass [14] (in the Maass-form case), and the classification follows from works of Gelbart-Jacquet [7] and Labesse-Langlands [12].

However, the proof contained here is quite different. It is also very concrete: one sees the fundamental units of quadratic fields arise in a remarkable hands-on fashion! On the other hand, this obscures some of the conceptual generality, and it may be difficult to carry this out for more general groups.

Chapter 4 is primarily technical in nature and discusses the generalization of this to number fields. We focus, in particular, on the case of a totally real field, but the discussion there is primarily intended to sketch that the same procedure can be done over any fields; in particular, the units do not intervene dangerously, as they sometimes do in analytic investigations of this nature.

Remark 1. The natural definition of a "dihedral" automorphic form is one that is associated to a representation of the Weil group with dihedral image. This is more general than requiring that the symmetric square have a pole, as a dihedral subgroup of $\mathrm{GL}(2, \mathbb{C})$ need not be conjugate to a subgroup of $O(2, \mathbb{C})$. The classification of these, more general, dihedral representations could be effected by this technique; one would incorporate a twist by a Dirichlet character.

1.3.2 Rankin-Selberg Convolutions and Converse Theorems

A second application, also suggested by Langlands is the following: rather than averaging the residue of $L(s, \mathrm{Sym}^2 f)$ over the spectrum, average the residue of $L(s, f \times \sigma)$, where σ is a Galois representation. Of course, it is not entirely clear what this means, since this L-function is not even well-defined! (At best, one can define it as a meromorphic function, using base change, for σ solvable.) Nevertheless, one might hope to obtain nontrivial information in the direction of modularity in this fashion.

It turns out that what one obtains from this is a version of the converse theorem for σ! To be exact, this gives an "analytic" version of the converse theorem, which implies the converse theorem in the form of "functional equations and analyticity of all twists implies modularity." It is of some interest as it shows how the trace formula (or the Petersson-Kuznetsov formula) leads naturally to a converse theorem. It would be of interest to see if this can be duplicated for other groups.

1.3.3 Higher Symmetric Powers

Finally, in Chapter 5 we discuss applying the same procedure to a *higher* symmetric power than the second. This, of course, also has a relation to Galois representations: for example, one expects that the forms f with $m(f, \mathrm{Sym}^{12}) = 1$ are precisely those f parameterized by an icosahedral Galois representation!

After briefly summarizing the reasons why the naive generalizations do not work – this was pointed out by Sarnak [17] – we then show how the same techniques can be used to improve on Duke's bound on the dimension of the space of holomorphic weight 1 forms.

We work over a number field in Chapter 5. A special case of what we prove is the following result over $\mathbb{Q}$:

Theorem 2. *(Special case of Chapter 5) Let χ be a Dirichlet character of modulus q. Let $S_1(q, \chi)$ be the $\mathbb{C}$-vector space of weight 1 holomorphic forms of level q and Nebentypus χ. Then, for all $\epsilon > 0$, one has the bound*

$$\dim\, S_1(q, \chi) \ll_\epsilon q^{6/7+\epsilon}$$

This sharpens a result of Duke, who proved the corresponding bound with $\frac{6}{7}$ replaced by $\frac{11}{12}$.

1.3.4 Related Work

Duke kindly brought to my attention the work of Mizumoto [16]. This relates to what is being done in Chapters 2 and 4. Indeed, Mizumoto is analytically continuing the symmetric square L-functions in the special case of holomorphic forms of full level over fields of class number 1. His method, when unwound, is similar to ours: the Poincaré series he uses is closely related to the Petersson formula that we use (or vice versa!), and he carries out a limit, encountering Kloosterman sums in a similar context. In Mizumoto's setting, one does not encounter the issue of a pole at all – the central point of this work – and the emphasis is quite different (in the holomorphic setting one does not encounter many of the difficulties associated with the infinite dimensional Maass-form type spaces where we work) but the basic ideas are similar.

1.4 Format of Thesis

The format of this thesis is as follows: Chapter 2 is devoted to carrying through the process outlined above, over $\mathbb{Q}$, and in Chapter 3 the relation of these ideas to converse theorems is outlined – not in maximal generality, but in a special case that indicates, essentially, how one obtains the conditions on L-functions that arise in the converse theorem out of the trace formula. In Chapter 4, we sketch the modifications necessary to work over a number field. This involves a derivation of the Petersson-Kuznetsov formula; forms of this do exist, but it is most convenient for our purpose to derive a particular form following the representation-theoretic ideas of Cogdell-Piatetski-Shapiro. In Chapter 5, we show the applicability of an *approximate* version of this technique, when one relaxes the requirement for exact results and merely aims for estimates. The consequence will be a generalization of a result of "Duke type" to number fields; it is, *even over* $\mathbb{Q}$, sharper than Duke's original bound. (This generalization – in the context of the "amplification method" – was independently and essentially simultaneously observed by P. Michel; see [15]).

The Appendix gathers together several points not treated in the text. It discusses, in terms of Langlands' philosophy and the translation to more concrete data, the expected parameterization of dihedral forms. It also contains a density result for Bessel transforms, a discussion of trace-type formulas and some results on partial sums of coefficients of L-series.

Chapters 2, 3 and 4 are, more or less, independent; although there are various cross-references, they are not essential for reading purposes. Chapter 5 depends to a slight extent on Chapter 4 for background, but is also essentially self-contained. Chapter 4 is also the most involved, technically and notationally, and it is probably better to read Chapter 2 to get a feel for the underlying idea.

There is an index of notation that precedes the bibliography, which may be of assistance in navigating this thesis.

Finally, a word on notation that will be used throughout this thesis: an ϵ which is otherwise undefined means "the formula is valid for any positive value of ϵ." For example, $f(x) \ll x^\epsilon$ means that the function f grows slower than any positive power of x. The implicit constant of the $\ll$, however, depends on ϵ.

Chapter 2

Dihedral Forms over $\mathbb{Q}$

2.1 Introduction

This Chapter carries through, in the context of forms over $\mathbb{Q}$, the procedure outlined in the introduction to isolate those "dihedral" forms f such that $L(s, f, \mathrm{Sym}^2)$ has a pole at $s = 1$.

The main result of the Chapter is the "trace formula" over dihedral forms: a linear functional that averages the residue of the symmetric square L-function over the spectrum. It is obtained in Proposition 3 and rewritten more conveniently in Equation 2.16 and Equation 2.18. This leads formally to the construction and classification of these dihedral forms: see Theorem 4 in Section 2.6. We only sketch the argument leading from the trace formulae to the classification, as it is relatively standard.

The main purpose of the Chapter is the proof of the Theorem using the limiting technique, and we have not attempted to make our proof as "minimalistic" as possible. In particular, we appeal to knowledge of integral representations of Rankin-Selberg L-functions and the symmetric square L-function; this does not compromise the result in any way, and streamlines the exposition.

As mentioned in the Introduction to this Thesis, this theorem is known, but the proof here is of an entirely different nature to previous ones.

In Section 2, the required trace formulae of Petersson and Kuznetsov are stated, along with integration transformation formulae (proved in Appendix.)

In Section 3, we carry out the limiting process, and obtain the resulting "trace formula over dihedral forms": see Equations 2.16 and Equation 2.18.

In Section 4, we briefly discuss the case (in the language of the Introduction to this thesis) of $\rho = \mathrm{St}$, the standard representation; we show how the same process as in Sections 2 and 3 shows that $m(f, \mathrm{St}) = 0$ for all forms (and, indeed, a still stronger result.)

In the final two sections, we sketch the passage from a trace formula to the classification Theorem 4. There is essentially only one non-formal part to this, which is a density assertion for the spectral test functions (which can be happily assumed without loss of continuity, but is derived in Section 6.3.2 of the Appendix.) However, there are other interesting issues which arise, most notably the contribution of the continuous spectrum to the answer.

(The Appendix contains various relatively well-known results that are relevant to this Chapter: the translation from classical to adelic language and various results on Bessel functions.)

2.2 Preliminaries

Let χ be a Dirichlet character to the modulus N. If N divides c, we define the Kloosterman sum $S_\chi(m, n, c)$ via:

$$\sum_{(\mathbb{Z}/c\mathbb{Z})^\times} \chi(x) e((mx + nx^{-1})/c)$$

in which χ is regarded as a character of $(\mathbb{Z}/c\mathbb{Z})^\times$, because N divides c. Here, as always, we define $e(\alpha) = e^{2\pi i \alpha}$.

We refer to Iwaniec's book [10], for a derivation of the Petersson and Kuznetsov formulae that we shall use. The notation we use is also based on Iwaniec's book.

Throughout this thesis, we will be using "smooth summation." That is, we will fix a C^∞ function $g(x)$ compactly supported in $(0,\infty)$ and of integral 1, and, given a sequence of numbers c_n, will usually use $\sum_n g(n/X)c_n$ rather than $\sum_{n<X} c_n$. This, for technical reasons, is very convenient. When it is not necessary to make the smooth function explicit, we shall use the notation $\sum_{n\sim X} c_n$, meaning the sum with an appropriate smooth weight function of integral 1; it should be thought of as a smoothed version of $\sum_{n<X} c_n$.

2.2.1 Petersson formula

We begin by stating the "classical" Petersson formula. Let $S_k(\Gamma_0(N),\chi)$ be the space of holomorphic cusp forms of weight k for the group $\Gamma_0(N)$, with Nebentypus χ. We must have $(-1)^k = \chi(-1)$; else, the space is trivial. For each form $f \in S_k(\Gamma_0(N),\chi)$, let $c_n(f)$ be the nth Fourier coefficient, and define:

$$a_n(f) = \sqrt{\frac{(k-2)!}{(4\pi)^{k-1}}}c_n(f)/n^{(k-1)/2}$$

Then:

$$\sum_f a_n(f)\overline{a_m(f)} = \delta_{mn} + 2\pi i^k \sum_{\substack{c\equiv 0 \bmod N \\ c>0}} \frac{1}{c} S_\chi(m,n;c) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) \tag{2.1}$$

The sum is over an orthonormal basis for $S_k(\Gamma_0(N),\chi)$, with respect to the Petersson inner product.

2.2.2 Petersson-Kuznetsov formula

This formula was generalized by Kuznetsov, to give a formula that includes Maass as well as holomorphic forms – in other words, the whole spectrum.

Let χ be a Dirichlet character to the modulus N.

$$\begin{aligned}\sum_f h_f(\varphi)a_n(f)\overline{a_m(f)} \quad &+ \sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h^{\pm}(t)\eta_{\mathfrak{c}}(n,1/2+it)\overline{\eta_{\mathfrak{c}}(m,1/2+it)}dt \\ = \quad &\sum_{\substack{c\equiv 0 (\mathrm{mod}\, N)\\ c>0}} \frac{1}{c}\varphi(\frac{4\pi\sqrt{|nm|}}{c})S_\chi(n,m;c) \end{aligned} \tag{2.2}$$

φ is a compactly supported function on $(0,\infty)$. The left hand summation ranges over an orthonormal basis for modular forms f with respect to $(\Gamma_0(N),\chi)$, holomorphic *and* Maass. The sum over $\mathfrak{c}$ is over cusps. The definitions of $h^{\pm}$ and $h_f(\varphi)$ are given below. The $a_n(f)$ and $\eta_{\mathfrak{c}}$ are the Fourier coefficients of cusp forms and Eisenstein series respectively, normalized so to be of size around 1.

To be precise, given a Maass cusp form f with L^2 norm 1 and eigenvalue $1/4+s^2$, one may write its Fourier expansion:

$$f(z) = \sum_{n\neq 0} \rho(n)W_s(nz)$$

where $W_s(x+iy) = 2\sqrt{y}K_{s-1/2}(2\pi|y|)e(x)$. One then defines:

$$a_n(f) = \left(\frac{4\pi|n|}{\cosh(\pi s)}\right)^{1/2}\rho(n)$$

The normalization for the Fourier coefficients of holomorphic forms is as in the Petersson formula. The normalization for the Eisenstein series is similar to that in the case of Maass cusp forms; we refer to Iwaniec for details in that case.

$h_f = h_f(\varphi)$ is a weight function, and the map between φ and $h_f(\varphi)$ is given by an appropriate integral transform that is explicated below; in fact, $h_f(\varphi)$ depends only on the Laplacian eigenvalue (or "infinity type") of f.

(It is convenient to think of the Petersson formula, Equation 2.1, as being a form of Equation 2.2, where $h_f = 1$ exactly for those forms f of weight k, and $\varphi(x) = 2\pi i^k J_{k-1}(x)$. This is not precisely accurate, as the Petersson formula involves an additional "diagonal term" δ_{mn}, which is closely related to the fact that this choice of φ is not compactly supported. However, this δ_{mn} will make no difference to our analysis: it will drop out in the limit.)

Let J_ν, K_ν denote the usual J- or K- Bessel function. The functions $h_f(\varphi)$ and $h^{\pm}(t)$ are given as follows:

1. When f is a holomorphic form of even weight k, it equals $h_f(\varphi) = h_k = i^k \int_0^\infty \varphi(x) J_{k-1}(x) x^{-1} dx$.

2. If f is a Maass form or Eisenstein series of eigenvalue $1/4 + t_f^2$ it varies, according to the sign of the product nm.

   (a) If $nm > 0$, then $h_f(\varphi) = h^+(t_f)$, where $h^+(t_f) = \int_0^\infty B_{2it_f}(x)\varphi(x) x^{-1} dx$, and $B_\nu(x) = (2\sin(\pi\nu/2))^{-1}(J_{-\nu}(x) - J_\nu(x))$.

   (b) If $nm < 0$, then $h_f(\varphi) = h^-(t_f) = \frac{4}{\pi}\cosh(\pi t_f) \int_0^\infty K_{2it_f}(x)\varphi(x) x^{-1} dx$.

The Kloosterman sum $S_\chi(m, n; c)$ is as defined before.

2.2.3 Integral transformation formulae

Since we will be applying Poisson summation, we will have need of the following integral transforms, which can be derived easily from the formulae for Fourier transforms of Bessel functions. These are derived in the Appendix.

First, some remarks on Fourier transform and its normalization. Let $f(x)$ be a function on $\mathbb{R}$; its Fourier transform is $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$. Then $f(x) =$

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{-ikx}dk$. With this normalization, the Fourier transform of the constant function 1 is the distribution $2\pi\delta$, where δ is the measure of mass 1 supported at 0, and the Fourier transform of $f(x)g(x)$ is $(1/2\pi)\hat{f}\star\hat{g}$, where $\star$ is convolution.

Let φ and $h^{\pm}$ be related as in the previous Subsection. Let $\Delta(\lambda)$ be the Fourier transform of $\varphi(x)/x$ and $\widehat{h^{\pm}}$ the Fourier transform of $h^{\pm}$. We have the formulae:

$$\widehat{h^{-}}(\lambda)=\widehat{h^{-}}(-\lambda)=\frac{1}{2}(\Delta(\sinh(\lambda/2))+\Delta(-\sinh(\lambda/2))) \tag{2.3}$$

$$\widehat{h^{+}}(\lambda)=\widehat{h^{+}}(-\lambda)=\frac{1}{2}(\Delta(\cosh(\lambda/2))+\Delta(-\cosh(\lambda/2))) \tag{2.4}$$

If k is odd, we also have the following, with $\varphi(x)=2\pi i^k J_{k-1}(x)$:

$$\frac{1}{2}(\Delta(\lambda)+\Delta(-\lambda))=\int_0^{\infty}\cos(\lambda x)\frac{\varphi(x)}{x}dx=\begin{cases}\frac{2\pi i^k}{k-1}\cos((k-1)\arcsin(\lambda)), & |\lambda|\le 1;\\ (-1)^{(k-1)/2}\frac{2\pi i^k}{k-1}e^{-(k-1)\cosh^{-1}(\lambda)}, & |\lambda|\ge 1\end{cases} \tag{2.5}$$

2.3 Isolation of dihedral representations

We will now prove the main result. The limiting process on the geometric side is carried out in the first subsection: see Proposition 3.

We will work in the following context. Let D be a fundamental discriminant. By this we mean that $|D|$ should be the discriminant of $\mathbb{Q}(\sqrt{D})$; it is possible D is negative. Let $\chi=\chi_D$ be the quadratic character $\left(\frac{D}{\cdot}\right)$, that is, the quadratic character associated to $\mathbb{Q}(\sqrt{D})$. It is a Dirichlet character to the modulus $|D|$. Let f be an integer and let $N=|D|f^2$. One expects to find "dihedral" forms, those whose symmetric square has a pole, on the group $\Gamma_0(N)$ with Nebentypus χ, and indeed we expect all dihedral forms will occur in this way as we vary N.

In the limiting process, the treatments for $D>0$ and $D<0$ vary slightly from each other. The exposition in the text will focus on the case $D > 0$ and will indicate the (very minor) changes needed for the case $D < 0$.

It is convenient to make the following:

Definition 1. *Define a constant* $c(N)$ *as follows:*

$$c(N) = \begin{cases} \left(\sqrt{N}\prod_{p|N}(1+1/p)\right)^{-1}, \, D > 0 \\ i\left(\sqrt{N}\prod_{p|N}(1+1/p)\right)^{-1}, \, D < 0 \end{cases}$$

It is not really a function only of N – it would probably be better denoted $c(N, \chi)$, but it will be clear from context what is going on, since when we use it we will be dealing with $D > 0$ and $D < 0$ separately.

Format of this section: In Subsection 2.3.1, the limiting process is carried out. In doing so, certain information about some local sums is required – we borrow the results from Subsection 2.3.3, where they are proved. The result for the limit L is in Equation 2.13. In Subsection 2.3.2, we translate the results from the "geometric side" function φ to being in terms of the "spectral side" functions $h^{\pm}$.

2.3.1 Geometric side

We will work in the space of modular forms for $(\Gamma_0(N), \chi)$, either applying the Petersson-Kuznetsov formula in the case where $D > 0$, or just the Petersson formula if $D < 0$; with this understood, the analysis proceeds identically for both cases, with φ an appropriate Bessel function in the latter case. If $D < 0$ we will choose an odd integer k, and will apply the Petersson formula to holomorphic forms of weight k.

Let g be a compactly supported, positive, C^∞ function on $(0, \infty)$. We will assume

$$\int_0^\infty g(x)dx = 1$$

Its precise behavior is unimportant; it is only used to truncate the sums smoothly and it will vanish from the analysis eventually.

We will be analyzing the asymptotic behavior of the sum

$$\lim_{X\to\infty}\left(\frac{1}{X}\sum_f h_f(\varphi)\sum_{n=1}^{\infty} g(n/X)a_{n^2}(f)\overline{a_m(f)}\right)+ \tag{2.6}$$
$$\left(\frac{1}{X}\sum_{n=1}^{\infty} g(n/X)\sum_{\mathfrak{c}}\frac{1}{4\pi}\int_{-\infty}^{\infty} h^{\pm}(t)\eta_{\mathfrak{c}}(n^2,1/2+it)\overline{\eta_{\mathfrak{c}}(m,1/2+it)}\right)$$

This sum is meant to average over the spectrum (weighted by the function function function function h_f) the residue of the symmetric square L-function.

The sign of $h^{\pm}$ is determined according to whether $m > 0$ or $m < 0$. The interpretation is slightly different if $D < 0$: in that case, there is no integral and the sum is over an orthonormal basis for $S_k(\Gamma_0(N), \chi)$. In other words, $h_f(\varphi)$ should be regarded as being supported (in the f aspect) on holomorphic forms of weight k.

We will often simply refer to the integral term as the "Continuous Spectrum Contribution." As will be seen in Subsection 2.5.4, it may be explicitly evaluated.

We will be assuming that $(m, N) = 1$, for convenience. We will denote the limit of Equation 2.6 (it exists, as we will prove!) by L. It depends on m, D, N and φ.

The Petersson-Kuznetsov formula shows that Equation 2.6 equals:

$$\frac{\sum_f h_f(\varphi)\sum_n g(n/X)a_{n^2}(f)\overline{a_m(f)} + (\text{Continuous Spectrum Contribution})}{X}$$
$$= \frac{1}{X}\sum_{N|c}\frac{1}{c}\sum_n g(n/X)\varphi(\frac{4\pi\sqrt{m}}{c}n)S_\chi(n^2,m,c) \tag{2.7}$$

If $D < 0$ we apply the Petersson formula in the form Equation 2.1. In that case, the formula is as above, but there does not exist a continuous spectrum contribution, the left-hand side sum is only over holomorphic forms of weight k, and $\varphi(x) = 2\pi i^k J_{k-1}(x)$. (As remarked previously, the contribution of the term δ_{nm} from Equation 2.1 vanishes in taking the limit as $X \to \infty$.)

We will need the following assumption on φ:

Hypothesis 1. There is a positive integer K so that $|\varphi(x)| \ll \min(x^{-1/2}, x^K)$, and, for all $k \leq K$, the kth derivative $|\varphi^{(k)}(x)| \ll_k 1$.

We will need K to be sufficiently large; just how large will come out from the proof. In particular, in the case of $D < 0$ and holomorphic forms of weight k, this proof will only apply for k sufficiently large; otherwise the Bessel function $J_{k-1}(x)$ will not satisfy Hypothesis 1.

We expand out the Kloosterman sum and split the n-sum into arithmetic progressions mod c, so we write $n = k + \lambda c$ with k defined mod c and $\lambda \in \mathbb{Z}$. Equation 2.6 then becomes:

$$\frac{1}{X}\sum_{N|c}\frac{1}{c}\sum_{\substack{k\in\mathbb{Z}/c\mathbb{Z}\\ x\in(\mathbb{Z}/c\mathbb{Z})^\times}}\chi(x)e(\frac{mx^{-1}+k^2x}{c})\sum_{n\in\mathbb{Z}, n\equiv k\,(c)} g(\frac{n}{X})\varphi(4\pi\sqrt{m}\frac{n}{c}) \tag{2.8}$$

We apply Poisson summation to the n sum; let ν be the Fourier-transform parameter. This gives:

$$\frac{1}{X}\sum_{N|c}\sum_{\nu\in\mathbb{Z}}\frac{1}{c^2}\left(\sum_{\substack{x\in(\mathbb{Z}/c\mathbb{Z})^\times\\ k\in(\mathbb{Z}/c\mathbb{Z})}}\chi(x)e(\frac{k^2x+mx^{-1}-\nu k}{c})\right)\left(\int_{-\infty}^{\infty} e(\frac{\nu x}{c})g(\frac{x}{X})\varphi(4\pi\sqrt{m}\frac{x}{c})dx\right) \tag{2.9}$$

It is now convenient to define the *local sum:*

$$\mathcal{A}(\nu; c, m) = \sum_{k\in(\mathbb{Z}/c\mathbb{Z}), x\in(\mathbb{Z}/c\mathbb{Z})^\times} e(\frac{k^2x+mx^{-1}-\nu k}{c})\chi(x) \tag{2.10}$$

At this point, some discussion of convergence in Equation 2.9 is necessary:

Proposition 1. *Suppose that $K \geq 4$. The double sum in Equation 2.9 converges absolutely. If one truncates the ν sum to the range $|\nu| \leq X^\delta$, for any fixed $\delta > \frac{2}{K-3}$, this does not affect the limit as $X \to \infty$.*

Proof. Repeated integration by parts shows that, if $\nu \neq 0$,

$$\int_{-\infty}^{\infty} e(\frac{\nu x}{c}) g(x/X) \varphi(4\pi\sqrt{m}\frac{x}{c}) dx \ll_k X|\frac{\nu}{c}|^{-k} \min(X,c)^{-k} = X|\nu|^{-k} \max(1, (\frac{c}{X})^k)$$

for each $k \leq K$. On the other hand, the assumed decay $\varphi(x) \ll \min(1, x^K)$ demonstrates that, using a crude absolute-value bound, $\int_x e(\frac{\nu x}{c}) g(x/X) \varphi(4\pi\sqrt{m}\frac{x}{c}) dx \ll X \min(1, (\frac{X}{c})^K)$. We put these estimates together, using the first bound for small c and the second bound for large c. Let $T \geq 1$ be a parameter to be determined. Using the trivial bound on the local sum $|\mathcal{A}(\nu; c, m)| \leq c^2$, we see that the total contribution of a given value of ν to Equation 2.9 is bounded by:

$$\frac{1}{X} \sum_c \left| \left(\int_x e(\nu x) g(\frac{x}{X}) \varphi(4\pi\sqrt{m}\frac{x}{c}) dx \right) \right| \ll |\nu|^{-K} \sum_{c<XT} \max(1, (\frac{c}{X})^K) + \sum_{c>XT} (\frac{X}{c})^K$$

$$\ll_K X(|\nu|^{-K} T^{K+1} + T^{-K+1}) \tag{2.11}$$

The optimal value is $T = \sqrt{\nu}$, which gives a bound of $O(X|\nu|^{1/2-K/2})$. On the other hand, for $\nu = 0$ the bound of $X\min(1, (\frac{X}{c})^k)$ shows the absolute convergence of the c sum.

Both assertions of the Proposition now follow easily. □

Now we return to our analysis, knowing that the range of the ν sum can be taken "not too large" compared to X. Substituting the definition of $\mathcal{A}(\nu; c, m)$, and replacing x by cx in the integral, Equation 2.9 becomes:

$$\frac{1}{X} \sum_{\nu \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{2\pi i \nu x} \left(\sum_{N|c} \frac{\mathcal{A}(\nu; c, m)}{c} g(\frac{cx}{X}) \right) \varphi(4\pi\sqrt{m}x) dx \tag{2.12}$$

The idea of the rest of the evaluation may be described as follows: for most values of ν, the inner sum, which is just a smoothed version of the sum $\sum_c \mathcal{A}(\nu; c, m)/c$, will exhibit cancellation. These values of ν will contribute only $o(X)$ to our sum;

their contribution vanishes as $X \to \infty$. For certain values of ν, however, the inner sum will not exhibit cancellation, and these will dominate in the limit as $X \to \infty$. These "special" values of ν will be identified in Section 2.3.3, and will correspond to solutions of Pell equations, and hence to units of quadratic fields. In particular, we can deduce the following:

Proposition 2. *Let x be an integer. We define $\delta(x)$ as follows:*

$$\begin{cases} \delta(x) = c(N), & \text{if } x = Ny^2 \text{ for } y \neq 0, y \in \mathbb{Z} \\ \delta(x) = c(N)/2, & \text{if } x = 0 \\ \delta(x) = 0, & \text{else} \end{cases}$$

Let $\alpha > 0$. Then there exists an absolute $A > 0$ so that:

$$\sum_{N|c} \frac{\mathcal{A}(\nu; c, m)}{c} g(c\alpha) = \frac{6}{\pi^2} \frac{\delta(\nu^2 - 4m)}{\alpha} + O_\epsilon((1 + |\nu|)^A \alpha^{-1/2-\epsilon})$$

Proof. This follows by expressing the right-hand side as an integral of the Mellin transform of g against the zeta-function $Z(s)$ associated to $\mathcal{A}(\nu; c, m)$, shifting contours, and using the properties of $Z(s)$ given in Theorem 3 (see Subsection 2.3.3). (See also the section 6.5 of the Appendix for some more details on this technique.) □

Denote by Δ the Fourier transform of $\varphi(x)/x$. Applying Proposition 1 and Proposition 2, we find that Equation 2.12 becomes, for any $\delta > \frac{2}{K-3}$,

$$\frac{6}{\pi^2} \sum_{\nu \in \mathbb{Z}} \delta(\nu^2 - 4m) \Delta(\frac{\nu}{2\sqrt{m}}) + O(\sum_{\nu < X^\delta} (1 + |\nu|)^A X^{-1/2+\epsilon})$$

So long as δ is sufficiently small (i.e. K is sufficiently large) the error term is $o(X)$. Substituting the definition of the function $\delta(x)$, this becomes:

Proposition 3. *The limit L of Equation 2.6 exists and equals:*

$$L = c(N)\frac{6}{\pi^2}\sum_{\nu\in\mathfrak{S}'}{}'\Delta(\frac{\nu}{2\sqrt{m}}) \tag{2.13}$$

Here $c(N)$ is given in Definition 1, $\mathfrak{S}'$ is the set of integers ν such that $\nu^2-4m = Ny^2$ for some integral y, Δ is the Fourier transform of $\varphi(x)/x$, and by $\sum'$ one mean the sum taken over $\nu \in \mathfrak{S}'$ but with the contribution of any $\nu^2 = 4m$ halved.

Remark 2. One can check that the constant A may be taken to be $1/2 + \epsilon$.

Checking the points at which we invoked Hypothesis 1, we see that it would have sufficed for our argument to take $K = 10$. In particular, this applies to the Bessel function $\varphi(x) = J_{k-1}(x)$ so long as $k \geq 11$; therefore, the arguments here apply to holomorphic forms of weight greater than or equal to 11.

Since, at the end of this Chapter, we indicate how to modify the procedure to apply to all holomorphic forms – including those of weight 1 – we do not aim for optimality here.

2.3.2 Translation in terms of $h^{\pm}$, h_k

In the previous section, the limit was computed in terms of φ; here we translate Equation 2.13 to a formula in terms of the spectral test functions $h^{\pm}, h_k$, incorporating also an explicit identification of the set $\mathfrak{S}'$ in terms of fundamental units.

It is important to note that the resulting formulae, *a priori*, hold only for those h^+ that are associated to φ satisfying Hypothesis 1; however, a density argument, discussed in the final section of this Chapter, extends it to all functions.

This difficulty does not occur in the holomorphic case. In view of the necessity for φ to satisfy the decay condition near 0 prescribed by Hypothesis 1, the Petersson-formula argument only applies to weights $k \gg 1$. However, as commented on in the Remark in the final section of this Chapter, this restriction can be removed by a more elaborate argument involving the Petersson-Kuznetsov formula and not merely the Petersson formula. The necessity of such an argument can perhaps be seen by noting that in the case of *weight 1* holomorphic forms, no Petersson formula exists at all (they are not even spectrally isolated).

Maass case ($D > 0$)

In this section we consider the case $D > 0$. We expect to find lifts of Grössencharacters of the quadratic field $\mathbb{Q}(\sqrt{D})$. Assume first that $m > 0$, so we are dealing with the transform $\varphi \to h^+$ and we wish to express the results in terms of h^+; this will be done by applying the transformation formula Equation 2.4.

The set $\mathfrak{S}'$ used in Proposition 3 is $\{\nu : \nu^2 - Df^2y^2 = 4m \text{ for some } y\}$. In other words, m is the norm of the element $\frac{1}{2}(\nu - \sqrt{N}y)$ in the quadratic order of discriminant N contained in $\mathbb{Q}(\sqrt{D})$. Let $\mathfrak{o}_{D,f}$ be this order in $\mathbb{Q}(\sqrt{D})$ of discriminant N; explicitly, it is $\mathbb{Z} + f\mathfrak{o}_D$, where $\mathfrak{o}_D$ is the ring of integers of $\mathbb{Q}(\sqrt{D})$. We also fix an embedding of $\mathbb{Q}(\sqrt{D})$ into $\mathbb{R}$ with respect to which we may speak of an element of $\mathbb{Q}(\sqrt{D})$ being "positive." Let $\mathfrak{o}_{D,f}^{(1)}$ be the set of units in $\mathfrak{o}_{D,f}$ that have norm 1.

We then have a map

$$\{x \in \mathfrak{o}_{D,f}, \text{Norm}(x) = m\} \to \mathfrak{S}' \tag{2.14}$$

given by $x \mapsto \text{tr}(x)$. One verifies that this map is surjective, and the fibre above ν has size 2 unless $\nu^2 = 4m$; in that case, it has size 1. This explicit identification of $\mathfrak{S}'$ permits a further simplification of Proposition 3.

Let X_m be a set of representatives, modulo $\mathfrak{o}_{D,f}^{(1)}$, for elements $x \in \mathfrak{o}_{D,f}$ such that $\text{Norm}(x) = m$. (We will choose each representative x to be *positive*; this can be done since we may replace x by $-x$.) In future sections we will often say *modulo units* when we mean *modulo units of positive norm*, i.e. *modulo* $\mathfrak{o}_{D,f}^{(1)}$.

Let ϵ_0 be a fundamental unit of $\mathfrak{o}_{D,f}^\times$. Set $\delta = 2$ if $\mathrm{Norm}(\epsilon_0) = -1$ and set $\delta = 1$ if $\mathrm{Norm}(\epsilon_0) = 1$. A generator for $\mathfrak{o}_{D,f}^{(1)}/\{\pm 1\}$ is then ϵ_0^δ. Again, we may assume that ϵ_0^δ is positive, changing the sign of ϵ_0 if not.

Now, for $\nu \in \mathfrak{S}'$, $\cosh^{-1}(\nu/2\sqrt{m}) = \log(\frac{\nu \pm \sqrt{\nu^2-4m}}{2\sqrt{m}})$. Using some elementary manipulation (we group together the terms involving ν and $-\nu$ and then apply Equation 2.4), we see that:

$$\sum_{\nu\in\mathfrak{S}'}{}' \Delta(\frac{\nu}{2\sqrt{m}}) = \sum_{x\in X_m}\sum_{k\in\mathbb{Z}} \widehat{h^+}(2(\log(\frac{x}{\sqrt{m}}) + k\log(\epsilon_0^\delta))) \tag{2.15}$$

Here $\widehat{h^+}$ is the Fourier transform of h^+. Notice at this point the importance of $\sum'$, which halves the contribution of $\nu = \pm 2\sqrt{m}$, as opposed to usual sum – it "accounts" for the varying fibre sizes in the map of Equation 2.14. Applying Poisson summation to each inner sum in Equation 2.15, we obtain

$$\frac{1}{2\delta\log(\epsilon_0)}\sum_{k\in\mathbb{Z}} 2\pi h^+(\frac{\pi k}{\log(\epsilon_0^\delta)}) \sum_{x\in X_m} e\left(k\frac{\log(x/\sqrt{m})}{\log(\epsilon_0^\delta)}\right)$$

We have thus proven:

Proposition 4. *Suppose $D > 0$ and $m > 0$. The limit L of Equation 2.6 exists and equals:*

$$L = c(N)\frac{6}{\pi\delta\log(\epsilon_0)}\sum_{k\in\mathbb{Z}} h^+(\frac{\pi k}{\delta\log(\epsilon_0)})\left(\sum_{x\in X_m} e\left(k\frac{\log(x/\sqrt{m})}{\log(\epsilon_0^\delta)}\right)\right) \tag{2.16}$$

Here $c(N)$ is as in Definition 1, ϵ_0 is a fundamental unit for the order $\mathfrak{o}_{D,f}$ of discriminant N, X_m is a set of representatives for elements of $\mathfrak{o}_{D,f}$ of norm m, modulo units in $\mathfrak{o}_{D,f}^{(1)}$, and δ is 1 or 2 according to whether $\mathrm{Norm}(\epsilon_0)$ *is* 1 *or* -1.

Now suppose $m < 0$. The set $\mathfrak{S}'$ now consists of ν so that $\nu^2 - Ny^2 = -4|m|$ for some y. For such a ν, $\sinh^{-1}(\nu/2\sqrt{|m|}) = \log(\frac{\nu\pm\sqrt{\nu^2-4m}}{2\sqrt{|m|}})$, the choice of sign depending on the sign of ν. Now let X_m be a set of (again positive) representatives of elements $\mathfrak{o}_{D,f}$ with norm m modulo units. Using Equation 2.3 rather than Equation 2.4, one sees that the limit L exists and equals:

$$L = c(N)\frac{6}{\pi\delta\log(\epsilon_0)}\sum_{k\in\mathbb{Z}} h^-(\frac{\pi k}{\delta\log(\epsilon_0)})\left(\sum_{x\in X_m} e\left(k\frac{\log(x/\sqrt{|m|})}{\log(\epsilon_0^\delta)}\right)\right) \tag{2.17}$$

Holomorphic forms, weight k

Now we consider the case $D < 0$. In other words, our spectral sum is over holomorphic forms, and we are hoping to find holomorphic forms associated to Grössencharacters of $\mathbb{Q}(\sqrt{D})$. Again, let $\mathfrak{o}_{D,f}$ be the order contained in $\mathbb{Q}(\sqrt{D})$ of discriminant N.

Again, the set $\mathfrak{S}'$ consists of ν such that $\nu^2 - Df^2y^2 = 4m$ for some y; this is now a finite set. Let $\tilde{X}_m$ be the set of elements $x \in \mathfrak{o}_{D,f}$ such that $\text{Norm}(x) = m$. We do *not* quotient by the action of the unit group (yet!)

$\tilde{X}_m$ maps to the set $\mathfrak{S}'$ via Trace : $\frac{\nu+fy\sqrt{D}}{2} \mapsto \nu$; the fibres have size 2 except for $\nu^2 = 4m$, where they have size 1.

For $\nu \in \mathfrak{S}'$, we must have $|\frac{\nu}{2\sqrt{m}}| \leq 1$, and we apply the integral transformation formulae 2.5 to Equation 2.13. With a little manipulation, one obtains:

$$\sum_{\nu\in\mathfrak{S}'}{}' \Delta(\frac{\nu}{2\sqrt{m}}) = (-1)^{(k-1)/2}\frac{2\pi i^k}{k-1}\frac{1}{2}\sum_{x\in\tilde{X}_m}\left(\frac{x}{\sqrt{m}}\right)^{k-1}$$

Since $x \in \tilde{X}_m \implies -x \in \tilde{X}_m$, the inner sum vanishes unless k is odd. From Equation 2.13, we deduce that the limit L equals:

$$L = -ic(N)\frac{6}{\pi(k-1)}\sum_{x\in\tilde{X}_m}\left(\frac{x}{\sqrt{m}}\right)^{k-1}$$

Let $\Lambda(f)$ be the group of units in $\mathfrak{o}_{D,f}$; let w_f be its cardinality. It is the cyclic group of w_f-th roots of unity. The set $\tilde{X}_m$ is closed under multiplication by $\Lambda(f)$ and the sum above vanishes unless k is congruent to 1 modulo w_f. Let X_m be the quotient of $\tilde{X}_m$ by units: so, as before, X_m is a set of representatives for elements of norm m modulo units.

Proposition 5. *Suppose $D < 0$ and we are computing in the space of weight k forms, $k \geq 11$. The limit L of Equation 2.6 exists and equals 0 unless $k \equiv 1 \bmod w_f$; in that case,*

$$L = -ic(N)\frac{6w_f}{\pi(k-1)} \sum_{x \in X_m} \left(\frac{x}{\sqrt{m}}\right)^{k-1} \tag{2.18}$$

Here $c(N)$ is as in Definition 1, X_m is a set of representatives for elements in the order $\mathfrak{o}_{D,f}$ of norm m modulo $\mathfrak{o}_{D,f}^\times$, and w_f is the number of roots of unity in $\mathfrak{o}_{D,f}$.

Note that here $D < 0$ and so $c(N)$ is i times a positive constant. Consequently L is real (as we expect).

2.3.3 Analysis of the local sum

We now turn to the analysis of the local sum:

$$\mathcal{A}(\nu; c, m) = \sum_{\substack{k \in \mathbb{Z}/c\mathbb{Z} \\ x \in (\mathbb{Z}/c\mathbb{Z})^\times}} e(\frac{-k\nu + k^2 x + m x^{-1}}{c})\chi(x) = \sum_{\substack{k \in \mathbb{Z}/c\mathbb{Z} \\ x \in (\mathbb{Z}/c\mathbb{Z})^\times}} e(x^{-1}\frac{k^2 - k\nu + m}{c})\chi(x)$$

The latter expression follows from the former via replacing k by kx^{-1}. Recall that χ is the quadratic character $\chi_D = \left(\frac{D}{\cdot}\right)$, although the following analysis will work equally well in general. Our treatment will be somewhat sketchy at the prime 2; the methods of Chapter 4 will treat that case in a more general setting.

We will prove the following:

Theorem 3. *Define $\delta(x)$, for $x \in \mathbb{Z}$, as in Proposition 2. Define, for $\nu, m \in \mathbb{Z}$, the Dirichlet series:*

$$Z(s) = \sum_{N|c} \frac{\mathcal{A}(\nu; c, m)}{c} c^{-s}$$

Then $Z(s)$ has meromorphic continuation to the complex plane. It is holomorphic in $\Re(s) > 1/2$, except for a possible simple pole at $s = 1$. This pole has residue $\frac{6}{\pi^2}\delta(\nu^2 - 4m)$; in particular, it exists if and only if $\nu^2 - 4m = Ny^2$, for some integral y.

The function Z has at most polynomial growth in vertical strips. This is uniform in ν, in the sense that for $\sigma > 1/2$, there are constants $A(\sigma)$ and $B(\sigma)$ so that

$$Z(\sigma + it) \ll (1 + |\nu|)^{A(\sigma)}(1 + |t|)^{B(\sigma)}$$

We begin by noting that $Z(s)$ has an Euler product:

Lemma 1. *The function $Z(s)$ decomposes as an Euler product $\prod_p Z_p(s)$, where the local Euler factor Z_p is given by:*

$$Z_p(s) = \sum_{k \geq v_p(N)} \mathcal{A}(\nu; p^k, \alpha)\chi'_p(p^k)^{-1}p^{-ks} \tag{2.19}$$

Here χ'_p is that part of the character χ that is supported prime to p; that is to say, χ'_p is a Dirichlet character to a modulus prime to p, and $\chi^{-1}\chi'_p$ is a Dirichlet character modulo a power of p.

Proof. Suppose $c = c_1c_2$, with c_1 and c_2 coprime, and decompose $\chi = \chi_1 \times \chi_2$ via the Chinese remainder theorem, so χ_i is a character to the modulus c_i. Then we have "twisted" multiplicativity:

$$\mathcal{A}(\nu; c_1c_2; m) = \chi_1([c_2]_{c_1}^{-1})\chi_2([c_1]_{c_2}^{-1})\mathcal{A}(\nu; c_1, m)\mathcal{A}(\nu; c_2, m)$$

Here $[c_1]_{c_2}$ refers to the residue class of c_1 modulo c_2, and similarly for $[c_2]_{c_1}$. An application of this proves the Lemma. □

Explicitly, one can write out χ'_p in our case as follows: if p is an odd prime dividing

D, then $\chi'_p(x) = \left(\frac{\epsilon_p D'}{x}\right)$, where $\epsilon_p = (-1)^{(p-1)/2}$ and $D' = D/p$. If $p = 2$ divides D, it is more convenient to write $\chi'_2(x)$ as $\left(\frac{x}{D'}\right)$, with $D' = D/D_2$ with D_2 the 2-part of D.

Lemma 2. *Let p be a prime; let $c = p^k$. Let l be the highest power of p that divides $\nu^2 - 4m$. (We define $l = \infty$ if $\nu^2 - 4m = 0$.) Let $g(p)$ be the Gauss sum $\sum_{x \bmod p} e(x^2/p)$. The value of $S = \mathcal{A}(\nu; c, m)$ is then as follows:*

1. *p is a prime not dividing $2D$.*
   - *(a) If $k \geq l + 2$, then $S = 0$.*
   - *(b) If $k = l + 1$, and k is odd, then $S = \left(\frac{(\nu^2-4m)/p^l}{p}\right) p^{(3k-1)/2}$. (We will not need k even).*
   - *(c) If $k \leq l$, then $S = 0$ if k is odd, and $p^{k/2}\phi(p^k) = p^{3k/2} - p^{3k/2-1}$ if k is even.*
2. *p divides D but not 2. This is rather similar to the above case, except with a parity inversion:*
   - *(a) If $k \geq l + 2$, then $S = 0$.*
   - *(b) If $k = l + 1$, and k is even. Then $S = \left(\frac{(4m-\nu^2)/p^l)}{p}\right) g(p)p^{3k/2-1}$. (We will not need the case k odd.)*
   - *(c) If $k \leq l$, then $S = 0$ if k is even, and $g(p)p^{(k-1)/2}\phi(p^k) = g(p)(p^{(3k-1)/2} - p^{(3k-3)/2})$ if k is odd.*
3. *$p = 2$. We will not explicitly need these results at $p = 2$; they are similar to those above, and whatever we need will be proven, in a somewhat greater level of generality, in Chapter 4.*

Proof. Suppose $p \neq 2$. One proceeds from the original definition (Equation 2.10) by completing the square:

$$\mathcal{A}(\nu; c, m) = \sum_{\substack{k \in \mathbb{Z}/c\mathbb{Z} \\ x \in (\mathbb{Z}/c\mathbb{Z})^\times}} e(\frac{x(k - \nu x^{-1}/2)^2 - \nu^2 x^{-1}/4 + m x^{-1}}{c}) \chi(x)$$

which gives, with $g(c) = \sum_{k \bmod c} e(k^2/c)$, the "Gauss sum mod c" (although c may not be a prime):

$$\mathcal{A}(\nu; c, m) = g(c) \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^\times} \left(\frac{x}{c}\right) e(\frac{x^{-1}(m - \nu^2/4)}{c}) \chi(x)$$

Here the symbol $\left(\frac{a}{b}\right)$ is extended from prime b by multiplicativity. The resulting sum is another Gauss-type sum, and the rest is careful book-keeping. □

Suppose first that $\nu^2 - 4m \neq 0$. Then each $Z_p(s)$ is a finite polynomial in p^{-s}, and for almost all p we have, from Equation 2.19,

$$Z_p(s) = 1 + \left(\frac{(\nu^2/4 - m)D}{p}\right) p^{-s} = \frac{1 - p^{-2s}}{1 - \left(\frac{(\nu^2/4-m)D}{p}\right) p^{-s}}$$

which coincides with the local factor, at p, of $L(s, \left(\frac{(\nu^2/4-m)D}{\cdot}\right))\zeta(2s)^{-1}$.

Now, suppose that $\nu^2 - 4m = 0$. Then, for all p the factor $Z_p(s)$ is a rational function of s, and for almost all p, the local factor $Z_p(s)$ equals

$$1 + \sum_{k \geq 2 \,\text{even}} (p^{k/2} - p^{k/2-1}) p^{-ks} = 1 + (1 - p^{-1}) \frac{p^{1-2s}}{1 - p^{1-2s}} = \frac{1 - p^{-2s}}{1 - p^{1-2s}}$$

which is the local factor, at p, of $\zeta(2s-1)/\zeta(2s)$.

This gives another piece of the Theorem:

Lemma 3. *The function $Z(s)$ has meromorphic continuation to the s-plane, and is holomorphic for $\Re(s) > 1/2$, with a possible pole at $\Re(s) = 1$ that occurs only if $(\nu^2 - 4m)D$ is a square.*

For $\sigma > 1/2$, there are constants $A(\sigma)$ and $B(\sigma)$, depending on N, σ, m – but not on ν – so that

$$Z(\sigma + it) \ll (1 + |\nu|)^{A(\sigma)}(1 + |t|)^{B(\sigma)}$$

Proof. The first claim is clear, since we know that $Z(s)$ is an Euler product that matches a well-understood function (either $L(s, \xi)/\zeta(s)$ or $\zeta(2s-1)/\zeta(2s)$, for an appropriate Dirichlet character ξ) at almost all places. (It is not hard to check that the "bad" places do not interfere with the conclusion.)

For the second claim, suppose, for instance, that we are in the case where $\nu^2 - 4m \neq 0$. Let ξ be the Dirichlet character $\left(\frac{D(\nu^2-4m)}{\cdot}\right)$. Let B (for "bad") denote the set of primes that divide $N(\nu^2 - 4m)$. Then:

$$Z(s) = \frac{L_B(s, \xi)}{\zeta_B(2s)} \prod_{p \in B} Z_p(s)$$

where, for instance, $\zeta_B(2s)$ denotes the Euler product for $\zeta(2s)$, omitting the primes in the set B.

For $\sigma > 1/2$, $|\zeta_B(2(\sigma + it))|^{-1} \ll C_1(\sigma)$, a function only of σ; this follows from the absolute convergence of the Euler product. As for $L_B(s, \xi)$, one may bound its growth by the Phragmen-Lindelöf principle; one obtains a bound $L_B(\sigma + it, \xi) \ll (1 + |\nu|)^{A(\sigma)}(1 + |t|)^{B(\sigma)}$, where the implicit constant does not depend on ν. Finally, using the trivial bound $\mathcal{A}(\nu; c, m) \leq c^2$ and the fact that $\mathcal{A}(\nu; p^k, m)$ vanishes if $k \geq v_p(\nu^2 - 4m) + 2$, we see that for *any* σ (not even necessarily satisfying $\sigma > 1/2$!) the product $\prod_{p \in B} Z_p(\sigma + it)$ is bounded by a polynomial in ν, with degree depending only on σ.

The argument must be modified when $\nu^2 - 4m = 0$; this is straightforward. □

Finally there is the issue of the pole at $s = 1$. The residue depends on the computation of $Z_p(1)$ for "ramified" p, that is, those p dividing N. Since we have already seen that $Z(s)$ has a pole at $s = 1$ only if $(\nu^2 - 4m)D$ is a square, we confine ourselves to that case.

The results about $Z_p(1)$ are contained in the following:

Lemma 4. *Suppose $(\nu^2 - 4m)D$ is a square. Then the value of $Z_p(1)$ is as follows. For $x \in \mathbb{Z}$, set $\gamma_x = 1$ if x is congruent to 1 mod 4, and $\gamma_x = i$ if x is congruent to 3 mod 4; for a prime p, define $\epsilon_p = \gamma_p^2 = (-1)^{(p-1)/2}$. Let $v_p(N)$ be the maximal power of p that divides N.*

1. *If p does not divide N, then $Z_p(1) = 1 + 1/p$.*

2. *If p odd divides N but not D, then $Z_p(1)$ vanishes unless $\nu^2 - 4m$ is divisible by $p^{v_p(N)}$; in that case, $Z_p(1) = p^{-v_p(N)/2}$.*

3. *If p divides D: Let $D' = D/p$. Then $Z_p(1)$ vanishes unless $\nu^2 - 4m$ is divisible by $p^{v_p(N)}$; in that case, $Z_p(1) = \gamma_p \left(\frac{\epsilon_p D'}{p}\right) p^{-v_p(N)/2}$.*

4. *If $p = 2$, then $Z_p(1)$ vanishes unless $\nu^2 - 4m$ is divisible by $p^{v_p(N)}$. In that case, it equals $p^{-v_p(N)/2}\gamma_{D/D_2}\left(\frac{D_2}{D/D_2}\right)$, where D_2 is the 2-part of D.*

Proof. These results are a consequence of Lemma 2 for $p \neq 2$. These sums will be dealt with in a more general context in Chapter 4, and that treatment will naturally include the case of residue characteristic 2; because of this, we will not prove the $p = 2$ result here. □

Corollary 1.

$$\prod_p Z_p(1) = \begin{cases} 1/\sqrt{N}, & D > 0 \\ i/\sqrt{N}, & D < 0 \end{cases}$$

Proof. The absolute value of this product is clearly $1/\sqrt{N}$. To compute the argument, we need to compute the product:

$$\prod_{p \neq 2, p|D} \gamma_p \left(\frac{\epsilon_p (D/p)}{p}\right) \cdot (\text{Contribution of } 2) \tag{2.20}$$

Let l be the number of primes congruent to 3 mod 4 that divide D. The product over odd primes dividing D of γ_p equals i^l, and the product over odd primes dividing D of $\left(\frac{\epsilon_p}{p}\right)$ is $(-1)^l$. Let D_2 be the 2-part of D. Applying quadratic reciprocity multiple times shows that

$$\prod_{p\neq 2, p|D} \left(\frac{D/p}{p}\right) = \left(\frac{D_2 \mathrm{sgn}(D)}{D/D_2}\right) q$$

where

$$q = \begin{cases} -1, & l \equiv 2, 3 \bmod 4 \\ 1, & \text{else} \end{cases}$$

Therefore, checking case by case, we see that:

$$\prod_{p\neq 2, p|D} \gamma_p \left(\frac{\epsilon_p (D/p)}{p}\right) = \begin{cases} \left(\frac{D_2 \mathrm{sgn}(D)}{D/D_2}\right), & l \text{ even} \\ -i\left(\frac{D_2 \mathrm{sgn}(D)}{D/D_2}\right), & l \text{ odd} \end{cases}$$

Finally, the contribution of 2 equals $\gamma_{D/D_2}\left(\frac{D_2}{D/D_2}\right)$. Some more case-by-case checking verifies the corollary. ☐

Lemma 5. *Suppose $(\nu^2 - 4m)D$ is a square. The residue of $Z(s)$ at $s = 1$ is $\frac{6}{\pi^2}c(N)$ if $\nu^2 - 4m \neq 0$, and $\frac{3}{\pi^2}c(N)$ if $\nu^2 - 4m = 0$.*

Proof. This comes from comparing $Z(s)$, Euler factor by Euler factor, with the quotient $\zeta(s)/\zeta(2s)$ if $\nu^2 - 4m \neq 0$, and with $\zeta(2s-1)/\zeta(s)$ if $\nu^2 - 4m = 0$. Since $Z(s)$ agrees with one or the other quotient at almost all places, all that is left for the computation of residue is to compute $Z_p(1)$ at bad places, which has already been done. ☐

2.4 The standard L-function

Here we briefly discuss a point noted in the Introduction: that the same method shows that the standard L-function of a modular form does not have a pole at $s = 1$. For simplicity, we shall do this on $\mathrm{SL}_2(\mathbb{Z})$; it will be clear that the argument will go through with a conductor or with a Nebentypus. Although this is simpler than the case of the symmetric square, we treat it only now; it is quite easy now that the technique has been established. This is also sketched in Sarnak [17].

Consider the sum:

$$L_{standard} = \lim_{X\to\infty}\left(\frac{1}{X}\sum_f h_f(\varphi)\sum_n g(n/X)a_n(f)\overline{a_m(f)}\right) + \tag{2.21}$$

$$\left(\frac{1}{X}\sum_n g(n/X)\sum_{\mathfrak{c}}\frac{1}{4\pi}\int_{-\infty}^{\infty} h^{\pm}(t)\eta_{\mathfrak{c}}(n,1/2+it)\overline{\eta_{\mathfrak{c}}(m,1/2+it)}\right) \tag{2.22}$$

Here, as before, one uses h^+ or h^- according to the sign of m.

As discussed in the Introduction, the evaluation of $L_{standard}$ can be regarded as a trace formula which spectrally averages the residue at $s = 1$ of the standard L-function.

For simplicity, we will consider the case where φ is compactly supported and all its derivatives are bounded. One can, with sharper analysis, replace this by weaker assumptions. Applying the Kuznetsov formula to Equation 2.21, and (as in the previous section) expanding the Kloosterman sums and splitting the n-sum into arithmetic progressions, we obtain:

$$L_{standard} = \lim_{X\to\infty}\frac{1}{X}\sum_{c\geq 1}\frac{1}{c}\sum_{\substack{k\in\mathbb{Z}/c\mathbb{Z}\\ x\in(\mathbb{Z}/c\mathbb{Z})^\times}} e(\frac{mx^{-1}+kx}{c})\sum_{n\equiv k(\mathrm{mod}\, c)} g(n/X)\varphi(4\pi\frac{\sqrt{mn}}{c}) \tag{2.23}$$

Let

$$\mathcal{A}_{standard}(\nu; m, c) = \sum_{\substack{k\in\mathbb{Z}/c\mathbb{Z} \\ x\in(\mathbb{Z}/c\mathbb{Z})^\times}} e(\frac{kx + mx^{-1}}{c} - \frac{k\nu}{c}) = \begin{cases} \phi(c)e(m\overline{\nu}/c), & (\nu, c) = 1 \\ 0, & \text{else} \end{cases} \tag{2.24}$$

It is the "local sum" that occurs in this situation. Here $\phi(c)$ is the Euler totient.

Applying Poisson sum to $L_{standard}$, we obtain, with ν as the argument of the Fourier transform:

$$L_{standard} = \lim_{X\to\infty} \frac{1}{X} \sum_{\nu\in\mathbb{Z}} \sum_{c\geq 1} \frac{\mathcal{A}_{standard}(\nu; m, c)}{c} \int_{-\infty}^{\infty} g(cx/X)\varphi(4\pi\sqrt{\frac{xm}{c}})e(x\nu)dx \tag{2.25}$$

We note that, since φ has compact support, the c that occur in the integral must be of order $\sqrt{X}$ (otherwise the product $g\varphi$ that occurs will be 0). In particular, the integral is effectively over an x-interval of length around $\sqrt{X}$, and additionally we expect to get significant cancellation from oscillatory nature of the integral.

Now $\mathcal{A}_{standard}(0, m; c) = 0$, so the term $\nu = 0$ does not contribute. On the other hand, each term with $\nu \neq 0$ can be estimated by repeated integration by parts; in fact, using the assumption that all the derivatives of φ are bounded, one obtains the estimate:

$$\int_{-\infty}^{\infty} g(cx/X)\varphi(4\pi\sqrt{\frac{xm}{c}})e(-x\nu)dx \ll_M |\nu|^{-M} X^{\frac{1-M}{2}}, \quad \nu \neq 0$$

Combining this with the estimate $|\mathcal{A}_{standard}(\nu; m, c)| \leq c$ and the fact that the c-sum in Equation 2.25 is of length $O(\sqrt{X})$, we see that $L_{standard} = 0$. (Indeed, we have shown that the expression whose limit defines $L_{standard}$ is actually $O(X^{-M})$ for all M. This error bound is closely related to the analytic continuation of the standard L-function to the entire plane.)

In a similar way, the analysis of Section 2.3 can be carried through to give results "close to" the analytic continuation of the symmetric square L-function. In the holomorphic setting, this is contained in Mizumoto [16]. It is unclear whether the analyticity of the symmetric square L-function in the general (Maass) case can be deduced, because of difficulty in isolating a single form. In the holomorphic case, the space is finite-dimensional and this issue does not arise.

2.5 From Trace Formula to Classification

The main work of the Chapter is already done, in deriving the trace formulae Equation 2.16 and Equation 2.18. In some sense, the rest of the Chapter is just book-keeping.

To convert these formulae into Theorem 4, stated in Section 2.6, one must first compute the expected contribution of quadratic fields, show that it matches what we have derived, and then formal arguments complete the proof.

The material that remains is essentially formal and computational, and we do not go through every detail, only giving the indication of how the computations are to be performed. We restrict ourselves to $(m, N) = 1$; although this suffices by strong multiplicity one, the difference in treating $(m, N) \neq 1$ is that one must make a more careful study of oldforms than that performed here.

We also restrict ourselves to the case $m > 0$, and therefore the spectral transform $h = h^+$; the analysis for $m < 0$ and h^- is actually slightly easier (since the the inversion of the Kontorovich-Lebedev transform $\varphi \to h^-$ does not involve Bessel functions at integral indices.)

We also discuss the contribution of the continuous spectrum, which must be explicitly removed from the formula. We state the result in general, and verify it in Subsection 2.5.4 in a relatively simple case, where $N = D$ is a prime congruent to 1 mod 4. It will be clear from this that the evaluation is a straightforward matter, but, of course, becomes more involved as the number of cusps increases.

2.5.1 Expected Answer

We first give a concrete interpretation of Theorem 4.

We must fix a good deal of notation. Refer also to Section 6.2 in the Appendix, which may clarify the need for these definitions; from the point of view of proving Theorem 4, one must first (with notation as in that Theorem) determine precisely which characters of $\mathbb{A}_K^\times/K^\times\mathbb{A}_\mathbb{Q}^\times$ give GL(2) forms of conductor dividing N, i.e. modular forms for $\Gamma_0(N)$.

D, f and $N = |D|f^2$ are fixed as in the start of Section 2.3. Set $K = \mathbb{Q}(\sqrt{D})$, let $\mathfrak{o}_{D,f}$ be the order in K of discriminant N, and let $C_{D,f}$ be the class group of $\mathfrak{o}_{D,f}$. Let $h_{D,f}$ be the cardinality of $C_{D,f}$.

Let $K_\infty = K \otimes \mathbb{R}$. $\mathbb{A}_K$ will denote the ring of adeles of K, and $\mathbb{A}_{K,f}$ the ring of finite adeles; we denote with a superscript $\times$ the corresponding idele rings. Let $\overline{\mathfrak{o}_{D,f}}$ be the closure of $\mathfrak{o}_{D,f}$ in $\mathbb{A}_{K,f}$, and let $U(f)$ be the units of $\overline{\mathfrak{o}_{D,f}}$; it is the product of the local group $U_v(f)$, where for v a finite place of K, $U_v(f)$ is the group of units in the K_v-closure of $\mathfrak{o}_{D,f}$.

$U(f)$ is an open compact subgroup of $\mathbb{A}_{K,f}^\times$, and one verifies that:

$$C_{D,f} = \mathbb{A}_{K,f}^\times/K^\times U(f) = \mathbb{A}_K^\times/K^\times K_\infty^\times U(f)$$

Let w_f be the number of roots of unity in $\mathfrak{o}_{D,f}$. Note that the characters of $\mathbb{A}_K^\times/K^\times U(f)$ of a prescribed infinity type (that is, a prescribed restriction to $K_\infty^\times$) are a principal homogeneous space for $\widehat{C_{D,f}}$, where the hat denotes "dual group."

Take ω to be a character of $\mathbb{A}_K^\times$ trivial on $K^\times U(f)\mathbb{A}_\mathbb{Q}^\times = K^\times U(f)\mathbb{R}^\times$. Theorem 4 claims that there should be a GL(2)-automorphic representation $\pi(\omega)$ over $\mathbb{Q}$ that is naturally associated to ω. We will translate this more concretely by identifying the corresponding newform; it will *a priori* be only a function on the upper half-plane, and its modularity will be deduced indirectly from our trace formulae Equation 2.16

and Equation 2.18.

We denote by ω_∞ the restriction of ω to $K_\infty^\times$. Let $\lambda_n(\omega)$ be the coefficient of n in Dirichlet series defined by the Hecke L-function $L(K, s, \omega)$. We then form the function f_ω on the upper half plane, defined as:

$$f_\omega(z) = \sum_n n^{(k_\omega - 1)/2} \lambda_n(\omega) e(nz) \quad (D < 0)$$

$$f_\omega(z) = \sum_n (\mathrm{sgn}(n)^{\epsilon_\omega}) \lambda_n(\omega) \sqrt{y} K_{it_\omega}(2\pi n y) e(nx) \quad (D > 0)$$

where, if $D > 0$, t_ω and ϵ_ω are defined in terms of ω_∞: ω_∞ is the character of $K_\infty^\times = \mathbb{R}^\times \times \mathbb{R}^\times$ so that $\omega_\infty(x, x^{-1}) = \mathrm{sgn}(x)^{\epsilon_\omega} |x|^{it_\omega}$. If $D < 0$, then k_ω is determined by ω_∞, namely, ω_∞ should be the character $z \mapsto z^{k_\omega - 1} \bar{z}^{-k_\omega + 1}$ of $\mathbb{C}^\times$.

This is, in concrete terms, the form f_ω whose L-function is expected to match that of ω, *including* factors at ∞. Note that $f_\omega = f_{\omega^{-1}}$.

We will define $\lambda_m(f_\omega) \equiv \lambda_m(\omega)$; we expect $\lambda_m(f_\omega)$ to be the mth Hecke eigenvalue of f_ω, (although at this stage we do not even know that f_ω is a modular form!)

Then, Theorem 4 states the following:

Concrete Expectation: For each $\omega \in \mathbb{A}_K^\times / \widehat{\mathbb{A}_Q^\times U(f)} K^\times$, f_ω is actually modular on $\Gamma_0(N)$ with Nebentypus χ_D, the quadratic character associated with $K = \mathbb{Q}(\sqrt{D})$. It is cuspidal precisely when $\omega \neq \omega^{-1}$. If $D > 0$ it is a Maass form with eigenvalue $1/4 + t_\omega^2$ and parity ϵ_ω; if $D < 0$ it is holomorphic with weight k_ω.

As one varies ω, the f_ω exhaust all newforms f of level dividing N, Nebentypus χ_D, and so that $L(s, f, \mathrm{Sym}^2)$ has a pole at $s = 1$. □

To prove this, we wish to demonstrate an equality of spectral sums, the left hand sum over cuspidal newforms with Nebentypus χ_D and level dividing N, and the right hand side over characters ω of $\mathbb{A}_K^\times / \mathbb{A}_Q^\times U(f) K^\times$ such that $\omega \neq \omega^{-1}$ (this condition is to ensure cuspidality):

$$\sum_{\substack{f:m(f,\mathrm{Sym}^2)=1\\ f\,\text{cusp. newform}}} h(t_f)\lambda_m(f) = \frac{1}{2} \sum_{\substack{\omega\in \mathbb{A}_K^\times/\widehat{\mathbb{A}_Q^\times U(f)K^\times}\\ \omega\neq\omega^{-1}}} h(t_\omega)\lambda_m(f_\omega) \quad (D>0) \tag{2.26}$$

and a similar statement for $D < 0$. Here h is a test function and, on the left hand side, f has eigenvalue $1/4 + t_f^2$. The $\frac{1}{2}$ on the right-hand side accounts for the fact that $f_\omega = f_{\omega^{-1}}$.

This, by essentially formal arguments, implies the modularity of f_ω, and then the final result Theorem 4: some of the details of this argument are given in the final section.

In this section, we will evaluate the right-hand side and do most of the work in evaluating the left-hand side. There are three issues that will arise:

1. The difference between the Fourier coefficients of the L^2-normalized form and the Hecke eigenvalues.

2. The contribution of the continuous spectrum to the limit of Equation 2.6.

3. The contribution of oldforms to the limit of Equation 2.6, since Equation 2.26 involves a sum only over newforms.

The most interesting is the second point: there exist Eisenstein series f so that $m(f, \mathrm{Sym}^2) = 2$; they correspond to characters ω such that $\omega = \omega^{-1}$. Since this multiplicity is larger than 1, one might expect some kind of divergence on the left hand side of Equation 2.6. Fortunately, this divergence is precisely balanced by the fact that this form is not spectrally isolated.

This point is also related to one encountered by Langlands in [13]. Namely, were one trying to carry out the analysis of this Chapter using the trace formula, one would have to deal with the contribution of the trivial representation; the Petersson-

Kuznetsov formula avoids this entirely. Nevertheless, even after removing the trivial representation we encounter considerable subtlety in dealing with the continuous spectrum.

2.5.2 Right-hand side: Grössencharacters

Maass Case

Suppose $D > 0$.

Here we compute the right hand side of Equation 2.26, except we will sum over all $\omega \in \mathbb{A}_K^\times/\widehat{K^\times U(f)}\mathbb{A}_\mathbb{Q}^\times$ – including those ω such that $\omega = \omega^{-1}$. We will remind ourselves of this by subscripting the sums with $\omega \in \mathrm{Cusp} \cup \mathrm{Eis}$–the sum both over "cuspidal" ω (those with $\omega \neq \omega^{-1}$) and "Eisenstein" ω (those with $\omega = \omega^{-1}$.) We have fixed D and f.

We start with the case $m = 1$. Let δ, ϵ_0 be as in Proposition 4. From the Appendix – see Subsection 6.2.2 – the t_ω are the integral multiples of $\frac{\pi}{\log(\epsilon_0^\delta)}$. They occur with multiplicity $2h_{D,f}/\delta$. It follows:

$$\sum_{\omega \in \mathrm{Cusp} \cup \mathrm{Eis}} h(t_\omega)\lambda_1(f_\omega) = \sum_{\omega \in \mathrm{Cusp} \cup \mathrm{Eis}} h(t_\omega) = \frac{2h_{D,f}}{\delta} \sum_{k \in \mathbb{Z}} h(\frac{\pi k}{\delta \log(\epsilon_0)})$$

Now, the generalization to general m is given by Equation 6.3 of the Appendix. Again, fix an embedding of $\mathbb{Q}(\sqrt{D})$ into $\mathbb{R}$, so that one can speak of a "positive" element. Let X_m be a set of positive representatives for elements $x \in \mathfrak{o}_{D,f}$ with $\mathrm{Norm}(x) = m$, modulo $\mathfrak{o}_{D,f}$-units of norm 1. Equation 6.3 implies:

$$\sum_{\omega \in \mathrm{Cusp} \cup \mathrm{Eis}} h(t_\omega)\lambda_m(f_\omega) = \frac{2h_{D,f}}{\delta} \sum_{k \in \mathbb{Z}} h(\frac{\pi k}{\delta \log(\epsilon_0)}) \sum_{x \in X_m} e(k\frac{\log(x/\sqrt{m})}{\log(\epsilon_0^\delta)}) \qquad (2.27)$$

Holomorphic Forms of Weight k

Suppose now $D < 0$; then $K = \mathbb{Q}(\sqrt{D})$ is imaginary quadratic; other notations are as before. The restriction of ω to $K_\infty = \mathbb{C}^*$ determines the expected weight of the holomorphic form f_ω: if $\omega|_{\mathbb{C}^\times}$ is the character $z \mapsto (z^r \bar{z}^{-r})$, then f_ω is expected to have weight $r+1$. In particular, one can check that there are $h_{D,f}$ distinct $\omega \in \mathbb{A}_K^\times / \widehat{K^\times \mathbb{A}_Q^\times U(f)}$ for which f_ω has weight k, as long as $k \equiv 1 \bmod w_f$. Otherwise, there are none.

Again, we apply Equation 6.3. We find, with X_m again a set of representatives for elements of norm m in $\mathfrak{o}_{D,f}$ modulo $\mathfrak{o}_{D,f}$-units of norm 1:

$$\sum_{\substack{f_\omega \text{weight } k \\ \omega \in \text{Cusp} \cup \text{Eis}}} \lambda_m(f_\omega) = \sum_{x \in X_m} \left(\frac{x}{\sqrt{m}} \right)^{k-1} \tag{2.28}$$

2.5.3 Left-Hand Side: Residues of Rankin-Selberg L-functions, newforms and oldforms

Here we will discuss the resolution of issues 1 and 3 from Section 2.5.1. This procedure is immensely simplified since we are assuming that $(m, N) = 1$. If one were consider $(m, N) \neq 1$, direct (and rather messy) computations would be necessary. As in the next section, we state the result in general, but only give a proof in a relatively simple example. The general case differs only in a greater notational complexity; since this point is not the main focus of the Chapter, we give only the idea.

Suppose M is an integer such that $D|M|N$ and g is a newform for the group $\Gamma_0(M)$ with Nebentypus $\chi = \left(\frac{D}{\cdot}\right)$. Suppose, for example, g is a holomorphic form; the reasoning goes through word-for-word with Maass forms. Then there is an associated subspace of forms Class(g) on $(\Gamma_0(N), \chi)$, namely that spanned by $g(lz)$ for l a divisor of $\frac{N}{M}$. The space of all holomorphic cusp forms for $(\Gamma_0(N), \chi)$ is then the orthogonal direct sum $\oplus_{M,g}$Class(g) as M and g vary. The point is to compute the contribution of each class Class(g) to Equation 2.6 in terms of g.

The result is:

Proposition 6. *Suppose* $(m, N) = 1$, *and take* M *such that* $D|M|N$. *For any cuspidal newform* g *for* $(\Gamma_0(M), \chi)$, *let* $\mathcal{B}(g)$ *be an orthonormal basis for* Class(g). *Then the contribution of* Class(g) *to Equation 2.6, that is, the sum*

$$\sum_{g_i \in \mathcal{B}(g)} \overline{a_m(g_i))} \lim_{X \to \infty} \frac{1}{X} \sum_{n \sim X} a_{n^2}(g_i)$$

exists, and is nonzero if and only $m(g, \mathrm{Sym}^2) = 1$. *(Here recall the meaning, from Section 2.2, of* $n \sim X$*; we avoid explicating it because of the unfortunate clash of notation with the form* g.*) In the latter case, it equals* $C_g \lambda_m(g)$, *where:*

$$C_g = \begin{cases} \frac{6}{\pi h_{D,f} \log(\epsilon_0)} c(N) & D > 0 \\ -i \frac{6 w_f}{(k-1)\pi h_{D,f}} c(N) & D < 0 \end{cases}$$

The notation is a little odd, as C_g is almost independent of the form g; it will therefore be regarded as a constant – we do not need to specify g, so long as it is known which case ($D > 0$ or $D < 0$) we are working in. This is the "miracle" that was discussed in Subsection 1.3.1. Note also that $h_{D,f}$ is the class number of the order $\mathfrak{o}_{D,f}$, and ϵ_0 a fundamental unit.

We are also appealing to the known theory of the symmetric square (although Rankin-Selberg theory is in fact sufficient, and so one never needs theta functions), which allows us to define the multiplicity $m(g, \mathrm{Sym}^2)$. (There is no circularity; that theory does not classify those f for which the multiplicity is nonzero.) It is possible to avoid this, making the treatment more self-contained; however, assuming it makes the exposition much smoother.

Proof. (Of Proposition, in a simple case): We will derive this in the following simple case: $N = Dp^2$, where p is a prime not dividing D, and g is a newform of level D. (In particular, normalization is so that $a_1(f) = 1$.) Define $g_0(z) = g(z) + \chi(p)g(p^2z) - a_p(g)g(pz)$. Then $g_0 \in \text{Class}(g)$ and its Fourier coefficients are given by:

$$a_n(g_0) = \begin{cases} a_n(g),\ (n,p) = 1 \\ 0,\ p|n \end{cases}$$

This follows directly from the Hecke relations. Further, Class(g) is spanned by $g_0, g_1 = g(pz)$ and $g_2 = g(p^2z)$; and, since the Fourier coefficients of g_0 vanish at multiples of p, $g_0 \perp g_1$ and $g_0 \perp g_2$. Since the coefficients of g_1 and g_2 are supported on integers divisible by p, they do not contribute to Equation 2.6 if $(m, N) = 1$. The contribution of Class(g) to Equation 2.6, therefore, equals the contribution of the *single* L^2-normalized form $\frac{g_0}{\sqrt{\langle g_0, g_0\rangle}}$.

To evaluate the contribution of g_0, we use the factorization:

$$\sum_{n=1}^{\infty} a_n^2(g_0)n^{-s} = a_1(g_0)\sum_{n=1}^{\infty} a_{n^2}(g_0)n^{-s} \sum_{(n,N)=1} \chi(n)n^{-s}$$

Combining this with the Rankin-Selberg theory on the upper half-plane, and Dirichlet's class number formula, we obtain the result of the proposition. (One can find enough details – for this purpose at least – on the Rankin-Selberg method in [10] for Maass forms, and [3] for holomorphic forms.) □

Remark 3. In general, it is merely a matter of detail to compute the contribution of Class(g). The point, however, is that for our application one can, by this "trick", circumvent any work and compute only the contribution of a particularly nice representative from Class(g).

This "trick" works in the form indicated only if each prime factor that divides N/M occurs to a power greater than 2. However, if this is not the case, a slight variant will suffice; for example, if g had been a newform of level $M = Dp$, then we could have taken $g_0 = g(z) - a_p(g)g(pz), g_1 = g(pz)$ and proceeded as above.

2.5.4 Contribution of the Continuous Spectrum

In this section, we sketch the computation of the *continuous spectrum* contribution to Equation 2.6. This can be done explicitly, given our understanding of Eisenstein series for GL(2).

Let ω be a character of $\mathbb{A}_K^\times / K^\times \mathbb{A}_\mathbb{Q}^\times U(f)$ which has order *two*: $\omega^{-1} = \omega$. Such a character must factor through the norm map $\mathbb{A}_K^\times \to \mathbb{A}_\mathbb{Q}^\times$; this is a consequence of Hilbert's Theorem 90.

For such an ω, the associated form f_ω is *not cuspidal*, and it does not contribute as part of the cuspidal spectrum in the Petersson-Kuznetsov as do its cuspidal "siblings." In fact, the symmetric square of f_ω has a *double* pole at $s = 1$, and it turns out that the contribution of f_ω comes from a "regularized contribution" of Eisenstein series. Note also that f_ω has eigenvalue 1/4.

In any case, it turns out that the continuous spectrum, as one might expect, contributes to Equation 2.6 in a way that corresponds exactly to the f_ω with $\omega = \omega^{-1}$:

Proposition 7. *Suppose $D > 0$. The continuous spectrum contribution L_{cts} to Equation 2.6, defined as:*

$$L_{cts} = \lim_{X \to \infty} \frac{1}{X} \sum_{n=1}^{\infty} g(n/X) \sum_{\mathfrak{c}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h^{\pm}(t) \eta_{\mathfrak{c}}(n^2, 1/2 + it) \overline{\eta_{\mathfrak{c}}(m, 1/2 + it)} \tag{2.29}$$

exists and equals

$$\frac{1}{2} h(0) C_g \sum_{\substack{\omega \in \widehat{\mathbb{A}_K^\times / \mathbb{A}_\mathbb{Q}^\times U(f) K^\times} \\ \omega = \omega^{-1}}} \lambda_m(f_\omega) \tag{2.30}$$

Here C_g is the constant of Proposition 6.

The proof of this Proposition is straightforward, at least in principle, as one can explicitly compute the coefficients of the Eisenstein series. For simplicity, we will content ourselves with working through the simplest nontrivial case: when $N = D$ is a prime congruent to 1 mod 4. In this case, one verifies (using Hilbert's Theorem 90) that the only character of $\mathbb{A}_K^\times/\mathbb{A}_\mathbb{Q}^\times U(f)K^\times$ of order 2 is the trivial one, $\omega = 1$, and the corresponding form f_1 is the Eisenstein series whose L-function coincides with $L(\mathbb{Q}(\sqrt{D}), s) = \zeta(s)L(s, \chi)$; thus the right-hand side reduces to $\frac{1}{2}h(0)C_g\lambda_m(f_1)$.

(In general, one will identify characters of $\mathbb{A}_K^\times/\mathbb{A}_\mathbb{Q}^\times U(f)K^\times$ with a certain collection of Dirichlet characters of $\mathbb{Q}$, and the argument that follows will have to be generalized to account for this and for the influence of multiple cusps.)

We will now show how to evaluate the continuous spectral contribution (2.29) in this case. We work on $(\Gamma_0(D), \chi)$.

Proof. (Sketch, for $D = N \equiv 1 \pmod 4$ a prime). There are two cusps; one can take as representatives the cusp $\mathfrak{c}_0$ at 0 and the cusp $\mathfrak{c}_\infty$ at ∞. One evaluates for $\mathfrak{c} = \mathfrak{c}_0$ the cusp at 0 (again, this is a simple computation from [10]):

$$\eta_{\mathfrak{c}_0}(n, 1/2 + it) = D^{-(1/2+it)} \frac{1}{\Lambda(\chi, 1 + 2it)} (\frac{4\pi}{\cosh(\pi t)})^{1/2} \sum_{ab=|n|} \chi(a)(a/b)^{it}$$

Λ is the *completed* Dirichlet L-function: $\pi^{-s/2}\Gamma(s)\sum_{n=1}^{\infty}\chi(n)/n^s$. (Recall that $\chi(-1) = 1$, that is, χ is "unramified at infinity.") As usual, we use $L(\chi, s)$ to denote the finite L-function $\sum_{n=1}^{\infty}\chi(n)/n^s$. We need to evaluate:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{\mathfrak{c}} \sum_{n\geq 1} g(n/X) \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)\eta_{\mathfrak{c}}(n^2, 1/2 + it)\overline{\eta_{\mathfrak{c}}(m, 1/2 + it)}dt \tag{2.31}$$

We compute this for $\mathfrak{c} = \mathfrak{c}_0$, and deduce from this the result when $\mathfrak{c} = \mathfrak{c}_\infty$. Note first that, in taking the product $\eta_{\mathfrak{c}}(n^2, 1/2+it)\overline{\eta_{\mathfrak{c}}(m, 1/2 + it)}$, the Γ factors from the

Λ-function cancel with the $\cosh(\pi t)$ term. Let

$$\eta'(m,s) = \sum_{ab=|m|} \chi(a)(a/b)^s$$

this is the "Hecke normalized" coefficient of this Eisenstein series. In particular $\eta'(1,s) = 1$. The $\mathfrak{c} = \mathfrak{c}_0$ contribution to Equation 2.31 then equals

$$\frac{1}{XD}\int_{-\infty}^{\infty} \overline{\eta'(m,1/2+it)} h(t) \frac{1}{|L(\chi,1+2it)|^2} \left(\sum_{n\geq 1} g(n/X) \sum_{ab=n^2} \chi(a)(a/b)^{it} \right) dt$$

Consider the inner-most sum in the above equation. Any pair (a,b) for which ab is a square can be written $a^*(j^2,l^2)$ with a^* squarefree, and we additionally require j coprime to D so that $\chi(a) \neq 0$. Substitute this into the above; we obtain:

$$\frac{1}{XD}\int_{-\infty}^{\infty} h(t) \frac{\overline{\eta'(m,1/2+it)}}{|L(\chi,1+2it)|^2} \left(\sum_{\substack{j,l,a^* \\ (j,D)=1}} g(a^*\frac{jl}{X}) \chi(a^*)(j/l)^{2it} \right) dt$$

We now introduce

$$H(t) = \overline{\eta'(m,1/2+it)} h(t)/|L(\chi,1+2it)|^2$$

Let $\hat{H}$ be the Fourier transform of H, so $\hat{H}(k) = \int_{-\infty}^{\infty} H(t)e^{ikt}$. The above expression equals:

$$\frac{1}{XD} \sum_{a^* \text{ squarefree}} \chi(a^*) \sum_{\substack{j,l \\ (j,D)=1}} g(a^*\frac{jl}{X}) \hat{H}(2\log(j) - 2\log(l)) \tag{2.32}$$

We see, without too much difficulty, that the numbers $\log(j) - \log(l)$ are becoming equidistributed on the real line, allowing us to approximate the innermost sum by an integral. To be precise, suppose first that j were not restricted to be coprime to D.

Then we would have, as $X \to \infty$,

$$\sum_{j \geq 1, l \geq 1} g(a^* \frac{jl}{X}) \hat{H}(2 \log(j) - \log(l)) \to \int_{x,y} g(a^* \frac{xy}{X}) \hat{H}(2 \log(x) - 2 \log(y)) dx dy$$

This naive approximation of a sum by an integral is not difficult to justify (one splits into sectors where j/l is approximately constant.)

In this integral, substitute $x = yt$; we obtain:

$$\int_{x,y} g(\frac{a^* y^2 t}{X}) \hat{H}(2 \log(t)) y dt dy = \frac{X}{2a^*} \int_t \frac{\hat{H}(2 \log(t))}{t} dt = \frac{\pi X}{2a^*} H(0)$$

When we restrict j to be coprime to D, we obtain an additional factor $(1 - 1/D)$, whence:

$$\lim_{X \to \infty} \frac{1}{X} \sum_{\substack{j,l \\ (j,D)=1}} g(a^* \frac{jl}{X}) \hat{H}(2 \log(j) - 2 \log(l)) = \frac{\pi H(0)}{2a^*} (1 - 1/D)$$

Subsitute this into Equation 2.32. (One must also establish appropriate uniformity in a^*, which amounts to controlling the contribution of large values of a^*. To do this, one can sum first over a^* and then over j, l, for a^* in the "large" range – that is, around X; this justifies the formal manipulations that we will perform.)

We find that the contribution of $\mathfrak{c} = \mathfrak{c}_0$ to Equation 2.31 is:

$$(\sum_{a \text{ squarefree}} \chi(a^*)/a^*)(1 - 1/D) H(0) \frac{\pi}{2D} = \frac{1}{D+1} \frac{3}{\pi} \frac{h(0) \overline{\eta'(m, 1/2)}}{L(1, \chi)}$$

(Here we have used that $\sum_{a^*} \chi(a^*)/a^* = \frac{L(1,\chi)}{\zeta(2)(1-1/D^2)}$.)

Finally, the total contribution of the continuous spectrum is twice this: there are two cusps, $\mathfrak{c}_\infty$ and $\mathfrak{c}_0$, and the functional equation of the Eisenstein series, in this case, essentially interchanges the two cusps, and it may be derived from this that they contribute identically. The total contribution of the continuous spectrum to the limit in Equation 2.6 is, therefore:

$$L_{cts} = h(0)\frac{6}{(D+1)\pi L(1,\chi)}\overline{\eta'(m,1/2)} = \frac{1}{2}h(0)C_g\sum_{ab=|m|}\chi(a) \qquad (2.33)$$

This confirms Proposition 7 in this case, since the right hand sum of Equation 2.30 is only over the character $\omega = 1$, and, indeed

$$\lambda_m(f_1) = \sum_{a||m|}\chi(a)$$

□

2.5.5 Case of full level

There is one case which behaves qualitatively differently, and we discuss it here. That is where one takes N to be a square and χ to be trivial. It is not necessary for the purpose of deriving the main Theorem, which is only concerned with cuspidal forms, but it is interesting to discuss. The limit L of Equation 2.6 is nonzero: no cuspidal spectrum contributes, but *all* of the continuous spectrum contributes.

The simplest instance is that of $N = 1$, that is, $\mathrm{SL}_2(\mathbb{Z})$. Here all the Eisenstein series $E(z,s)$ that occur have $m(\mathrm{Sym}^2, E(z,s)) = 1$. (One might regard them as the forms that are associated to the *split* quadratic algebra over $\mathbb{Q}$.)

Again, the spectral side of the Petersson-Kuznetsov formula involves the continuous spectrum term (there is only one cusp and we consequently drop the subscript $\mathfrak{c}$):

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\eta(n,1/2+it)\overline{\eta(m,1/2+it)}dt$$

in this case, with ξ the completed Riemann zeta function $\pi^{-s/2}\Gamma(s)\zeta(s)$, we have:

$$\eta(n,s) = \frac{1}{\xi(1+2it)}\left(\frac{4\pi}{\cosh(\pi t)}\right)^{1/2}\sum_{ab=|n|}(a/b)^{it}$$

Let $\eta'(n,s) = \sum_{ab=n}(a/b)^s$; then the Dirichlet series $\sum_n \eta'(n,it)/n^z$ equals $\zeta(z+it)\zeta(z-it)$, and the Dirichlet series $\sum_n \eta'(n^2,it)/n^z$ equals $\zeta(z+2it)\zeta(z-2it)\zeta(z)/\zeta(2z)$. In particular, computing the residue at $z=1$ gives the asymptotic of $\sum_{n\sim X}\eta'(n^2,it))$; from this we easily deduce that for t fixed:

$$\lim_{X\to\infty}\frac{1}{X}\sum_{n\sim X}\eta(n^2,1/2+it)\overline{\eta(1,1/2+it)} = (4\pi)\frac{6}{\pi^2}$$

By the techniques of the appendix (section on L-functions), one sees that this is valid with an error term of the form $O((1+|t|)^A X^{-1/2+\epsilon}$; in particular, there is enough uniformity to allow the interchange of integral and limit in the continuous spectral contribution, as long as $h(t)$ is of reasonable decay.

The contribution of the continuous spectrum to Equation 2.6 (see Equation 2.31 for the explicit form), with $m=1$, is therefore equal to $6\hat{h}(0)/\pi^2$, where $\hat{h}$ is the Fourier transform of h.

More generally,

$$\lim_{X\to\infty}\frac{1}{X}\sum_{n\sim X}\eta(n^2,1/2+it)\overline{\eta(m,1/2+it)} = 4\pi\frac{6}{\pi^2}\sum_{ab=m}(a/b)^{it}$$

from where we see that the contribution of the continuous spectrum to Equation 2.6 is:

$$\frac{6}{\pi^2}\sum_{ab=m}\hat{h}(\log(b)-\log(a)) \tag{2.34}$$

This matches Equation 2.13 in the case of $N=1$: In this case, the set $\mathfrak{S}'$ consists of numbers such that ν^2-4m is a square; such ν consist precisely of $a+b$, where

$ab = m$. For such a ν, we find $\cosh^{-1}(\nu/2\sqrt{m}) = \log(b) - \log(a)$. The constant $c(N)$ was not really defined in this case, but in order to make Proposition 2 valid we set $c(N) = 1$; with that, we see that Equation 2.13 agrees with the limit derived "directly" in Equation 2.34.

2.6 Putting it all together!

The main Theorem on the classification of forms with $m(f, \mathrm{Sym}^2) = 1$ is the following:

Theorem 4. *The cuspidal representations π (equivalently cuspidal newforms f) on* $\mathrm{GL}(2)/\mathbb{Q}$*, such that $m(f, \mathrm{Sym}^2) = 1$, correspond to pairs of distinct Grössencharacters $\{\omega, \omega^{-1}\}$ of quadratic field extensions $K/\mathbb{Q}$ with the property that $\omega|\mathbb{A}_{\mathbb{Q}}^{\times}$ is trivial. The map is $\omega \mapsto f_\omega$; the form f_ω associated to ω is characterized by the matching of L-functions, $L(f_\omega, s) = L(K, \omega, s)$. The conductor of $\pi(\omega)$ is* $\mathrm{Disc}(K)\mathrm{Norm}(\mathfrak{f}_\omega)$*, where $\mathfrak{f}_\omega$ is the conductor of ω.*

The point of this section is to make absolutely clear how the results of this Chapter actually constitute a proof of this Theorem: the construction and classification of dihedral forms. It is, in some sense, "standard": a trace formula constitutes a "classification", but there are certain issues of density of function spaces that need to be discussed before this is asserted.

We are not proving the Theorem in complete generality; for instance, we do not compute the Hecke eigenvalues of f_ω at integers dividing the level.

At the very outset, note that any cusp form with $m(f, \mathrm{Sym}^2) \geq 1$ must be self-dual; in particular, its central character must be quadratic. There is no loss, then, in working on $(\Gamma_0(N), \chi)$ for $N = D_\chi f^2$ (one can always enlarge N if necessary).

As remarked earlier, we deal only with the case of computing Fourier coefficients with $m > 0$, the other case being essentially the same, *mutatis mutandis*.

Maass Forms ($D > 0$)

Firstly, in Proposition 3, and its translation Equation 2.16, we have derived a spectral sum formula, over dihedral forms – but only over those test functions $h = h^+$ that arise from φ of compact support. The limit formula we have shown is of the form:

$$\lim_{X\to\infty} \sum_f h(t_f) \frac{1}{X} \sum_{n\sim X} a_{n^2}(f)\overline{a_m(f)} = \cdots$$

and we wish to extract from this formula the Hecke eigenvalues and Laplacian eigenvalue of all forms with $\frac{1}{X}\lim_{X\to\infty}\sum_{n\sim X} a_{n^2}(f) \neq 0$. The first problem is the delicate issue of the interchange of sum and limit, since the class of test functions h is somewhat limited.

In dealing with this issue, we encounter a somewhat unfortunate (from a philosophical point of view) difficulty: we must appeal to the theory of the symmetric square or the Rankin-Selberg L-function, and, unlike Section 2.5.3, cannot be easily avoided. To be precise, we need an estimate, for some $M > 0$, that if f is a cusp form of eigenvalue $1/4 + t_f^2$,

$$\frac{1}{X} \sum_{n\sim X} a_{n^2}(f) \ll (1 + |t_f|)^M \tag{2.35}$$

that is *uniform* in t_f. This is easily done by appeal to the theory of the symmetric square L-function, *or*, if one prefers, one may use only the theory of the Rankin-Selberg L-function together with very mild information on the growth of $\zeta(s)^{-1}$ along $\Re(s) = 1$.

There is no circularity in this appeal; however, it would be preferable to be able to deduce the required estimate directly from the limiting formula. It is possible that this could be done with careful analysis of the allowable spectral functions h.

In any case, we proceed, assuming the theory of the symmetric square or Rankin-Selberg theory together with information on $\zeta(s)$. Using the methods discussed in Section 6.5 of the Appendix, one can obtain the required estimate Equation 2.35, and, at this point, we may appeal to Theorem 7 of the Appendix, which is essentially a density result stating that one gets "a large number" of spectral test functions h^+ from φ of compact support (and, in particular, from φ satisfying Hypothesis 1. This could also be proved more directly, since Hypothesis 1 is significantly weaker than compact support; nevertheless, the result in the Appendix is of independent interest.) The hypotheses of the Theorem follow from Equation 2.35. The Theorem is phrased abstractly, but, in our context, it shows that one may invert the summation and limit and convert Equation 2.16 to the statement:

$$\sum_f h(t_f)\left(\lim_{X\to\infty}\frac{1}{X}\sum_{n\sim X} a_{n^2}(f)\overline{a_m(f)}\right) + L_{cts} = \tag{2.36}$$

$$c(N)\frac{6}{\pi\delta\log(\epsilon_0)}\sum_{k\in\mathbb{Z}} h\left(\frac{\pi k}{\delta\log(\epsilon_0)}\right)\left(\sum_{x\in X_m} e\left(k\frac{\log(x/\sqrt{m})}{\log(\epsilon_0^\delta)}\right)\right) \tag{2.37}$$

We have already evaluated L_{cts} and, if one groups the f of the left-hand sum into "classes" indexed by newforms, we can sum $\lim_{X\to\infty}\frac{1}{X}\sum_n a_{n^2}(f)\overline{a_m(f)}$ over each class. (See Proposition 6 and Proposition 7.) Substituting and using Equation 2.27, we obtain:

$$C_g\sum_{\substack{f \text{ cusp.newform}\\ m(f,\mathrm{Sym}^2)=1}} h(t_f)\lambda_m(f) + \frac{1}{2}h(0)C_g\sum_{\omega\in\mathrm{Eis}}\lambda_m(f_\omega) = \tag{2.38}$$

$$\frac{C_g h_{D,f}}{\delta}\sum_{k\in\mathbb{Z}} h\left(\frac{\pi k}{\delta\log(\epsilon_0)}\right)\left(\sum_{x\in X_m} e\left(k\frac{\log(x/\sqrt{m})}{\log(\epsilon_0^\delta)}\right)\right) \quad = \frac{1}{2}C_g\sum_{\omega\in\mathrm{Cusp}\cup\mathrm{Eis}} h(t_\omega)\lambda_m(f_\omega)$$

where we have used Equation 2.27, and the sum over f is over cuspidal newforms of level dividing N.

Removing the Eisenstein contribution, we therefore obtain the required matching:

$$\sum_{\substack{f \text{ cusp.newform}\\ m(f,\mathrm{Sym}^2)=1}} h(t_f)\lambda_m(f) = \frac{1}{2}\sum_{\omega\in\mathrm{Cusp}} h(t_\omega)\lambda_m(f_\omega)$$

It follows, in particular, that the set of t_f are exactly the set of t_ω. For a fixed $t_0 \in \mathbb{R}$, we obtain:

$$\sum_{\substack{t_f = t_0 \\ m(f,\mathrm{Sym}^2)=1}} \lambda_m(f) = \frac{1}{2} \sum_{\substack{\omega \in \mathrm{Cusp} \\ t_\omega = \pm t_0}} \lambda_m(f_\omega)$$

We are now in a "finite dimensional" situation and may apply Proposition 18 of the Appendix, with the set S' consisting of $\{\omega \in \mathrm{Cusp}, t_\omega = t_0\}$, and $b_m^\alpha = \lambda_m(f_\omega)$.

Proposition 18 shows, then, that the set of f on the left hand side are in bijection with pairs $\{\omega, \omega^{-1}\}$, and their coefficients $\lambda_m(f)$ match, under this bijection, the $\lambda_m(f_\omega)$, at least for m coprime to N.

When one puts this result together for all D and f, one obtains the main Theorem when one restricts f to weight 0 Maass forms.

Holomorphic Case ($D < 0$)

The procedure for holomorphic forms of weight $k \gg 1$ is even easier, as one does not need to invoke Theorem 7 or even Section 6.5; nor does one need to deal with the continuous spectrum. It is quite simple to see, as the space is finite dimensional, that Equation 2.18 and Proposition 18 of the Appendix imply directly the existence of holomorphic forms associated to quadratic Grössencharacters, and that these exhaust holomorphic forms with $m(f, \mathrm{Sym}^2) = 1$.

Remark 4. We note that the restriction on weight is unnecessary, if one is willing to do a little more work. The case of weight 1 is the most delicate. Here is the idea of the argument, which we will not explicate: rather than apply the Petersson formula directly to forms of low weight, we apply the combined Petersson-Kuznetsov formula (see, for example, [6]), which includes a sum over Maass forms of weight 1. This allows us to use a spectral function φ of compact support again, and one can use *exactly* the same argument as was used for Maass forms above. This is a curious validation of the principle that it is beneficial to consider things in larger families than the obvious ones!

Chapter 3

Converse Theorems

3.1 Introduction

This chapter is an attempt to analyze Langlands' idea, suggested in *Beyond Endoscopy* and mentioned in the Introduction to this thesis, of "twisting by a Galois representation."

Again, we shall begin by discussing the method in the most general setting to fix the idea of what we are trying to do. Let F be a number field and fix $\sigma : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\mathbb{C})$, a complex Galois representation.

We have already seen – at least over $\mathbb{Q}$, and the general case is sketched in Chapter 4 – that one may isolate, using a limiting process in the trace formula, those automorphic representations π such that $L(s, \mathrm{Sym}^2\pi)$ has a pole. Similarly, given π, one may define, at least in an appropriate half-plane, the product L-function $L(s, \pi \times \sigma)$. It is then meaningful to ask whether one may, using similar ideas, isolate those forms for which $L(s, \pi \times \sigma)$ has a pole. Of course, in contrast to the previous setting, this set of forms has at most one element – it will be the form π that is parameterized by the Galois representation $\tilde{\sigma}$, if such a form exists at all! One hopes, then, to be able to deduce modularity of σ by carrying out this limiting process.

Evidently, one will need some non-formal information about σ in order to carry out this procedure. This chapter is meant to demonstrate that: *the information required to carry out this technique is essentially equivalent to that information required by the converse theorem.* In other words, conditions that are equivalent to the analyticity and functional equations of various twisted L-functions attached to σ will come directly out of the trace formula!

Although one can produce some variants on the usual converse theorem that reflect the analytic nature of this approach – see Subsection 3.2.3 – this technique does not seem to have great applicability. Nevertheless, it seems of value in the context of this method, and points to a natural interface between it and the "standard" techniques of automorphic L-functions.

3.2 Limiting Process

Let $(b_n)_{n=1}^{\infty}$ be a sequence of numbers; we are interested in the existence of a modular form with the coefficients b_i (appropriately normalized). We will work this out in the simplest case, when we expect this form to be holomorphic of "large enough" even weight k on $\mathrm{SL}_2(\mathbb{Z})$. Here it is proven for $k \geq 6$. We will assume that for all $\epsilon > 0$, we have $b_n = O(n^{\epsilon})$ – by the Ramanujan-Petersson conjecture, we certainly don't expect to find a form with these coefficients otherwise!

(The assumption on weight is clumsy and unfortunate, and perhaps could be removed, but the main point here is to outline the method and not on technical perfection. The generalization to $\Gamma_0(N)$ is almost immediate.)

We are attempting to understand how the assertion: "The functional equations and appropriate analytic properties of the L-series b_n/n^s and all its twists imply that there is an automorphic form with these coefficients," may be derived from the trace formula.

The notion of "twisting" we will need is "twisting by additive characters" rather than the usual "twisting by multiplicative characters." (These notions are essentially equivalent; but additive characters arise more naturally from the Petersson-Kuznetsov formula.)

For a fixed weight k holomorphic form f, one can form the limit

$$\lim_{X \to \infty} \frac{1}{X} \sum_{n<X} a_n(f) b_n$$

(If the b_n were indeed the Fourier coefficients of a form g, we would expect this limit to be proportional to $\langle f, g \rangle$.) This limit should be understood as representing the residue of a Rankin-Selberg L-function. Recall also that the Fourier coefficients are normalized (see Subsection 2.2.1).

We will sum this over the spectrum and will analyze:

$$S(m; X) = \sum_{f} \overline{a_m(f)} \sum_{n \sim X} b_n a_n(f) \tag{3.1}$$

by using the Petersson formula. The sum is over an orthonormal basis for holomorphic forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, notations being as in Subsection 2.2.1.

First let us see how one can deduce, from the behavior of this sum, the existence of a form F with Fourier coefficients b_n.

Proposition 8. *Suppose there exists a constant C such that, for all m, the limit $\lim_{X \to \infty} S(m; X)/X$ equals $C b_m$. Then there exists a holomorphic form of weight k with Fourier coefficients (b_m).*

Proof. Firstly, by taking an appropriate sum over m with some weights α_m, consider:

$$\sum_m \alpha_m S(m; X) = \sum_f \left(\sum_m \alpha_m \overline{a_m(f)} \right) \sum_{n \sim X} b_n a_n(f)$$

Now, by choosing the α_m correctly, we can "isolate a single form". (That is, given a form f_0, we can extend it to an orthonormal basis $\{f_0, f_1, \ldots, f_d\}$, and then choose the α_m so that $\sum_m \alpha_m \overline{a_m(f_i)} = 1$ if $i = 0$ and $= 0$ if $i \geq 1$.)

This shows that

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n\sim X} b_n a_n(f)$$

exists for every form f. Call it $c(f)$.

Now, form the sum $F = \sum_f c(f)\overline{f}$, the sum ranging over an orthonormal basis and $\overline{f}$ the form whose Fourier coefficients are the complex conjugate of those of f. Then the mth Fourier coefficient of F is given by:

$$a_m(F) = \sum_f c(f)\overline{a_m(f)} = \lim_{X\to\infty} \sum_f \frac{1}{X} \sum_{n\sim X} b_n a_n(f)\overline{a_m(f)}$$

By assumption, this equals Cb_m. Therefore, F/C has normalized Fourier coefficients a_m equal to b_m. □

In the definition of $S(m;X)$, we include a weight function $g(n/X)$ to quantify the range $n \sim X$ over which we sum n. Applying the Petersson formula, Equation 2.1, to evaluate $S(m;X)$, we find

$$S(m;X) = 2\pi i^k \sum_{c=1}^{\infty} \sum_{n=1}^{\infty} \frac{S(n,m,c)b_n}{c} g(n/X) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) + b_m g(m/X) \qquad (3.2)$$

The final term $b_m g(m/X)$ is the "diagonal" contribution. It vanishes for X sufficiently large. We will denote it as Diag; it is essentially irrelevant, as it does not affect $\lim_{X\to\infty} S(m;X)/X$.

Remark 5. One may truncate the c-sum in Equation 3.2 at $c = X^{\frac{k-1}{2k-3}+\epsilon}$ without affecting the computation of $\lim_{X\to\infty} \frac{S(m;X)}{X}$.

Indeed, take $\beta > 1$, to be fixed later. Recall that we assumed $b_n = O(n^\epsilon)$, and truncate the sum at $c = O(X^{\beta/2})$. In doing so, the remainder term is of size

$$X^{1+\epsilon} \sum_{c > X^{\beta/2}} \frac{1}{\sqrt{c}} \left(\frac{\sqrt{X}}{c} \right)^{k-1} = X^{1+\epsilon} \cdot X^{(k-1)/2} X^{-\beta(k-3/2)/2}$$

In particular, if $\beta > (k-1)/(k-3/2)$, the error in so truncating c will not affect the computation of the limit $\lim_{X\to\infty} S(m;X)/X$.

Now expand out the Kloosterman sum in Equation 3.2:

$$S(m;X) = 2\pi i^k \sum_{c=1}^{\infty} \frac{1}{c} \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^\times} e(mx^{-1}/c) \sum_{n=1}^{\infty} b_n e(nx/c) g(n/X) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) + \text{Diag} \tag{3.3}$$

It is, at this stage, not clear how one might establish an asymptotic $S(m;X) \sim b_m X$. In particular, one sees that the coefficient of *each* b_n, for $n \ll X$, is of size possibly as large as $\sqrt{X}$. The key is that the inner summation that occurs above may be analyzed by means of Voronoi-type summation formulae which are equivalent to the functional equation; this enables the sum to be transformed into a form that "emphasizes" the coefficient of b_m. We discuss this in the next section.

3.2.1 Voronoi-type summation formulae

The sums over n in Equation 3.3, are highly reminiscent of the following "Voronoi-type" summation formulae, first introduced for $b_m = d(m)$, the divisor function. In some sense, the only point of this Chapter is to point out this resemblance!

We first introduce the relevant integral transform. Let $g(x)$ be a C^∞ function with "sufficiently rapid decay" near 0 and infinity (a condition of the form $g(x) \ll \min(x^l, x^{-l})$ for some fixed l will suffice, and this will certainly be the case in our application). Let q be a positive integer. Define the following transform:

$$\hat{g}_q(y) = \frac{2\pi i^k}{q} \int_0^\infty g(x) J_{k-1}(\frac{4\pi\sqrt{xy}}{q}) dx \tag{3.4}$$

The Voronoi-type summation formula we are about to state is a "smooth version" of the functional equation for an L-function.

Proposition 9. *Let d, q be coprime natural numbers. Let $L(d,q;s)$ be the additively twisted L-function $\sum_{m=1}^{\infty} b_m e(dm/q) m^{-s}$. Suppose that $\Lambda(d,q;s) = (2\pi)^{-s} q^s \Gamma(\frac{k-1}{2} + s) L(d,q;s)$ satisfies the "standard functional equation" $\Lambda(d,q;s) = \Lambda(\bar{d},q;1-s)$, where $\bar{d}$ is the inverse of d modulo q. (This is the functional equation it would satisfy if the b_m came from a holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.) Suppose, additionally, that $L(d,q;s)$ has at most polynomial growth in vertical strips, and has a finite number of poles at $s = \rho_i$ with residue $\mathrm{Res}_{s=\rho_i} L(d,q;s) = R(\rho_i)$. Then*

$$\sum_{m=1}^{\infty} b_m e(dm/q) g(m) = \sum_{m=1}^{\infty} b_m e(-\bar{d}m/q) \hat{g}_q(m) + E \tag{3.5}$$

where, if G is the Mellin transform of g, the "error term" E is given by

$$E = (2\pi i) \sum_{\rho_i \text{pole}} R(\rho_i) G(\rho_i)$$

Proof. (Sketch only!) Let $G(s) = \int_0^{\infty} g(x) x^{s-1} dx$. Then, we have, for some fixed $\sigma \gg 1$,

$$\sum_{m=1}^{\infty} b_m e(dm/q) g(m) = \sum_m b_m e(dm/q) \int_{\Re(s)=\sigma} G(s) m^{-s} ds = \int_{\Re(s)=\sigma} G(s) L(d,q;s) ds$$

Now, one shifts the line of integration to a $\sigma \ll 0$. In doing so, one picks up an "error term" $2\pi i \sum_i R(\rho_i) G(\rho_i)$ as one crosses the poles of the L-function. At this point, when one applies the functional equation to the integral and inverse Mellin transformation, one obtains the function $\hat{g}_q$ of above: the point is that, on the Mellin transform side, multiplication by a quotient of Γ-functions corresponds to convolution with an appropriate Bessel function. $\square$

3.2.2 Analysis of $S(m;X)$

We will now assume the:

Hypothesis 2. For each pair of coprime integers d and q, the additively twisted L-function $L(d,q;s) = \sum_m b_m e(dm/q) m^{-s}$ associated with the sequence (b_m) has analytic continuation to the complex plane, and $\Lambda(d,q;s) = (2\pi)^{-s} q^s \Gamma(\frac{k-1}{2}+s) L(d,q;s)$ satisfies $\Lambda(d,q;s) = \Lambda(\bar{d},q;1-s)$.

We now show that this information suffices to complete the analysis of Equation 3.3. We apply the transformation formula 3.5 to the inner sum, with $q = c$. We obtain:

$$S(m;X) = (4\pi^2) \sum_{y=1}^{\infty} \sum_{c=1}^{\infty} b_y f_c(y) \left(\sum_{x \in (\mathbb{Z}/c\mathbb{Z})^\times} e((m-y)x^{-1}/c) \right) + \text{Diag} \tag{3.6}$$

where

$$f_c(y) = \int_0^\infty g(xc^2/X) J_{k-1}(4\pi\sqrt{xm}) J_{k-1}(4\pi\sqrt{xy}) dx \tag{3.7}$$

The idea is now as follows. We expect, in Equation 3.6, only the term with $y = m$ to contribute in the limit, because for $y \neq m$ the Bessel functions that occur in the above integral are "out of phase" with each other and so $f_c(y)$ is small.

The term with $y = m$ will contribute a constant multiple of $b_m X$. All other terms will (in the limit) contribute $o(X)$, and we will be left with the asymptotic $S(m;X) \sim C b_m X$, for a constant C that will be explicitly determined.

The key is the estimation of $f_c(y)$, for y not equal to m.

Proposition 10. *If y is an integer, and $y \neq m$, one has the bound* $\sum_{c=1}^{\infty} |f_c(y)| = O_\epsilon(X^{1/2+\epsilon} y^{-k/2+\epsilon})$.

Proof. To estimate it, we split into two cases: $c^2 < XT$ and $c^2 > XT$, where T is a parameter to be determined later. In the former case, we integrate by parts many times; in the latter case, we use the asymptotic for $J_{k-1}(4\pi\sqrt{xm})$ near 0. Optimizing the parameter T will give the result.

The purpose of integration by parts is really to utilize the oscillation in the Bessel functions; it only manifests itself for large values of the argument, whence the need to split into two ranges. We will use the following standard formulas:

$$\int_x x^{k+1} J_k(\alpha x) dx = \frac{1}{\alpha} x^{k+1} J_{k+1}(\alpha x)$$

$$\frac{d}{dx} x^{-k} J_k(x) = x^{-k} J_{k+1}(x)$$

Thus $\int_x x^{k/2} J_k(\sqrt{x}) dx = 2x^{(k+1)/2} J_{k+1}(\sqrt{x})$, $(x^{-k/2} J_k(\sqrt{x}))' = \frac{1}{2} x^{-(k+1)/2} J_{k+1}(\sqrt{x})$.

First, the integration-by-parts estimation. We write the integral as $f_c(y) = \int_0^\infty u(x)v(x)dx$, where

$$u(x) = \frac{1}{x^{-(k-1)/2}} g(xc^2/X) J_{k-1}(4\pi\sqrt{xm}), \quad v(x) = x^{(k-1)/2} J_{k-1}(4\pi\sqrt{xy})$$

We will differentiate u and integrate v.

The integral of v is $\frac{1}{2\pi\sqrt{y}} x^{k/2} J_k(4\pi\sqrt{xy})$ while the derivative of u, slightly rewritten, is:

$$\frac{c^2}{X} g'(xc^2/X) x^{-(k-1)/2} J_{k-1}(4\pi\sqrt{xm}) + \frac{1}{2}(4\pi\sqrt{m})^{-1} g(xc/X^2) x^{-(k/2)} J_k(4\pi\sqrt{xm})$$

We can repeat this as many times as we like. 4π and $\sqrt{m}$ are harmless constants that can be ignored. After t rounds of integration by parts the integral is a sum of several terms; each of them is of the form, for some $0 \leq l \leq t$:

$$\left(\frac{c^2}{X}\right)^l g^{(l)}(xc^2/X) x^{-(k-1+t-l)/2} J_{k-1+t-l}(4\pi\sqrt{xm}) \cdot \left(\frac{1}{y^{t/2}} x^{(k-1+t)/2} J_{k-1+t}(4\pi\sqrt{xy})\right)$$

which we write in the form:

$$\frac{1}{y^{t/2}}\left(\frac{c^2}{X}\right)^{l/2}\frac{g^{(l)}(xc^2/X)}{(xc^2/X)^{-l/2}}J_{k-1+t-l}(4\pi\sqrt{xm})J_{k-1+t}(4\pi\sqrt{xy})$$

Therefore, $f_c(y)$ is the sum of certain quantities I_j, where each I_j is an integral of the form:

$$\frac{1}{y^{t/2}}\left(\frac{c^2}{X}\right)^{l/2}\int_0^\infty dx\frac{g^{(l)}(xc^2/X)}{(xc^2/X)^{l/2}}J_{k-1+t-l}(4\pi\sqrt{xm})J_{k-1+t}(4\pi\sqrt{xy}) \tag{3.8}$$

To estimate this integral, put $h(x) = g^{(l)}(x)/x^{l/2}$, and we must integrate $h(xc^2/X)$ against a product $J_{k_1}(a\sqrt{x})J_{k_2}(b\sqrt{x})$, with $a \neq b$. To do this, we integrate by parts and note that the integral $\int_0^X J_{k_1}(a\sqrt{x})J_{k_2}(b\sqrt{x})dx$ is very close to *bounded* in X. Namely, for $|a-b| \gg 1$,

$$\int_0^X J_{k_1}(a\sqrt{x})J_{k_2}(b\sqrt{x})dx \ll_{k_1,k_2} X^\epsilon$$

Given $\delta > 0$, this estimate is uniform in $|a-b| \geq \delta$. This may be proved by using the asymptotic at ∞ for the Bessel functions, in terms of sines and cosines. Indeed, one can drop the X^ϵ without much extra effort. It is at this point alone in the proof that we use the assumption that $m \neq y$.

When differentiating $h(xc^2/X)$, we get $\frac{c^2}{X}h'(xc^2/X)$; it is of size c^2/X and supported in an interval of length X/c^2. Therefore, we obtain a bound, uniform in a, b for $|a-b|$ bounded away from 0:

$$\int_0^\infty h(xc^2/X)J_{k_1}(a\sqrt{x})J_{k_2}(b\sqrt{x})dx \ll_{k_1,k_2} X^\epsilon$$

In view of Equation 3.8, this implies:

$$f_c(y) = O\left(\frac{X^\epsilon}{y^{t/2}}\max(1, \left(\frac{c^2}{X}\right)^{t/2})\right)$$

Of course, this bound gets worse as c increases. We apply it for $c^2 < XT$. For $c^2 > XT$, we use the fact that the Bessel function $J_{k-1}(z)$ is $O(z^{k-1})$ for z small. Since $g(xc^2/X)$ is supported in x around X/c^2, the integrand in Equation 3.7 is bounded by $O((X/c^2)^{(k-1)/2})$, and it is being integrated over an interval of length about X/c^2, so:

$$f_c(y) = O\left(\left(\frac{X}{c^2}\right)^{(k+1)/2}\right)$$

Consequently, the sum $\sum_c |f_c(y)|$ is bounded by a constant multiple of:

$$X^\epsilon \sum_{c^2<XT} y^{-t/2} \max(1, \left(\frac{c^2}{X}\right)^{t/2}) + \sum_{c^2>XT} (\frac{X}{c^2})^{(k+1)/2}$$

which gives

$$X^{1/2+\epsilon}\left(\frac{T^{(t+1)/2}}{y^{t/2}} + \frac{1}{T^{k/2}}\right)$$

Taking T to be slightly less than y (that is, $y^{1-\delta}$ for appropriately chosen δ) and t sufficiently large, we obtain a bound for $\sum_c |f_c(y)|$ of $O(\frac{X^{1/2+\epsilon}}{y^{k/2-\epsilon}})$, as desired. □

Now, we split the sum for $S(m;X)$ into $y \neq m$ and $y = m$:

$$S(m;X) = (4\pi^2) \sum_{y\geq 1} \sum_{c\geq 1} b_y f_c(y) \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^\times} e((m-y)x^{-1}/c) + \text{Diag}$$

$$= 4\pi^2 \left(b_m \sum_c f_c(m)\phi(c) + \sum_{y\neq m} b_y \sum_{c=1}^{\infty} f_c(y) \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^\times} e((m-y)x^{-1}/c) \right) + \text{Diag}$$

Here, $\phi(c)$ is Euler's totient function.

The inner sum $\sum_{x\in(\mathbb{Z}/c\mathbb{Z})^\times} e(x^{-1}(m-y)/c)$ is a Ramanujan sum that is bounded, for fixed m, by $|y|^{1+\epsilon}$ – absolute in c. Using Proposition 10 and the assumption that $|b_y| \ll y^\epsilon$, we see that the sum of all terms for $y \neq m$ is bounded:

$$\left| \sum_{y\neq m} b_y \sum_c f_c(y) \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^\times} e((m-y)x^{-1}/c) \right| \ll X^{1/2+\epsilon} \sum_y \frac{1}{y^{(k-2-\epsilon)/2}}$$

In particular, as long as $k \geq 5$ the series converges, and we see the contribution of all terms $y \neq m$ is bounded by $O(X^{1/2+\epsilon})$; in particular, their contribution does not affect the limit $\lim_{X\to\infty} S(m;X)/X$.

Now, $f_c(m) = \int g(xc^2/X) J_{k-1}(4\pi\sqrt{xm})^2 dx$, so

$$\sum_{c=1}^{\infty} f_c(m)\phi(c) = \int_0^{\infty} J_{k-1}(4\pi\sqrt{xm})^2 (\sum_{c=1}^{\infty} \phi(c) g(xc^2/X))$$

Lemma 6. *There is a $\delta > 0$ so that*

$$\sum_{c=1}^{\infty} \phi(c) g(xc^2/X) du = \frac{3}{\pi^2}\frac{X}{x}\int_0^{\infty} g(x)dx + O_\delta((X/x)^{1-\delta})$$

Proof. This is proved by the usual technique of expressing it as an integral of the Mellin transform of g against $\sum_{m=1}^{\infty} \phi(m)m^{-s}$ and shifting contours; note that

$$\sum_{m=1}^{\infty} \phi(m)m^{-s} = \zeta(s-1)/\zeta(s)$$

is analytic for $\Re(s) > 1$, except for a simple pole at $s = 2$. □

It follows that

$$\sum_{c=1}^{\infty} f_c(m)\phi(c) = \frac{3}{\pi^2} X \int_0^{\infty} x^{-1} J_{k-1}(\sqrt{x})^2 dx + o(X) = \frac{3}{\pi^2(k-1)} X + o(X)$$

whence $S(m;X) \sim \frac{12}{k-1} b_m X$. Applying Proposition 8, we see that we have established, under Hypothesis 2, the modularity of (b_m). (One can also verify that the constant $C = \frac{12}{k-1}$ is correct: that is, the same as should be obtained from the Rankin-Selberg method.)

3.2.3 Allowing Poles for the Twists

The flavor of this proof of a converse theorem is quite analytic, of course. Here we present a variant of the converse theorem; although it has no application that I can see, it does at least illustrate that this method is fundamentally different to integral-representation techniques. The idea: one can prove modularity of (b_m) even if the twisted L-functions have poles, *so long as those poles are "sparse enough" and the residues "small enough."* In other words, "analyticity on average" is sufficient to establish modularity. One can also easily formulate variants.

Again, the hypotheses of the Theorem below are quite crude and can certainly be improved.

Theorem 5. *Let b_m, $m \geq 1$, be a sequence of complex numbers, satisfying $|b_m| \ll m^\epsilon$ for all $\epsilon > 0$, and k an even integer, $k \geq 6$.*

Given d, c coprime, denote by $L(d, c, s)$ the twisted L-function $\sum_{m=1}^{\infty} b_m e(dm/c) m^{-s}$, and define $\Lambda(d, c, s) = (2\pi)^{-s} c^s \Gamma(\frac{k-1}{2} + s) L(d, c, s)$.

Suppose that $\Lambda(d, c, s)$ has analytic continuation and satisfies the standard functional equation $\Lambda(d, c, s) = \Lambda(\overline{d}, c, 1 - s)$; further, suppose that $L(d, c, s)$ has at most polynomial growth along vertical strips and has a finite set of poles on the line $\Re(s) = 1/2$; let $\mathcal{P}(L(d, c, s))$ denote this set of poles, and, for each pole ρ, let $R(\rho)$ be the residue at $s = \rho$ of $L(d, c, s)$.

Let $r = (\beta + 1)/\beta = \frac{2k - 5/2}{k-1}$, where β is the constant of Remark 5. Suppose that one has, for some $\epsilon > 0$ and for all T,

$$\sum_{\substack{c<T \\ d \bmod c}} \sum_{\rho \in \mathcal{P}(L(d,c,s))} |R(\rho)| \ll T^{r-\epsilon} \tag{3.9}$$

Here T is being allowed to go to infinity. Then the b_m are the coefficients of a weight k holomorphic form for $\mathrm{SL}_2(\mathbb{Z})$.

Note that the assumption that all the poles lie on $\Re(s) = 1/2$ is not necessary; however, weakening this correspondingly weakens the exponent of the Theorem. The assumption on finiteness of poles is also certainly not necessary, and is included only to avoid a more complicated discussion of decay conditions along vertical lines.

Proof. We proceed as before, but now, once we apply the Petersson formula and the functional equation to compute $S(m; X)$ (as in Equation 3.6) we obtain an "error term," arising when one shifts a contour past a pole of $L(d, c; s)$. The size of this error term E is easily bounded, using Proposition 9; it is:

$$E \ll X^{\epsilon} \sum_{(c,d)} \frac{1}{c} \left| \sum_{\rho \in \mathcal{P}(L(d,c,s))} G_c(\rho) R(\rho) \right| \tag{3.10}$$

where

$$G_c(s) = \int_0^{\infty} g(x/X) J_{k-1}(4\pi\sqrt{mx}/c) x^{s-1} dx = c^{2s} \int_0^{\infty} g(c^2x/X) J_{k-1}(4\pi\sqrt{mx}) x^{s-1} dx$$

Consider the pole ρ of $L(d, c, s)$; note that by assumption $\Re(\rho) = 1/2$. Applying the naive bound that $J_{k-1}(x)$ is uniformly bounded in x, we see that the contribution of ρ to E in Equation 3.10 is bounded by a constant multiple of:

$$X^{\epsilon} |\frac{R(\rho)}{c}| \cdot |c^{2\rho} \int_x g(c^2x/X) x^{\rho-1} dx| \ll X^{1/2+\epsilon} \frac{|R(\rho)|}{c}$$

where the implicit constants are absolute in d, c, ρ and X. Thus:

$$E \ll X^{1/2+\epsilon} \sum_{d,c,\rho \in \mathcal{P}(L(d,c;s))} \frac{1}{c} |R(\rho)|$$

Now Remark 5 shows that one need only sum over $c < X^{(k-1)/(2k-3)}$. The assumption, Equation 3.9, on $R(\rho)$ then shows:

$$E \ll X^{1/2+\epsilon} \sum_{\substack{d,c<X^{(k-1)/(2k-3)} \\ \rho \in \mathcal{P}(L(d,c;s))}} \frac{1}{c}|R(\rho)| \ll X^{1-\epsilon}$$

Thus, under the hypothesis of the theorem, one still obtains the asymptotic $S(m;X) \sim Cb_m X$, and therefore the modularity of the sequence (b_m) follows as before. □

Finally, note that the exponent r of the Theorem, defined as $r = \frac{2k-5/2}{k-1}$, is less than 2 but satisfies $r \to 2$ as $k \to \infty$. The exponent 2 is suggestive, as the sum of Equation 3.9 is over $O(T^2)$ Dirichlet series, and so it is perhaps the best one could establish by this method.

Chapter 4

Dihedral forms over number fields

4.1 Introduction

In this chapter, we will sketch how the classification of dihedral forms, already discussed over $\mathbb{Q}$, goes through over a number field. This is, by and large, a question of notational and not intrinsic difficulty; the difficulty one might expect, from the units, can be seen (and overcome) equally in the case of the totally real field and the general case. Because of this notational complexity, the stress of this chapter will be to be focus on a particular case, to illustrate that these difficulties can be overcome; we do not unravel the resulting formulas with such completeness as over $\mathbb{Q}$. We will work in the context of forms that arise from a totally real quadratic extension of a totally real field. This is the generalisation of the $D > 0$ case of Chapter 2.

However, the core of the limiting process, as presented in Subsections 4.3.2 - 4.3.4, is valid in complete generality: in particular, the final answer Equation 4.20 is essentially valid over a number field, with some minor modifications needed to deal with complex places. We then rephrase this answer in terms of the spectral test function in Equation 4.23 (and Equation 4.24), and this can be seen to agree with the expected answer computed in Equation 4.47. (As over $\mathbb{Q}$, this "matching of trace formulas" translates into the classification theorem for the relevant forms.)

The base field will be denoted by F. We will not give details as completely as over $\mathbb{Q}$; in particular, we will take relative care with the inclusion of constants that pertain to the field such as h_F (the class number), D_F (the discriminant) and R_F (the regulator); certain other constants (that in any case depend only on the degree of F) will be hidden in the normalizations.

There are certain forms of the Kuznetsov formula available in the literature. We have included a derivation of a formula apposite to our purposes in the final Section 4.6; in particular, it is a GL_2 formula, and we follow the representation-theoretic approach of Cogdell and Piatetski-Shapiro in that we combine the contribution of "holomorphic" and "non-holomorphic" forms. We postpone this derivation to the end of the Chapter, and for reading purposes one may accept the statement of the formula and follow the reasoning without reading these sections.

Throughout this chapter we will not touch on the continuous spectrum contribution at all; it will not even be made explicit in the Kuznetsov formula.

Notations: We denote by $\mathfrak{o}_F$ the ring of integers of F, by C_F its class group, and by d the degree $[F:\mathbb{Q}]$. We will fix A, a set of representatives for ideal classes modulo principal ideals; then $|A| = |C_F| = h_F$. We denote by F_∞ the tensor product $F \otimes \mathbb{R}$, and by $F_{\infty,+}$ the "totally positive" elements of this tensor product. v will always denote a place of F and F_v the completion at that place. If v is a place of F, we write $v|\infty$ if v is archimedean. We have $F_\infty = \prod_{v|\infty} F_v = \prod_{v|\infty} \mathbb{R}$; we will fix an ordering of the places so that we can freely identify F_∞ with $\mathbb{R}^{[F:\mathbb{Q}]}$. By default, if f is a function on $\mathbb{R}$, then we use the same symbol to define the function on $F_\infty = \mathbb{R}^{[F:\mathbb{Q}]}$ that is defined "place-by-place", i.e. for $x = (x_v) \in \mathbb{R}^{[F:\mathbb{Q}]}$, we set $f(x) = (f(x_v)) \in \mathbb{R}^{[F:\mathbb{Q}]}$.

$\mathbb{A}_F$ will be the ring of adeles of F, and $\mathbb{A}_{F,f}$ will be the ring of *finite* adeles – the restricted product, over all finite places v, of F_v. If v is a finite place, $\mathfrak{o}_{F,v}$ or just $\mathfrak{o}_v$ is the ring of local integers. $\mathfrak{o}_F^*$ denotes the group of units in $\mathfrak{o}_F$; similarly $\mathfrak{o}_F^{*,+}$ is the group of totally positive units. $\widehat{\mathfrak{o}_F}$ will be the completion of $\mathfrak{o}_F$ with respect to the topology defined by ideals; it is identified with the maximal compact subring $\prod_v \mathfrak{o}_{F,v}$ of $\mathbb{A}_{F,f}$. We will also denote by $U_{F,v} = U_v = \mathfrak{o}_{F,v}^\times$ the maximal compact subgroup of F_v. The units of $\widehat{\mathfrak{o}_F}$ are then $\prod_{v \text{ finite}} U_v$. q_v will be the cardinality of the residue field of $\mathfrak{o}_v$.

For any fractional ideal $\mathfrak{b}$, we denote by $[\mathfrak{b}]$ a generator if one exists. We say that an idele $x_{\mathfrak{b}} \in \mathbb{A}_{F,f}^\times \subset \mathbb{A}_F^\times$ *corresponds to* $\mathfrak{b}$ if it has the same valuation at all (finite) places, and write $(x_{\mathfrak{b}}) = \mathfrak{b}$. Via this, we can choose a subset of $\mathbb{A}_{F,f}$ corresponding to the ideals in A; with this choice, A is a subset of $\mathbb{A}_{F,f}^\times$ and of $\mathbb{A}_F^\times$. If we need to make the distinction clear, we denote by $\pi_{\mathfrak{a}}$ the element of $\mathbb{A}_{F,f}$ corresponding to $\mathfrak{a} \in A$. The normalization of measures that is used is stated in the final section (start of Section 4.6.)

One other notational point: our symbol Norm will denote "absolute norm" and always take positive values, unless otherwise specified. (For example, when taking the norm from a quadratic extension K to F we will write $\mathrm{Norm}_{K/F}$.) For example, the number of an integral ideal $\mathfrak{I}$ will be the cardinality $|\mathfrak{o}_F/\mathfrak{I}|$. For $x \in F$ we will denote by $\mathrm{Norm}(x)$ the norm of the ideal generated by x, i.e., the absolute value of $\mathrm{Norm}_{F/\mathbb{Q}}(x)$. Similarly, for $x \in F_\infty$, we will denote by $\mathrm{Norm}(x)$ the *absolute value* of the "usual" norm of x, i.e. $|\det(x)|$ when x acts on F_∞ by multiplication.

Z will denote the center of GL_2. N will denote the group of unipotent upper triangular matrices, that is, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, with $x \in \mathbb{A}_F$. If x is restricted to lie in F, we obtain a discrete subgroup denoted by N_F. An (additive) character of $\mathbb{A}_F$ then determines a character of N; thus, if ψ is a character of $\mathbb{A}_F$, we will use ψ to denote the corresponding character of N without further comment.

4.2 Translation from adelic to classical

Essentially, the Kuznetsov-type formula used will be quite classical: we will work on a quotient of a real group by an arithmetic subgroup, or at least a finite union of such objects. In particular, the space is *disconnected*. The aim of this section is to cover the translation between the adelic and the classical picture, which is not as clear as over $\mathbb{Q}$.

Fix an integral ideal $\mathfrak{I}$ of F, and a possibly fractional ideal $\mathfrak{a}$. We define $\Gamma_0(\mathfrak{I};\mathfrak{a})$ to be the group of $\mathrm{GL}_2(F)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \in \mathfrak{I}\mathfrak{a}^{-1}$, $b \in \mathfrak{a}$, $a, d \in \mathfrak{o}_F$ and $ad - bc \in \mathfrak{o}_F^\times$; we define $\Gamma_0(\mathfrak{I})$ to be $\Gamma_0(\mathfrak{I};\mathfrak{o}_F)$. $K_0(\mathfrak{I})$ will be the closure of $\Gamma_0(\mathfrak{I})$ in $\mathrm{GL}_2(\mathbb{A}_{F,f})$. If v is a finite place of F, $K_{0,v}(\mathfrak{I})$ will be the topological closure of $\Gamma_0(\mathfrak{I})$ in $\mathrm{GL}_2(F_v)$; then $K_0(\mathfrak{I}) = \prod_{v\,\text{finite}} K_{0,v}(\mathfrak{I})$.

We fix an additive character ψ of $\mathbb{A}_F/F$: we take it to be the composite of the "standard character" of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ (the character which is $e(x)$ when restricted to $\mathbb{R} = \mathbb{Q}_\infty$) with trace. By restriction, it gives a character of F_∞ trivial on $\mathfrak{o}_F$. Let $\mathfrak{d}^{-1}$ be the conductor of ψ, that is, the dual of $\mathfrak{o}_F$ with respect to the character of F_∞ it induces; it is a fractional ideal, and its inverse $\mathfrak{d}$ is the usual different.

4.2.1 Classical setting

Take a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$; let χ be its central character. Let χ_∞ be the restriction of χ to $F_\infty^\times$, and χ_f the restriction of χ to $\mathbb{A}_{F,f}$. Then χ_f, by restriction, determines a character of $\widehat{\mathfrak{o}_{F,f}}^\times$, and hence of a "Dirichlet character" of $\mathfrak{o}_{F,f}$. (This is just the generalization of the fact that Grössencharacters of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ give Dirichlet characters of $\mathbb{Z}$.)

Let $\mathfrak{I}$ be the conductor of π; it is an integral ideal of F, divisible by the conductor of χ, and is defined in such a way that each local component π_v, for v finite, has a one dimensional space of vectors transforming by χ_f under the action of $K_{0,v}(\mathfrak{I})$. (χ_f

determines a character of $K_{0,v}(\mathfrak{I})$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \chi_f(a)$.)

Choosing a vector in the space of π with this $K_0(\mathfrak{I})$-transformation property, we obtain a "newform" associated to π. (Strictly, to merit the name "newform" one should normalize the ∞-type also.) We now wish to identify the space on which this form "lives" as a real locally symmetric space (with multiple connected components, of course).

From strong approximation for SL_2, we see that $\mathrm{SL}_2(F)\mathrm{SL}_2(F_\infty)$ is dense in $\mathrm{SL}_2(\mathbb{A}_F)$. Observe that the determinant map $K_0(\mathfrak{I}) \to \mathbb{A}_{F,f}^\times$ is surjective. In particular, these two facts imply that $\mathrm{GL}_2(F)\mathrm{GL}_2(F_\infty)K_0(\mathfrak{I})$ exhausts the set of elements of $\mathrm{GL}_2(\mathbb{A}_F)$ with determinant in $F^\times F_\infty^\times \mathbb{A}_{F,f}^\times$.

There is a well defined map $\underline{\det} : \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I}) \to \mathbb{A}_F^\times / F^\times \prod_v U_{F,v} = C_F$. We choose representatives from each fibre: for each $\mathfrak{a} \in A$, which we can regard as an element $\pi_\mathfrak{a}$ of $\mathbb{A}_{F,f}$, we define $\delta_\mathfrak{a} = \mathrm{diag}(\pi_\mathfrak{a}, 1)$. Using $\delta_\mathfrak{a}$ as a basepoint, we identify the fibre of $\underline{\det}$ containing $\delta_\mathfrak{a}$ with the quotient of $\mathrm{GL}_2(F_\infty)$ by $\Gamma_0(\mathfrak{I};\mathfrak{a})$. Therefore:

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I}) = \amalg_{\mathfrak{a}\in A} \mathcal{X}_\mathfrak{a} \tag{4.1}$$

where $\mathcal{X}_\mathfrak{a}$ is the fibre above the ideal class of $\mathfrak{a}$, and is identified with the quotient $\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty)$. Forms that transform under $K_0(\mathfrak{I})$ by χ_f correspond to forms that transform under $\Gamma_0(\mathfrak{I};\mathfrak{a})$ by χ_f^{-1}. (Note that χ_f determines a character of $\Gamma_0(\mathfrak{I};\mathfrak{a})$ via: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_f(a)$.) The inversion occurs in passing from adelic to classical.

Let $\mathrm{Fun}_\chi(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I}))$ be the space of functions on $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)$ transforming by χ under the right action of $K_0(\mathfrak{I})$ and under χ_∞ by the center of $\mathrm{GL}_2(F_\infty)$. Let $\mathrm{Fun}_\chi(\mathcal{X}_\mathfrak{a})$ be the space of functions on $\mathrm{GL}_2(F_\infty)$ which transform under the center $Z(F_\infty)$ of $\mathrm{GL}_2(F_\infty)$ by the character χ_∞ and transform under (the left action of) $\Gamma_0(\mathfrak{I};\mathfrak{a})$ by χ_f^{-1}. (That is, for f in this space, $\gamma \in \Gamma_0(\mathfrak{I};\mathfrak{a})$, and $g \in \mathrm{GL}_2(F_\infty)$, we require $f(\gamma g) = \chi_f(\gamma)^{-1} f(g)$.)

By a mild abuse of notation, we will describe functions belonging to $\mathrm{Fun}_\chi(\mathcal{X}_\mathfrak{a})$ as being functions "on" $\mathcal{X}_\mathfrak{a}$; strictly, they take values in the line bundle associated to the character $\chi_f(\gamma)^{-1}$ of $\Gamma_0(\mathfrak{I};\mathfrak{a})$.

Then we have an equality, deduced by the same reasoning as Equation 4.1:

$$\mathrm{Fun}_\chi(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I})) = \oplus_{\mathfrak{a}\in A} \mathrm{Fun}_\chi \mathcal{X}_\mathfrak{a} \tag{4.2}$$

Definition 2. *We denote by* $L^2_\chi(\mathfrak{I})$ *– or merely by* L^2_χ *when* $\mathfrak{I}$ *is clear – the space of* L^2 *functions in either of the above function spaces.*

Here the notion of L^2-function is with respect to a measure derived in the natural fashion from a Haar measure on the group.

Finally, we note that the left hand side of Equation 4.2 decomposes further:

$$\mathrm{Fun}_\chi(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I})) = \oplus_{\chi'} V_{\chi'}$$

Here χ' ranges over characters of $Z(\mathbb{A}_F)$, and $V_{\chi'}$, for such a χ', is the component which transforms under $Z(\mathbb{A}_F)$ by χ'. Now, if $V_{\chi'} \neq 0$, then χ' must agree with χ on $F^\times F^\times_\infty \prod_v U_{F,v}$; in particular, χ' must be a twist of χ by a character of C_F.

Remark 6. The adelic operation of twisting a form by a character ω of C_F amounts, classically, to multiplying its restriction to $\mathcal{X}_\mathfrak{a}$ by $\omega(\mathfrak{a})$.

To rephrase, *forms on our (disconnected) symmetric space* $\amalg_{\mathfrak{a}\in A}\mathcal{X}_\mathfrak{a}$ correspond to *automorphic forms on* $\mathrm{GL}(2)$ *over* F*, with conductor dividing* $\mathfrak{I}$ *and central character an unramified twist of* χ.

Of course, one might with to study forms with a specified central character; the difference is rather like the difference between, in the setting of holomorphic modular forms on the upper half-plane, studying $S_k(\Gamma_1(N))$ and $S_k(\Gamma_0(N),\chi)$; studying the former amounts to studying all the latter together. In our context, one could study the space of forms with a specified central character by considering bundle-valued forms on a smaller space; but it is simpler, though less precise, to stay in our current level of generality.

4.2.2 Whittaker models and Fourier coefficients

We now study in more detail how to extract the coefficients of the L-series of π from the Fourier coefficients of the corresponding newform. For each representation π, we will choose a semi-canonical representative in the corresponding space, and the Fourier coefficients of this representative will determine the L-function of π. In particular, we will clarify the choice of "normalization" for the Fourier coefficients.

We now fix a choice of maximal compact in $\mathrm{GL}_2(F_\infty)$; its connected component is a product of SO(2)s, and when we refer to $\mathrm{SO}(2, F_\infty)$ we mean this particular choice. We can identify SO(2) with $\{z \in \mathbb{C} : |z| = 1\}$, and its irreducible representations are just the $\sigma_k : z \mapsto z^k$, for $k \in \mathbb{Z}$; we say σ_k is *of weight* k.

Definition 3. *Let* π_∞ *be a (unitary) irreducible representation of* $\mathrm{GL}_2(F_\infty)$*; we can write* $\pi_\infty = \otimes_{v|\infty}\pi_v$*. We say a vector* f *in the space of* π_∞ *is of* minimal SO(2)-weight *if* $f = \otimes f_v$*, and each* f_v *transforms under* SO(2) *by* σ_w*, where* $w \in \mathbb{Z}$ *is a minimal* SO(2)*-weight that occurs in* π_v *– that is, if* σ_k *is another* SO(2)*-representation occurring in* π_v*, we have* $|k| \geq |w|$*.*

Note that weights w and $-w$ can occur in the same representation, so such a minimal weight vector is not necessarily unique, even up to scaling. It is, however, unique up to scaling if the representation is unramified. Also, each SO(2)-isotypic subspace is one dimensional.

With notations as in the previous section, choose ϕ_0 a vector in the space of π which transforms under $K_0(\mathfrak{I})$ by χ_f and is also of minimal weight for π_∞. In particular, this function ϕ_0 is a pure tensor.

Let $\mathbf{f}$ be the corresponding function on $\amalg_{\mathfrak{a}}\mathcal{X}_{\mathfrak{a}}$; by this, we mean the function $\mathbf{f}$ with restriction $\mathbf{f}_{\mathfrak{a}}$ to $\mathcal{X}_{\mathfrak{a}}$, where $\mathbf{f}_{\mathfrak{a}}(g_\infty) = \phi_0(\delta_{\mathfrak{a}} g_\infty)$ for $g_\infty \in \mathrm{GL}_2(F_\infty)$. (For each $\mathfrak{a} \in A$, recall that $\delta_{\mathfrak{a}}$ was the matrix $\mathrm{diag}(\pi_{\mathfrak{a}}, 1)$.)

Let $\delta(\alpha, \beta)$ be the diagonal matrix with entries α and β. The field or ring in which α and β lie will be clear from usage. The function ϕ_0 has a Fourier expansion in terms of its associated Whittaker function $W_{\phi_0}(g) = \int_{N_F \backslash N} \phi_0(ng)\psi(n)dn$, which is the analogue of the classical Fourier expansion. Namely, for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$, we have:

$$\phi_0(g) = \sum_{\alpha \in F^\times} W_{\phi_0}(\delta(\alpha, 1)g)$$

Since ϕ_0 is a pure tensor, the Whittaker function W_{ϕ_0} has a product decomposition $W_{\phi_0} = \prod_v W_{\phi_0,v}$, where the product runs over all places v. We'll denote by $W_{\phi_0,\infty}$ the product over all archimedean v. We can consequently write, for $g_\infty \in \mathrm{GL}_2(F_\infty)$,

$$\mathbf{f}_{\mathfrak{a}}(g_\infty) = \phi_0(\delta_{\mathfrak{a}} g_\infty) = \sum_{\alpha \in F^\times} \left(W_{\phi_0,\infty}(\delta(\alpha, 1)g_\infty) \prod_{\text{finite } v} W_{\phi_0,v}(\delta(\pi_{\mathfrak{a}}\alpha, 1)) \right)$$

It is not difficult to see that, for v a finite place, $W_{\phi_0,v}(\mathrm{diag}(X, 1))$ vanishes for $X \notin \mathfrak{d}_v^{-1}$ (the inverse of the local different; the largest ideal on which ψ_v is trivial). The point is that the character ψ ramifies exactly when F_v ramifies over $\mathbb{Q}_p$, p a prime of $\mathbb{Q}$ below v. Set $a_{\mathbf{f}}^{un}(\mathfrak{a}, \alpha) = \prod_{v \text{ finite}} W_{\phi_0,v}(\delta(\pi_{\mathfrak{a}}\alpha, 1))$. The "un" superscript stands for "unnormalized". Then $a_{\mathbf{f}}^{un}(\mathfrak{a}, \alpha)$ vanishes for $\alpha \notin \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. We can therefore write this expansion in the form:

$$\mathbf{f}_{\mathfrak{a}}(g_\infty) = \sum_{\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}} a_{\mathbf{f}}^{un}(\mathfrak{a}, \alpha) W_{\phi_0,\infty}(\delta(\alpha, 1)g_\infty)$$

To give a better definition of the Fourier coefficient than that implicit in the above Equation, we need a normalization for the ∞-Whittaker functions; we also include a factor so that the resulting coefficients are uniformly (even in the field F) of size around 1.

Definition 4. *Let $\mathbf{f} = (\mathbf{f}_{\mathfrak{a}})_{\mathfrak{a}\in A}$ be a function belonging to $\oplus_{\mathfrak{a}\in A}\mathrm{Fun}_{\chi}(\mathcal{X}_{\mathfrak{a}})$, so that each $\mathbf{f}_{\mathfrak{a}}$ lies in an irreducible $\mathrm{GL}_2(F_\infty)$-representation and is of minimal $\mathrm{SO}(2)$-weight. (A priori, the $\mathrm{GL}_2(F_\infty)$-representations associated to $\mathbf{f}_{\mathfrak{a}}, \mathbf{f}_{\mathfrak{a}'}$ for $\mathfrak{a} \neq \mathfrak{a}'$ do not have to be isomorphic; they will always be, however, in the context we use.)*

We define the (normalized) Fourier coefficients $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ of $\mathbf{f}$ so that for $\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}$ and $g_\infty \in \mathrm{GL}_2(F_\infty)$,

$$\mathbf{f}_{\mathfrak{a}}(g_\infty) = \mathbf{f}(\begin{pmatrix} \pi_{\mathfrak{a}} & 0 \\ 0 & 1 \end{pmatrix} g_\infty) = \frac{1}{D_F^{1/2}} \sum_{\alpha\in\mathfrak{a}^{-1}\mathfrak{d}^{-1}} a_{\mathbf{f}}(\mathfrak{a}, \alpha) \frac{W_\infty(\delta(\alpha, 1)g_\infty)}{\sqrt{\mathrm{Norm}(\alpha\mathfrak{a})}} \tag{4.3}$$

where W_∞ is normalized by the requirements that it be a Whittaker function associated to the $\mathrm{GL}_2(F_\infty)$-representation in which $\mathbf{f}_{\mathfrak{a}}$ lies; it has the same $\mathrm{SO}(2)$-weight as $\mathbf{f}_{\mathfrak{a}}$; $\int_{x\in F_\infty^\times} |W_\infty(x, 1)|^2 d^\times x = 1$, and, if $x \in \mathbb{R}^+$ is regarded "diagonally" as an element of F_∞, then as $x \to \infty$, we require $W_\infty(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix})$ be "asymptotically" real and positive. (This last requirement is totally arbitrary and we insert it merely to fix W_∞; one could very comfortably do without it.) We set $a_{\mathbf{f}}(\mathfrak{a}, \alpha) = 0$ if $\alpha \notin \mathfrak{d}^{-1}\mathfrak{a}^{-1}$.

Remark 7. We have, if $\epsilon \in \mathfrak{o}_F^*$ is a unit, the equality $a_{\mathbf{f}}(\mathfrak{a}, \alpha) = a_{\mathbf{f}}(\mathfrak{a}, \alpha\epsilon)$; this follows since $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Remark 8. $\mathbf{f}$ is a function on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/K_0(\mathfrak{I})$, and we can as well speak of the Fourier coefficients with respect to a different choice of representatives for C_F (i.e. we can change the set A). Suppose $\gamma \in F$, and let (γ) be the fractional ideal generated by γ. Then the relation between different sets of Fourier coefficients is expressed by:

$$a_{\mathbf{f}}(\mathfrak{a}(\gamma), \alpha\gamma^{-1}) = a_{\mathbf{f}}(\mathfrak{a}, \alpha) \tag{4.4}$$

This is a simple consequence of the definitions above and will be necessary in what follows.

We define the normalized coefficients of the L-series as follows: if $\mathfrak{q}$ is an ideal, choose $\mathfrak{a} \in A$ in the same ideal class as $\mathfrak{q}\mathfrak{d}^{-1}$. Let $[\mathfrak{q}\mathfrak{d}^{-1}\mathfrak{a}^{-1}]$ be a generator for the principal ideal $\mathfrak{q}\mathfrak{d}^{-1}\mathfrak{a}^{-1}$. Choose $\mathfrak{a}_0$ in the ideal class of $\mathfrak{d}^{-1}$. Then define:

$$\lambda_{\mathbf{f}}(\mathfrak{q}) = \frac{a_{\mathbf{f}}(\mathfrak{a}, [\mathfrak{q}\mathfrak{d}^{-1}\mathfrak{a}^{-1}])}{a_{\mathbf{f}}(\mathfrak{a}_0, [\mathfrak{d}^{-1}\mathfrak{a}_0^{-1}])} \tag{4.5}$$

We then define the L-series of $\mathbf{f}$ to be:

$$L(s, \mathbf{f}) = \sum_{\mathfrak{q}} \lambda_{\mathbf{f}}(\mathfrak{q})\mathrm{Norm}(\mathfrak{q})^{-s}$$

where the sum is taken over integral ideals of F. With this normalization, the values $\lambda_{\mathbf{f}}(\mathfrak{q})$ are the "Hecke eigenvalues" of the form $\mathbf{f}$.

Let $C_{\mathbf{f}}$ be the constant so that $a_{\mathbf{f}}(\mathfrak{a}, \alpha) = C_{\mathbf{f}}\lambda_{\mathbf{f}}(\mathfrak{a}\alpha\mathfrak{d})$. If $\mathbf{f}$ has L^2-norm one, then it is closely related to the reciprocal of a special value of the adjoint L-function.

Remark 9. There is a twist by $\mathfrak{d}$ that occurs. This is a rather irritating fact, and it is possible to "twist" the definitions so this does not occur. We have chosen not to do this; the price will be the introduction of $\mathfrak{d}$ at a later stage in the argument.

4.3 Limiting Process

We will repeat the procedure of Chapter 2. However, at various points we will make simplifying assumptions, which we will summarize here for clarity. We indicate the reason for each simplification, and how to remove it.

1. We will use a slightly weakened form of the Kuznetsov formula. It corresponds to the h^+-form over $\mathbb{Q}$. The disadvantage of this is that the resulting formula fails to differentiate between *even* and *odd* forms. This is essentially for notational simplicity; it amounts to assuming that the "geometric test function" φ of Equation 4.6 is supported on the *totally positive elements* of $F_\infty^\times$. The general case amounts to removing this restriction on support; as over $\mathbb{Q}$, the Bessel transforms that occur in the general case are slightly different.

2. Rather than classifying the forms with a specified central character χ, we classify those forms whose central character is a *unramified twist* of a fixed character χ. This introduces extra complexity in the answer; we use it because it makes the Kuznetsov formula quite simple. It simplifies the analysis somewhat, as it makes the field F "look" as if it has class number 1.

3. We will be computing a sum of the form – with an appropriate weight function to make the sum convergent –

$$\sum_{m(\mathbf{f},\mathrm{Sym}^2)=1} \lambda_{\mathbf{f}}(\mathfrak{m})$$

for certain ideals $\mathfrak{m}$. We will take $\mathfrak{m}$ to be principal and prime to the conductor $\mathfrak{I}$. The first assumption is almost completely trivial and the general case is the same argument almost word-for-word; the final result, as we note, is valid for any $\mathfrak{m}$. The second assumption is analogous to the assumption $(m, N) = 1$ of Chapter 2.

4.3.1 Unramified twists of quadratic extension

Since we are analyzing matters in a context that lumps together automorphic forms whose central characters differ by a class group character, we will need to understand the corresponding phenomenon for quadratic field extensions.

Consider a quadratic extension $K_0 = F(\sqrt{D})$, where D is an element of F. Let χ_{K_0} be the character of $\mathbb{A}_F^\times$ associated by class field theory: that is, the unique character of $\mathbb{A}_F^\times$ trivial on $\mathrm{Norm}(\mathbb{A}_{K_0}^\times) \cdot F^\times$.

Let $\widehat{C_F}$ be the dual of the class group of F, and $\widehat{C_F}(2)$ the 2-torsion of this group. If one twists χ_{K_0} by an element of $\widehat{C_F}(2)$, one obtains another Grössencharacter of order 2; this must, by class field theory, correspond to a quadratic field extension K', that is ramified at the same places (and with the same discriminant) as K_0.

We will let S be the set of fields K' that arise from $K_0 = F(\sqrt{D})$ by twisting in this fashion. S can be characterized as: the set of fields whose associated Grossencharacter $\chi_{K'}$ agrees with χ_{K_0} on principal ideals. Every field in S is of the form $F(\sqrt{D\kappa})$ where κ is totally positive and $D\kappa$ has even valuation at all finite places; this condition is not, however, sufficient, on account of conditions at primes over 2. A quadratic extension of F corresponds to an element of $F^*/(F^*)^2$, and, by abuse of notation, we will also let S denote the set of classes in $F^*/(F^*)^2$ corresponding to fields in S. Note that all fields in S have the same infinity type (for example, if one is totally real, than they all are.)

4.3.2 Kuznetsov's formulas

We will be working on the space $L^2_\chi(\mathfrak{I})$ of functions introduced in Definition 2. We will quote the formula from Subsection 4.6.3. We also refer to the start of Section 4.6 for a discussion of the normalizations of the various quantities (measures and Fourier coefficients in particular) that occur. The formula at the end of the chapter is a computation on $L^2(\Gamma\backslash\mathrm{GL}_2(F_\infty))$, for Γ a group of the form $\Gamma_0(\mathfrak{I};\mathfrak{a})$; by applying it to the $\mathcal{X}_\mathfrak{a}$ we easily obtain the result stated below. See, in particular, Proposition 15. A similar formula, although not including the contribution of holomorphic forms, may be found in Bruggeman-Miatello [2]; their formula is, in a certain sense, better for the particular case we carry through in this Chapter.

$L^2_\chi(\mathfrak{J})$ decomposes as a sum and integral of $\mathrm{GL}_2(F_\infty)$ representations. For each such representation, we fix once and for all a vector $\mathbf{f}$ of minimal weight for $\mathrm{SO}_2(F_\infty) = \prod_{\infty|v} \mathrm{SO}_2(\mathbb{R})$ (see Definition 3) and of L^2-norm 1. The resulting set of "representative" forms will be called $\mathcal{B}$. All the sums over $\mathbf{f}$ are over $\mathbf{f} \in \mathcal{B}$; we will not make this explicit.

For each form $\mathbf{f}$, one has a set of Fourier coefficients $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$, where $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}$ and $\mathfrak{a} \in A$. (see Definition 4). Let φ be a function compactly supported on F_∞^+: the notation means "totally positive elements of F_∞", so φ is really a function on $(\mathbb{R}^+)^d$. It has a corresponding spectral transform $f \mapsto h_{\mathbf{f}}(\varphi)$, depending on the ∞-type of $\mathbf{f}$; the general formula is given in the Subsection 4.6.3, but the case we will need (which in any case is enough to see what is going on) is quoted at the end of this Section.

We will always be assuming φ to be of "product type", meaning it is a product of functions on F_v^+ as v ranges over archimedean places of F. (This is a harmless assumption; we can approximate any function arbitrarily well by linear combinations of such "basic" test functions.)

Take $\mathfrak{a}_1, \mathfrak{a}_2 \in A$, and $\alpha_i \in \mathfrak{a}_i^{-1}\mathfrak{d}^{-1}$ such that the product $\alpha_1\alpha_2$ is totally positive. (This is not really a restriction – see Remark 10 below). As usual, $\delta_{\mathfrak{a}_1,\mathfrak{a}_2}$ is the symbol with value 1, if $\mathfrak{a}_1 = \mathfrak{a}_2$, and zero otherwise. Then the Kuznetsov formula over F, or rather a version of the Bruggeman-Miatello formula, states:

$$\sum_{\mathbf{f}\in\mathcal{B}} a_{\mathbf{f}}(\mathfrak{a}_1, \alpha_1)\overline{a_{\mathbf{f}}(\mathfrak{a}_2, \alpha_2)} h_{\mathbf{f}}(\varphi) + \text{Continuous Spectrum Contribution} = \delta_{\mathfrak{a}_1,\mathfrak{a}_2} \cdot \tag{4.6}$$

$$\left(\sum_{c \in \mathfrak{J}\mathfrak{a}_1^{-1} - \{0\}} \sum_{\epsilon \in \mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2} \frac{1}{\mathrm{Norm}(c\mathfrak{a}_1)} KS_\chi(\alpha, \epsilon\alpha'; c)\varphi(\frac{\sqrt{\alpha\alpha'\epsilon}}{c}) \right)$$

Here, one choose the totally positive square root of $\alpha\alpha'\epsilon$. The "Kloosterman sum"

KS_χ is defined as follows:

$$KS_\chi(\alpha, \alpha'; c) = \sum_{d \in (\mathfrak{o}_F/(c\mathfrak{a}_1))^\times} \psi(\frac{\alpha d + \alpha' d^{-1}}{c})\chi_f(d) \tag{4.7}$$

Strictly, it depends on $\mathfrak{a}_1$, but this dependence will always be clear from context.

Remark 10. If there exists a unit $\epsilon \in \mathfrak{o}_F^*$ so that $\alpha_1\alpha_2\epsilon$ is totally positive, then one can replace (say) α_1 by $\alpha_1\epsilon$ and apply the formula unchanged. If there does not exist such a unit, the left-hand side sum is *zero*.

Remark 11. Note that the sum of Equation 4.6 vanishes if $\mathfrak{a}_1 \neq \mathfrak{a}_2$. This is as one expects: indeed, one can twist any occurring form $\mathbf{f}$ by a class group character, and considering the sub-sum over a complete set of twists of a single form shows that if $\mathfrak{a}_1 \neq \mathfrak{a}_2$ the sum must vanish. Similarly, Remark 10 follows by using twists by the *narrow* class group.

Remark 12. In the normalization that we have chosen, the average size of coefficients $a_\mathbf{f}$ is about $\frac{1}{\sqrt{h_F D_F}}$.

Remark 13. Throughout this chapter, there will be no explicit discussion of the continuous spectrum contribution. The expression is very similar to that over $\mathbb{Q}$, and the treatment is also identical.

The transform $\varphi \leftrightarrow h_\mathbf{f}$

We discuss $h_\mathbf{f}(\varphi)$ a little more. Suppose that $\mathbf{f}$ belongs to the space of a $\mathrm{GL}_2(F_\infty)$ representation, π_∞. Then one may define a certain "Bessel function" J_{π_∞} on $F_\infty^\times$, that describes the action of a Weyl group element on the Kirillov model – see Equation 4.52 or [4]. We define the transform $h_\mathbf{f}$ via:

$$h_\mathbf{f}(\varphi) = 2(\frac{1}{2\pi})^{[F:\mathbb{Q}]} \int_{u \in F_\infty^\times} \varphi(u) \frac{1}{\mathrm{Norm}(u)} J_{\pi_\infty}(u^2) d^\times u$$

We now make this explicit in the case of main interest. First, note that the spectrum of $L^2_\chi(\Gamma\backslash \mathrm{GL}_2(F_\infty))$ is more complicated than that encountered in Chapter 2: one can have representations that are "holomorphic" at some places and "principal series" at other places. We will only explicitly discuss the transform above when π_∞ is given by a product of principal series representations – the "generic" case.

One canonically identifies $\mathrm{GL}_2(F_\infty)$ with $\prod_{v|\infty} \mathrm{GL}_2(F_v) = \mathrm{GL}_2(\mathbb{R})^{[F:\mathbb{Q}]}$. For $\mathrm{sgn} \in \{0,1\}$, let $\pi(t, \mathrm{sgn})$ be the representation of $\mathrm{GL}_2(\mathbb{R})$ that has trivial central character and which corresponds to Maass forms of eigenvalue $1/4 + t^2$ and parity determined by sgn. In other words, it is that representation unitarily induced from the character $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto |xy^{-1}|^{it}(x|x|^{-1}y|y|^{-1})^{\mathrm{sgn}}$ of the diagonal torus.

Given $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^{[F:\mathbb{Q}]}$, and $\mathbf{sgn} \in \{0,1\}^{[F:\mathbb{Q}]}$, define a representation of $\mathrm{GL}_2(F_\infty)$ given by $\pi(\mathbf{t}, \mathbf{sgn}) = \otimes_{i\in[F:\mathbb{Q}]}\pi(t_i, \mathrm{sgn}_i)$.

Now, if $\mathbf{f}$ belongs to a representation $\pi(\mathbf{t}, \mathbf{sgn})$, $h_{\mathbf{f}}(\varphi)$ is given by a function of $\mathbf{t}$ alone, denoted $h_\varphi(\mathbf{t})$, or simply $h(\mathbf{t})$. (In other words, the formula 4.6 as stated does not distinguish the parity of Maass forms. In order to do so, one would use test functions φ that were not merely supported on $F_{\infty,+}$.) The function $h(\mathbf{t})$ may be explicitly understood by means of the following integral transformation formula, which is the precise analogue of Equation 2.4 from Chapter 2:

Let $\widehat{h_\varphi}(k)$, for $k \in \mathbb{R}^{[F:\mathbb{Q}]}$, be the Fourier transform of $\mathbf{t} \mapsto h(\mathbf{t})$; by this we mean the integral $\int_{\mathbf{t}} h(\mathbf{t})e^{2\pi i(k_1t_1+\ldots k_dt_d)}d\mathbf{t}$. Similarly define $\Delta(k)$, to be the Fourier transform of $\varphi(|x|)/\mathrm{Norm}(x)$, where $\mathrm{Norm}(x) : F_\infty = \mathbb{R}^{[F:\mathbb{Q}]} \to \mathbb{R}$ is the map taking the absolute value of the product of all factors, and, as regards $\varphi(|x|)$, "absolute value" is defined place by place on F_∞.

Let $\mathbf{cosh} : \mathbb{R}^{[F:\mathbb{Q}]} \to \mathbb{R}^{[F:\mathbb{Q}]}$ be the map $(x_1, \ldots, x_d) \to (\cosh(x_1), \ldots, \cosh(x_d))$. Then, for $k \in \mathbb{R}^{[F:\mathbb{Q}]}$,

$$\widehat{h_\varphi}(k) = \frac{1}{2^{[F:\mathbb{Q}]-1}}\Delta(2\mathbf{cosh}(\pi k)) \tag{4.8}$$

4.3.3 Constructing the sum to pick out dihedral forms

Let K_0 be a totally real quadratic extension of F, and let S be as in Section 4.3.1. Let $\chi = \chi_{K_0}$ be the associated Grössencharacter. Let $\mathfrak{D}_{K_0/F}$ be the discriminant of K_0 over F; it is an integral ideal of F. Let $\mathfrak{f}$ be an integral ideal of F, and define the ideal $\mathfrak{I} = \mathfrak{D}_{K_0/F}\mathfrak{f}^2$. We will pick out dihedral forms in the space $L^2_\chi(\mathfrak{I})$. We also, for $K \in S$, define $\mathfrak{o}_{K,\mathfrak{I}}$ to be the order in K given by $\mathfrak{o}_F + \mathfrak{f}\mathfrak{o}_K$. It has relative discriminant $\mathfrak{I}$.

Note that $\mathfrak{I}$ and $L^2_\chi(\mathfrak{I})$ remain unchanged if we replace K_0 by K, for any $K \in S$. In reality, we are really working with the equivalence class of fields in S and not with a particular one.

Now, we wish to pick out forms $\mathbf{f}$ for which the symmetric square L-function $L(s, \mathbf{f}, \mathrm{Sym}^2)$ has a pole at $s = 1$. This is the same as asking about the pole at $s = 1$ of:

$$\sum_{\mathfrak{q}} \lambda_{\mathbf{f}}(\mathfrak{q}^2)/\mathrm{Norm}(\mathfrak{q})^s$$

where the sum is taken over all integral ideals $\mathfrak{q}$ of F.

We split this according to the ideal class to which $\mathfrak{q}$ belongs. Letting $\mathfrak{o}_F^*$ denote the multiplicative group of units in F, we see (using Equation 4.5 and Remark 8) that the above series has a pole at $s = 1$ if and only if the following one does:

$$\sum_{\mathfrak{a} \in A} \sum_{\alpha \in \mathfrak{a}^{-1}/\mathfrak{o}_F^*} a_{\mathbf{f}}(\mathfrak{a}^2\mathfrak{d}^{-1}, \alpha^2)/\mathrm{Norm}(\alpha\mathfrak{a})^{-s}$$

Here $\mathfrak{a}^{-1}/\mathfrak{o}_F^*$ means elements of $\mathfrak{a}^{-1}$ modulo multiplication by units of $\mathfrak{o}_F$. (Note that we have finally introduced a twist by $\mathfrak{d}$ in the ideal class representatives. As was seen in Remark 8, changing ideal class representatives is not a problem. Although it is unfortunately a little confusing to have to do so, it is inevitable at some point that the different will enter.)

Fix an ideal $\mathfrak{m}$; we will sum the coefficient $\lambda_{\mathbf{f}}(\mathfrak{m})$ over all those $\mathbf{f}$ whose symmetric square has a pole. This is the analogue of what was done over $\mathbb{Q}$. By Remark 11, the sum over $\mathbf{f}$ of $\lambda_{\mathbf{f}}(\mathfrak{q}^2)\lambda_{\mathbf{f}}(\mathfrak{m})$ will vanish for those $\mathfrak{q}$ so that $\mathfrak{q}^2$ is not in the same ideal class as $\mathfrak{m}$. For simplicity in our derivation, we will assume that $\mathfrak{m}$ is principal; the final formula will be valid in general, however. In particular, since we are assuming that $\mathfrak{m}$ is principal, then $\mathfrak{q}^2$ must be principal for the sum to be nonvanishing.

As before, let g be a C^∞ function compactly supported on $(0,\infty)$ and with integral 1. Denote by [ ] a generator of an ideal; if the ideal is not principal, we discard the term from the sum. Then, we wish to form the following spectral sum:

$$\Sigma(X) = \sum_{\mathbf{f}} \sum_{\mathfrak{a}\in A} h_{\mathbf{f}}(\varphi) \sum_{\alpha\in\mathfrak{a}^{-1}/\mathfrak{o}_F^*} g(\text{Norm}(\alpha\mathfrak{a})/X) a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{a}^2, \alpha^2)\overline{a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{a}^2, [\mathfrak{a}^{-2}\mathfrak{m}])} + \text{Continuous Spectrum Contribution} \tag{4.9}$$

We can split the sum $\Sigma(X)$ into ideal classes as:

$$\Sigma(X) = \sum_{\mathfrak{a}} \Sigma_{\mathfrak{a},[\mathfrak{a}^{-2}\mathfrak{m}]}(X) \tag{4.10}$$

where, for $\mathfrak{a} \in A$ and $\mu = [\mathfrak{a}^{-2}\mathfrak{m}] \in \mathfrak{a}^{-2}$, $\Sigma_{\mathfrak{a},\mu}(X)$ isolates the contribution of one particular $\mathfrak{a} \in A$:

$$\Sigma_{\mathfrak{a},\mu}(X) = \sum_{\mathbf{f}} h_{\mathbf{f}}(\varphi) \sum_{\alpha\in\mathfrak{a}^{-1}/\mathfrak{o}_F^*} g(\text{Norm}(\alpha\mathfrak{a})/X) a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{a}^2, \alpha^2)\overline{a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{a}^2, \mu)} + \text{C.S.C.} \tag{4.11}$$

Here, for convenience of typesetting, we have abbreviated "Continuous Spectrum Contribution" to "C.S.C."

We will regard μ as fixed and suppress it as a subscript until the end, when we recover the sum $\Sigma(X)$. We also define $\Sigma_{\mathfrak{a},\mu}(X)$ to be zero if $\mathfrak{a}^{-2}\mathfrak{m}$ does not have a totally positive generator.

As before, we wish to show that $\lim_{X\to\infty} \Sigma_{\mathfrak{a}}(X)/X$ exists. By modifying μ by a unit (see Remark 10), we can assume it is totally positive. Then, applying Equation 4.6, we find the following formula for $\Sigma_{\mathfrak{a}}(X)$:

$$\Sigma_{\mathfrak{a}}(X) = \sum_{\substack{\alpha\in\mathfrak{a}^{-1}/\mathfrak{o}_F^* \\ \epsilon\in\mathfrak{o}_F^{*;+}/(\mathfrak{o}_F^*)^2}} g(\frac{\mathrm{Norm}(\alpha\mathfrak{a})}{X}) \sum_{c\in\mathfrak{J}\mathfrak{d}\mathfrak{a}^{-2}-\{0\}} \frac{1}{\mathrm{Norm}(c\mathfrak{d}^{-1}\mathfrak{a}^2)} KS_\chi(\alpha^2,\epsilon\mu;c)\varphi(\frac{\sqrt{\mu\epsilon\alpha^2}}{c}) \tag{4.12}$$

Note that the "implicit ideal" in KS_χ (see comment after Equation 4.7) is $\mathfrak{a}^2\mathfrak{d}^{-1}$. Since we will be holding $\mathfrak{a}$ fixed until the very end, the fact that this varies with $\mathfrak{a}$ will hopefully not cause difficulty.

The limit $\lim_{X\to\infty} \Sigma_{\mathfrak{a}}(X)/X$ exists, and we will eventually evaluate it; see (4.23).

4.3.4 Poisson Summation

We will be holding $\mathfrak{a}$ fixed in this Subsection until almost the end – Equation 4.23. Thus, we will introduce definitions that are dependent on $\mathfrak{a}$, but, to avoid even more notational complexity than is already present, we will not explicitly mark the $\mathfrak{a}$ dependence in these definitions unless it is necessary to do so.

Important Notation: We introduce a very convenient notation. In the sum Equation 4.12, c is an element of $\mathfrak{J}\mathfrak{d}\mathfrak{a}^{-2}$. We will denote by $\mathfrak{c}$ the integral ideal $\mathfrak{d}^{-1}\mathfrak{a}^2(c)$; thus $\mathfrak{c}$ is an integral ideal that is divisible by $\mathfrak{J}$. This notation will be used throughout the rest of the Chapter. Therefore, for example, the $\mathrm{Norm}(c\mathfrak{d}^{-1}\mathfrak{a}^2)$ that features above can be replaced by $\mathrm{Norm}(\mathfrak{c})$. Although the relation between c and $\mathfrak{c}$ depends on $\mathfrak{a}$, we will be holding $\mathfrak{a}$ fixed whenever we use this, as remarked above.

Rather than sum over $\alpha \in \mathfrak{a}^{-1}/\mathfrak{o}_F^*$ and $c \in \mathfrak{J}\mathfrak{d}\mathfrak{a}^{-2}$, we can sum over $\alpha \in \mathfrak{a}^{-1}$ and $c \in (\mathfrak{J}\mathfrak{d}\mathfrak{a}^{-2} - \{0\})/\mathfrak{o}_F^*$; one checks that we have the correct invariance in the summand to carry this out. (Note in particular that, if ϵ is a unit, then $KS_\chi(\epsilon^2 A, B; \epsilon C) = KS_\chi(A, B; C)$.) The sum over c is now converted into a sum over *ideals* $\mathfrak{c}$, which are divisible by $\mathfrak{I}$ and are in the same class as $\mathfrak{d}^{-1}\mathfrak{a}^2$. Since $\mathfrak{a}^2$ must be principal (see the comments preceding Equation 4.9), $\mathfrak{c}$ must be in the same class as $\mathfrak{d}^{-1}$. Define $k(x) = \varphi(|x|)$, where "absolute value" is defined place by place on $F_\infty = \mathbb{R}^d$. We will also write $\mathfrak{a} \sim \mathfrak{b}$ if $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class. We then get:

$$\Sigma_{\mathfrak{a}}(X) = \sum_{\substack{\mathfrak{c}\sim\mathfrak{d}^{-1} \\ \mathfrak{I}|\mathfrak{c}}} \frac{1}{\mathrm{Norm}(\mathfrak{c})} \left(\sum_{\substack{\alpha\in\mathfrak{a}^{-1} \\ \epsilon\in\mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2}} g(\mathrm{Norm}(\alpha\mathfrak{a})/X) k(\frac{\alpha\sqrt{\epsilon\mu}}{c}) KS_\chi(\alpha^2, \epsilon\mu; c) \right)$$

Here observe that the inner sum is dependent only on the ideal class of c, that is, only dependent on $\mathfrak{c}$. One sees that $\alpha \mapsto KS_\chi(\alpha^2; \epsilon\mu, c)$ is invariant under $\alpha \mapsto \alpha + \lambda$, if $\lambda \in \mathfrak{c}\mathfrak{a}^{-1}$; thus, we may split the α sum into congruence classes modulo $c\mathfrak{d}^{-1}\mathfrak{a} = \mathfrak{c}\mathfrak{a}^{-1}$. One obtains:

$$\Sigma_{\mathfrak{a}}(X) = \sum_{\substack{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1} \\ \epsilon\in\mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2}} \frac{1}{\mathrm{Norm}(c)} \sum_{x\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} KS_\chi(x^2; \mu\epsilon, c) \left(\sum_{\alpha\equiv x \bmod (\mathfrak{c}\mathfrak{a}^{-1})} g(\mathrm{Norm}(\alpha\mathfrak{a})/X) k(\alpha\frac{\sqrt{\mu\epsilon}}{c}) \right)$$

Note that on account of the compact support of k, c is restricted to a region $\mathrm{Norm}(c) \sim X$. Now, if f is a function on $F \otimes \mathbb{R}$, a is an element of F and $\mathfrak{p}$ is a fractional ideal, one may evaluate – the "twisted" Poisson summation formula –

$$\sum_{\lambda\in\mathfrak{p}} f(a+\lambda) = \frac{1}{\mathrm{vol}(F_\infty/\mathfrak{p})} \sum_{\nu\in\mathfrak{p}^{-1}\mathfrak{d}^{-1}} \hat{f}(\nu)\psi(-a\nu)$$

where $\hat{f}(\nu) = \int_{F\otimes\mathbb{R}} f(x) e(\mathrm{tr}(x\nu)) dx$.

The previous sum then becomes, applying this summation formula with $\mathfrak{p} = c\mathfrak{d}^{-1}\mathfrak{a} = \mathfrak{a}^{-1}\mathfrak{c}$, and denoting by $\hat{gk}$ the Fourier transform of the function $x \mapsto$

$g(\mathrm{Norm}(x\mathfrak{a})/X)k(\sqrt{\mu\epsilon}x/c)$:

$$\sum_{\substack{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1}\\ \epsilon\in\mathfrak{o}_F^+/(\mathfrak{o}_F^*)^2}} \frac{1}{\mathrm{Norm}(\mathfrak{c})\mathrm{vol}(F_\infty/c\mathfrak{d}^{-1}\mathfrak{a})}\left(\sum_{\nu\in\mathfrak{a}^{-1}} \hat{gk}(\nu c^{-1}) \sum_{x\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} KS_\chi(x^2;\mu\epsilon,c)\psi(-\frac{\nu x}{c})\right) \tag{4.13}$$

Now

$$\hat{gk}(\nu c^{-1}) = \int_{F_\infty} g(\frac{\mathrm{Norm}(x)\mathrm{Norm}(\mathfrak{a})}{X})k(\frac{\sqrt{\mu\epsilon}}{c}x)\psi(x\nu/c)dx \tag{4.14}$$

$$= \mathrm{Norm}(c)\int_{F_\infty} g(\frac{\mathrm{Norm}(x)\mathrm{Norm}(\mathfrak{a}c)}{X})k(\sqrt{\mu\epsilon}x)\psi(x\nu)dx \tag{4.15}$$

Define $\Psi_{c,\epsilon}(\nu)$ as follows:

$$\Psi_{c,\epsilon}(\nu) = \int_{F_\infty} g(\mathrm{Norm}(x)\mathrm{Norm}(\mathfrak{a}c)/X)k(\sqrt{\mu\epsilon}x)\psi(x\nu)dx$$

and define $\mathcal{A}_\mathfrak{a}(\nu;\mathfrak{c},\mu\epsilon) = \sum_{x\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} KS_\chi(x^2,\mu\epsilon,c)\psi(-\frac{\nu x}{c})$, the local sum that occurs in Equation 4.13. This sum is analyzed in the next two sections, and the results relevant to us are summarized in Proposition 12 and Corollary 2; we will continue assuming the results from there.

We substitute into Equation 4.13 the definitions of $\Psi_{c,\epsilon}$, $\mathcal{A}_\mathfrak{a}$ and the fact that $\mathrm{vol}(F_\infty/\mathfrak{o}_F) = D_F^{1/2}$. Then:

$$\Sigma_\mathfrak{a}(X) = \frac{D_F^{1/2}}{\mathrm{Norm}(\mathfrak{a})} \sum_{\substack{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1}\\ \epsilon\in\mathfrak{o}_F^{*;+}/(\mathfrak{o}_F^*)^2}} \left(\sum_{\nu\in\mathfrak{a}^{-1}} \frac{\mathcal{A}_\mathfrak{a}(\nu;\mathfrak{c},\mu\epsilon)}{\mathrm{Norm}(\mathfrak{c})}\Psi_{c,\epsilon}(\nu)\right) \tag{4.16}$$

Fix a norm $||\cdot||$ on F_∞, considered as an $\mathbb{R}$-vector space. Now, the function $\Psi_{c,\epsilon}$ is the Fourier transform of a function of compact support and with reasonable derivatives, so it decays rapidly, uniformly in c for $\mathrm{Norm}(c) \sim X$. That is, it will satisfy a uniform estimate $\Psi_{c,\epsilon}(\nu) \ll X||\nu||^{-M}$. By an argument parallel to that of Chapter 2, one sees that the double sum above is absolutely convergent, and it is possible to truncate the above sum to $||\nu|| \ll X^{\delta}$, for any $\delta > 0$ (but δ fixed as $X \to \infty$.) As in Chapter 2, only certain values of ν will contribute in the limit; namely, those ν such that inner sum of Equation 4.16 does not manifest cancellation.

Expanding $\Psi_{c,\epsilon}(\nu)$, we may write Equation 4.16 as:

$$\Sigma_{\mathfrak{a}}(X) = \frac{D_F^{1/2}}{\text{Norm}(\mathfrak{a})} \sum_{\substack{\nu\in\mathfrak{a}^{-1} \\ \epsilon\in\mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2}} \int_{F\otimes\mathbb{R}} k(x\sqrt{\mu\epsilon})\psi(x\nu) \tag{4.17}$$

$$\left(\sum_{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1}} g(\frac{\text{Norm}(x)\text{Norm}(c\mathfrak{a})}{X}) \frac{\mathcal{A}_{\mathfrak{a}}(\nu;\mathfrak{c},\mu\epsilon)}{\text{Norm}(\mathfrak{c})} \right) dx \tag{4.18}$$

The following Proposition expresses the behavior of the innermost "local sum" above. It may be deduced from Corollary 2 of the next section.

Proposition 11. *Let C be the constant defined in Proposition 12. Recall the definition of the set of "twisted fields" S from Section 4.3.1, and recall that S is identified with a set of classes in $F^*/(F^*)^2$, whence it is meaningful to ask whether an element of F lies in S . Note that $|S| = |\widehat{C_F}(2)|$, the cardinality of the 2-torsion in $\widehat{C_F}$. Fix a norm $||\cdot||$ on F_∞. Fix $\mathfrak{a} \in A$. For $x \in F$, define*

$$\delta_{\mathfrak{a}}(x) = \begin{cases} h_F^{-1}C & x \neq 0, x \in \mathfrak{I}\mathfrak{a}^{-2}, x \in S; \\ \frac{|\widehat{C_F}(2)|}{2} h_F^{-1}C & x = 0; \\ 0 & \text{else} \end{cases}$$

Then, for $\alpha \in \mathbb{R}$, for some $A > 0$, and for all $\epsilon > 0$,

$$\sum_{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1}} g(\frac{\text{Norm}(\mathfrak{c})\alpha}{X}) \frac{\mathcal{A}_{\mathfrak{a}}(\nu;\mathfrak{c},\mu)}{\text{Norm}(\mathfrak{c})} = \delta_{\mathfrak{a}}(\nu^2 - 4\mu)\frac{X}{\alpha} + O_\epsilon((1+||\nu||)^A (X/\alpha)^{1/2+\epsilon})$$

Applying this with $\alpha = D_F \text{Norm}(\mathfrak{a})^{-1}\text{Norm}(x)$, and using the assumption that k is compactly supported, and the fact, remarked after Equation 4.16, that we can truncate the ν-sum to $||\nu|| \ll X^\delta$, we obtain:

$$\Sigma_{\mathfrak{a}}(X) = D_F^{-1/2} X \sum_{\substack{\nu \in \mathfrak{a}^{-1} \\ \epsilon \in \mathfrak{o}_F^{*;+}/(\mathfrak{o}_F^*)^2}} \delta_{\mathfrak{a}}(\nu^2 - 4\mu\epsilon) \int_{F\otimes\mathbb{R}} k(x\sqrt{\mu\epsilon}) \frac{1}{\mathrm{Norm}(x)} \psi(x\nu) dx + O_\epsilon(X^{1/2+\epsilon}) \tag{4.19}$$

Now $\int_{F\otimes\mathbb{R}} k(x\sqrt{\mu\epsilon}) \frac{1}{\mathrm{Norm}(x)} \psi(x\nu) dx = \Delta(\frac{\nu}{\sqrt{\mu\epsilon}})$, where Δ is the Fourier transform of $k(x)/\mathrm{Norm}(x)$. We would also like an arithmetic parameterization of ν such that $\delta_{\mathfrak{a}}(\nu^2 - 4\mu\epsilon) \neq 0$; this is given by the

Lemma 7. *Fix $\mu \in \mathfrak{a}^{-2}$. Let $\mathfrak{S} = \{(x, K),\ K \in S,\ x \in \mathfrak{a}^{-1}\mathfrak{o}_{K,\mathfrak{I}},\ \mathrm{Norm}_{K/F}(x) = \mu\}$, where $\mathfrak{o}_{K,\mathfrak{I}}$ is the order of K defined at the start of Subsection 4.3.3, and let $\mathfrak{S}' = \{\nu \in \mathfrak{a}^{-1} : \delta_{\mathfrak{a}}(\nu^2 - 4\mu) \neq 0\}$. Then there is a map $\mathfrak{S} \to \mathfrak{S}'$, given by $x \mapsto \mathrm{tr}_{K/F}(x)$, which is surjective; the fibre above ν has size 2 unless $\nu^2 = 4\mu$, in which case it has size $|S| = |\widehat{C_F(2)}|$.*

Proof. This is not entirely formal, but straightforward, and we omit the details. □

Equation 4.19 becomes:

$$\lim_{X\to\infty} \frac{\Sigma_{\mathfrak{a}}(X)}{X} = \frac{h_F^{-1} D_F^{-1/2} C}{2} \sum_{K\in S} \sum_{\epsilon \in \mathfrak{o}_F^{*;+}/(\mathfrak{o}_F^*)^2} \left(\sum_{x \in \mathfrak{a}^{-1}\mathfrak{o}_{K,\mathfrak{I}} : \mathrm{Norm}_{K/F}(x) = \mu\epsilon} \Delta(\frac{\mathrm{tr}_{K/F}(x)}{\sqrt{\mu\epsilon}}) \right) \tag{4.20}$$

We transfer this to a statement in terms of h, using the transformation formula Equation 4.8. We need to first fix a notation.

For each $K \in S$, let σ_K be the nontrivial Galois automorphism of K over F. For each place v of F, let τ_v be the corresponding embedding of F into $\mathbb{R}$ and choose an embedding σ_v of K into $\mathbb{R}$ extending τ_v. Then, for $x \in K$, define $\mathbf{log}(x) = (\log(|\sigma_v(x)/\sigma_K\sigma_v(x)|))_v$, an element of $\mathbb{R}^{[F:\mathbb{Q}]}$. This is perhaps not the most obvious choice for the meaning of **log**; the point of this definition is that it is trivial for $x \in F$. Also note that a different choice of σ_v changes the sign of some co-ordinates; this will be irrelevant. Using Equation 4.8 and some manipulation we find that, for $x \in K$

with $\mathrm{Norm}_{K/F}(x) = \mu\epsilon$,

$$\Delta(\frac{\mathrm{tr_{K/F}}(x)}{\sqrt{\mu\epsilon}}) = 2^{[F:\mathbb{Q}]-1}\hat{h}(\frac{1}{2\pi}\mathbf{log}(x)) \tag{4.21}$$

It is convenient (for the eventual purpose of matching trace formulae) to make a further translation of the result. We define $\mathfrak{o}'_{K,\mathfrak{J}}$ to be the elements of $\mathfrak{o}_{K,\mathfrak{J}}$ with norm in $\mathfrak{o}_F^{*,+}$. Note that $\mathfrak{o}_F^* \subset \mathfrak{o}'_{K,\mathfrak{J}}$. Let $X_{K,\mathfrak{a},(\mu)}$ be a set of representatives, modulo the action of $\mathfrak{o}'_{K,\mathfrak{J}}$, for elements $x_0 \in \mathfrak{a}^{-1}\mathfrak{o}_{K,\mathfrak{J}}$ with totally positive norm and so that $(\mathrm{Norm}_{K/F}(x_0)) = (\mu)$, i.e. $\mathrm{Norm}_{K/F}(x_0) = \mu\epsilon$ for some $\epsilon \in \mathfrak{o}_F^{*,+}$.

One now obtains:

$$\lim_{X\to\infty} \frac{\Sigma_{\mathfrak{a}}(X)}{X} = 2^{[F:\mathbb{Q}]-2} h_F^{-1} D_F^{-1/2} C \sum_{K\in S} \left(\sum_{x\in X_{K,\mathfrak{a},(\mu)}} \sum_{\epsilon'\in\mathfrak{o}'_{K,\mathfrak{J}}/\mathfrak{o}_F^*} \hat{h}(\frac{1}{2\pi}\mathbf{log}(x\epsilon')) \right) \tag{4.22}$$

This expression is valid even if $\mathfrak{m}$ is not principal.

We finally sum over ideals $\mathfrak{a}$ according to Equation 4.10 in order to recover $\Sigma(X)$. We therefore set, for $\mathfrak{a} \in A$, $X_{K,\mathfrak{a},\mathfrak{m}}$ to be a set of representatives, modulo the action of $\mathfrak{o}'_{K,\mathfrak{J}}$, for elements $x_0 \in \mathfrak{a}^{-1}\mathfrak{o}_{K,\mathfrak{J}}$ with totally positive norm and generating $\mathfrak{a}^{-2}\mathfrak{m}$. Then

$$L = \lim_{X\to\infty} \frac{\Sigma(X)}{X} = 2^{[F:\mathbb{Q}]-2} h_F^{-1} D_F^{-1/2} C \sum_{K\in S} \sum_{\mathfrak{a}\in A} \left(\sum_{x\in X_{K,\mathfrak{a},\mathfrak{m}}} \sum_{\epsilon'\in\mathfrak{o}'_{K,\mathfrak{J}}/\mathfrak{o}_F^*} \hat{h}(\frac{1}{2\pi}\mathbf{log}(x\epsilon')) \right) \tag{4.23}$$

To make the expression look somewhat nicer, we now introduce the constant:

$$V_{\mathfrak{J}} = 2^{2-[F:\mathbb{Q}]} h_F D_F^{3/2} \zeta(F,2) \mathrm{Norm}(\mathfrak{J}) \prod_{\mathfrak{q}|\mathfrak{J}} (1 + \mathrm{Norm}(\mathfrak{q})^{-1})$$

which is, up to a constant bounded in terms of $[F : \mathbb{Q}]$, the volume of the space $\amalg_{\mathfrak{a}\in A}\mathcal{X}_{\mathfrak{a}}$. (This is with respect to the normalization of measure defined in the Appendix.) Recall also the definition of $C_{\mathfrak{f}}$ (after Remark 8), so that $a_{\mathfrak{f}}(\mathfrak{a}, \alpha) = C_{\mathfrak{f}} \lambda_{\mathfrak{f}}(\mathfrak{a}\alpha\mathfrak{d})$.

Finally, note that (for any $K \in S$) the *absolute* discriminant of $\mathfrak{o}_{K,\mathfrak{J}}$ is given by $\mathrm{Disc}(\mathfrak{o}_{K,\mathfrak{J}}) = D_F^2 \mathrm{Norm}(\mathfrak{J})$; we denote it by $D_{K,\mathfrak{J}}$. Therefore $\sqrt{\mathrm{Norm}(\mathfrak{J})} = \sqrt{D_{K,\mathfrak{J}}} D_F^{-1}$. It is also worth noting that one expects $|C_{\mathfrak{f}}|^2$ to be of size about $\frac{\sqrt{D_F}}{V_{\mathfrak{J}}}$. (This is not a simple consequence of anything stated so far. We state it as a fact so the reader has more feeling for the order of magnitude of the quantities.)

With some manipulation, we obtain the final answer, phrased in terms of Hecke coefficients:

$$\lim_{X \to \infty} \left(\frac{\sum_{\mathfrak{f}} h_{\mathfrak{f}}(\varphi) |C_{\mathfrak{f}}|^2 \overline{\lambda_{\mathfrak{f}}(\mathfrak{m})} \sum_{\mathfrak{q}} g(\mathrm{Norm}(\mathfrak{q})/X) \lambda_{\mathfrak{f}}(\mathfrak{q}^2) + \text{C.S.C}}{\sum_{\mathfrak{q}} g(\mathrm{Norm}(\mathfrak{q})/X)} \right)$$
$$= \frac{\sqrt{D_F}}{V_{\mathfrak{J}}} \sqrt{\frac{D_{K,\mathfrak{J}}}{D_F}} \left(\sum_{K \in S} \sum_{\substack{\mathfrak{a} \in A \\ x \in X_{K,\mathfrak{a},\mathfrak{m}}}} \sum_{\epsilon' \in \mathfrak{o}'_{K,\mathfrak{J}}/\mathfrak{o}_F^*} \hat{h}\left(\frac{1}{2\pi} \mathbf{log}(x\epsilon')\right) \right) \tag{4.24}$$

4.4 Computation of the local sums

The aim of this section is to evaluate the local sums that arose in the previous section. The results of this section were already used in the derivation of the limit formula Equation 4.24.

4.4.1 Translation of local sums to adelic integrals

The local sums that have been encountered in Equation 4.13 will be most conveniently expressed (and evaluated) in terms of adelic integrals; that is the purpose of what follows. This will make clear the multiplicativity, and, more importantly, makes more clear (at least to me) what happens at bad places, such as those dividing 2. Unfortunately, there will be a considerable notational burden in this section; it seems unavoidable.

We will continue to assume that χ is the Grössencharacter associated to the totally real quadratic extension K_0 over F. We use, as before χ_f for the induced character of $(\mathfrak{o}_F/\mathfrak{f}_\chi)^\times$ obtained by restricting χ to elements of $\mathbb{A}_{F,f}^\times$ that are of valuation 0 everywhere; occasionally, when it is clear, we will merely use χ in this case. Note that, since K is totally real, χ is trivial on $F_\infty^\times$ (although we indicate at the point where this is used how it can be removed).

Fix $\mathfrak{a} \in A$, $\nu \in \mathfrak{a}^{-1}$, $\mu \in \mathfrak{a}^{-2}$. c will be an element of $\mathfrak{I}\mathfrak{a}^{-1}$, and we continue with the notational convention introduced at the start of Subsection 4.3.4, namely, $\mathfrak{c}$ denotes the ideal $\mathfrak{d}^{-1}\mathfrak{a}^2(c)$.

We then wish to analyze the sum

$$\mathcal{A}_\mathfrak{a}(\nu;\mathfrak{c},\mu) = \sum_{m\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} KS_\chi(m^2,\mu;c)\psi(-\frac{m\nu}{c}) \tag{4.25}$$

$$= \sum_{x\in(\mathfrak{o}/\mathfrak{c})^\times}\sum_{m\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} \psi(\frac{m^2x+\mu x^{-1}-m\nu}{c})\chi_f(x)$$

$$= \sum_{x\in(\mathfrak{o}/\mathfrak{c})^\times}\sum_{m\in\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{c}} \psi(x^{-1}\frac{m^2+\mu-m\nu}{c})\chi_f(x)$$

The sums may be replaced by integrals over $\hat{\mathfrak{o}}_F$, the completion of $\mathfrak{o}_F$ with respect to ideals; it is the maximal compact subring of $\mathbb{A}_{F,f}$, or, in other words, the product $\prod_{v\text{ finite}}\mathfrak{o}_{F,v}$. The group of units $\hat{\mathfrak{o}}_F^\times$ is the product $\prod_{v\text{ finite}}\mathfrak{o}_{F,v}^\times$. Let $\pi_\mathfrak{a}$ be an idele corresponding to $\mathfrak{a}$. We obtain–*using measures that assign mass* 1 *to the domains of integration, which we denote by* $d^{(1)}$, denoting by $\phi(\mathfrak{c})$ the "Euler phi" function $|(\mathfrak{o}/\mathfrak{c})^\times|$, and considering χ_f as a character on $\hat{\mathfrak{o}}_F^\times \subset \mathbb{A}_F^\times$,

$$\mathcal{A}_\mathfrak{a}(\nu;\mathfrak{c},\mu) = \mathrm{Norm}(\mathfrak{c})\phi(\mathfrak{c})\int_{x\in\hat{\mathfrak{o}}_F^\times}\int_{m\in\pi_\mathfrak{a}^{-1}\hat{\mathfrak{o}}_F}\psi(x^{-1}\frac{m^2+\mu-m\nu}{c})\chi_f(x)d^{(1)}xd^{(1)}m$$

We may, since χ is trivial on $F_\infty^\times \subset \mathbb{A}_F^\times$, harmlessly insert a factor $\chi_f(c)$. (Note this insertion would in fact be valid *in general*, since the definition of the Kloosterman sum, in the Equation 4.50 derived at the end of this Chapter, involves a factor $\chi_\infty(c)$.) For $A, B \in \mathbb{A}_{F,f}^\times$, we write $(A) = (B)$ if the associated fractional ideals are equal, that is, $AB^{-1} \in \hat{\mathfrak{o}}_F^\times$. With this notation:

$$\mathcal{A}_\mathfrak{a}(\nu; \mathfrak{c}, \mu) = \mathrm{Norm}(\mathfrak{c})\phi(\mathfrak{c}) \int_{z \in \mathbb{A}_{F,f}^\times, (z)=(c)} \int_{m \in \pi_\mathfrak{a}^{-1}\widehat{\mathfrak{o}_F}} \psi(z^{-1}(m^2 - m\nu + \mu)\chi(z) d^{(1)}z d^{(1)}m$$

Since χ is quadratic, $\chi(\pi_\mathfrak{a}^2) = 1$, so:

$$\mathcal{A}_\mathfrak{a}(\nu; \mathfrak{c}, \mu) = \mathrm{Norm}(\mathfrak{c})\phi(\mathfrak{c}) \int_{(z)=(c\pi_\mathfrak{a}^2)} \int_{m \in \widehat{\mathfrak{o}_F}} \psi(z^{-1}(m^2 - m\pi_\mathfrak{a}\nu + \pi_\mathfrak{a}^2\mu)\chi(z) d^{(1)}z d^{(1)}m$$

This motivates the definition of the adelic integral.

Definition 5. *Let $\mathfrak{c}'$ be any ideal divisible by $\mathfrak{d}$, and $\nu', \mu' \in \mathbb{A}_F$. We define $I(\nu'; \mathfrak{c}'; \mu')$ to be this integral:*

$$\mathrm{Norm}(\mathfrak{c}'\mathfrak{d}^{-1})\phi(\mathfrak{c}'\mathfrak{d}^{-1}) \int_{(z)=\mathfrak{c}'} \int_{m \in \widehat{\mathfrak{o}_F}} \psi(z^{-1}(m^2 - m\nu' + \mu'))\chi(z) d^{(1)}z d^{(1)}m \qquad (4.26)$$

where the measures are Haar measures that assign mass 1 *to the domain of integration. Note that ν' and μ' do not have to be elements of F. Also note that we will only ever apply this where the ideal $\mathfrak{c}'$ "contains" the ramification of both χ and ψ; we do not need to make this precise for now.*

Therefore, the sum $\mathcal{A}_\mathfrak{a}(\nu; \mathfrak{c}, \mu)$ defined in Equation 4.25 is related by $\mathcal{A}_\mathfrak{a}(\nu; \mathfrak{c}, \mu) = I(\nu\pi_\mathfrak{a}; \mathfrak{c}\mathfrak{d}, \mu\pi_\mathfrak{a}^2)$ – where, as throughout, we are using the convention that $\mathfrak{c} = \mathfrak{d}^{-1}\mathfrak{a}^2(c)$. Note that it is invariant under $(\nu, \mu) \to (\nu u, \mu u^2)$ for $u \in \widehat{\mathfrak{o}_F}^\times$.

4.4.2 Zeta functions

To understand the average behavior of $\mathcal{A}_\mathfrak{a}(\nu; \mathfrak{c}, \mu)$ with respect to $\mathfrak{c}$, we introduce a "zeta series", or, at least one for the corresponding integral I. It is defined in Definition 6, and the relation to I is clarified in Lemma 8; the main result is Proposition 12, and we finally translate back to the local sum $\mathcal{A}_{\mathfrak{a}}(\nu;\mathfrak{c},\mu)$ in Corollary 2. The bulk of the proof of the Proposition is contained in the next Subsection.

We will fix $\nu',\mu' \in \mathbb{A}_F^\times$ so that there exist elements $\nu_0,\mu_0 \in F^\times$ and $\iota \in \mathbb{A}_F^\times$ so that $\nu' = \nu_0\iota, \mu' = \mu_0\iota^2$. This is, in particular, the case needed to apply the results to $\mathcal{A}_{\mathfrak{a}}(\nu;\mathfrak{c},\mu) = I(\nu\pi_{\mathfrak{a}};\mathfrak{c}\mathfrak{d},\mu\pi_{\mathfrak{a}}^2)$. In particular, one can speak of $F(\sqrt{\nu'^2-4\mu'})$: by it we mean the quadratic field $F(\sqrt{\nu_0^2-4\mu_0})$, or equivalently the unique quadratic extension of F which, for every place v of F, agrees with $F_v(\sqrt{\nu'^2_v-4\mu'_v})$. (As usual, by ${\nu'}_v$ we mean the component at v of the adele ν'; it is thus an element of F_v.) For a finite place v, by $v(\nu'^2-4\mu')$ we mean the valuation $v(\nu'^2_v-4\mu'_v)$. We also note that if $\nu'^2-4\mu'$ is zero at one place it is zero everywhere.

We now introduce the zeta series. Lemma 8 and Equation 4.28 will make concrete the relation between I and this zeta function. We also use ν',μ' for the arguments of I to continue to emphasize that they need not lie in the field F, but only in $\mathbb{A}_F$.

It is also convenient to make this definition of the zeta-function for an arbitrary Grössencharacter ω. It will eventually be applied for ω an unramified twist of the original χ_{K_0}.

For each finite place v, fix Haar measure dx on F_v so that the measure of $\mathfrak{o}_v$ is 1, and define measure on $F_v^\times$ so that the measure of $\mathfrak{o}_v^\times$ is $(1-q_v^{-1})$. With these normalizations:

Definition 6. *Let ω be a Grossencharacter of $\mathbb{A}_F^\times/F^\times$ (not necessarily quadratic). Let ω_v,ψ_v be the restrictions of the Grossencharacter ω and the additive character ψ to F_v. We define the local factor $Z_v(\omega_v,s)$ via:*

$$Z_v(\omega_v,s) = \sum_{r\geq v(\mathfrak{I})} Z_{v,r}q_v^{r(1-s)} \tag{4.27}$$

where for $r \geq 0$ integral,

$$Z_{v,r} = \begin{cases} \int_{v(x)=r+v(\mathfrak{d})} \int_{m \in \mathfrak{o}_{F,v}} \psi_v(x^{-1}(m^2 - m\nu' + \mu'))\omega_v(x)dxdm, \; r \geq 1 \\ \frac{1}{(1-q_v^{-1})} \int_{v(x)=v(\mathfrak{d})} \int_{m \in \mathfrak{o}_{F,v}} \psi_v(x^{-1}(m^2 - m\nu' + \mu'))\omega_v(x)dxdm, \; r = 0 \end{cases}$$

Finally, we define:

$$Z(\omega, s) = \prod_{v \text{ finite}} Z_v(\omega_v, s)$$

Note that $Z_{v,0} = 0$ unless ω is unramified; in that case, it equals $\omega(\mathfrak{d}_v)$.

The following Lemma is now an easy consequence of the definitions.

Lemma 8. *With this definition,*

$$\sum_{\mathfrak{I}|\mathfrak{c}} I(\nu'; \mathfrak{c}\mathfrak{d}, \mu')\mathrm{Norm}(\mathfrak{c})^{-1-s} = Z(\chi_{K_0}, s)$$

We are interested (see Proposition 11) in the behavior of $\mathcal{A}_{\mathfrak{a}}(\nu; \mathfrak{c}, \mu) = I(\nu\pi_{\mathfrak{a}}; \mathfrak{c}\mathfrak{d}, \mu\pi_{\mathfrak{a}}^2)$ when $\mathfrak{c} \sim \mathfrak{d}^{-1}$. Therefore, we use characters of C_F to "pick out" the ideal class of the different:

$$\begin{aligned} \sum_{\mathfrak{I}|\mathfrak{c}, \mathfrak{c}\sim\mathfrak{d}^{-1}} \frac{I(\nu'; \mathfrak{c}\mathfrak{d}, \mu')}{\mathrm{Norm}(\mathfrak{c})} \mathrm{Norm}(\mathfrak{c})^{-s} &= \frac{1}{h_F} \sum_{\chi' \in \widehat{C_F}} \chi'(\mathfrak{d}) \left(\sum_{\mathfrak{I}|\mathfrak{c}} \frac{I(\nu'; \mathfrak{d}\mathfrak{c}, \mu')}{\mathrm{Norm}(\mathfrak{c})} \mathrm{Norm}(\mathfrak{c})^{-s} \chi'(\mathfrak{c}) \right) \\ &= \frac{1}{h_F} \sum_{\chi' \in \widehat{C_F}} Z(\chi\chi', s) \qquad (4.28) \end{aligned}$$

We summarize the results we will prove about $Z(\omega, s)$ in the following Proposition. Note it is important to separate the case $\nu'^2 - 4\mu' = 0$. The results are true in a good deal more generality than that in which we state them; we restrict ourself considerably for ease of proof.

Proposition 12. *Let ω be an unramified twist of χ_{K_0} (such as occurs in Equation 4.28), so $\omega = \chi_{K_0}\chi'$ for $\chi' \in \widehat{C_F}$.*

The function $Z(\omega, s)$ extends to a meromorphic function on the complex plane. Let $\chi_{\nu'^2-4\mu'}$ be the Grössencharacter of F that is associated to the quadratic extension $F(\sqrt{\nu'^2 - 4\mu'})$.

Suppose $\nu'^2 - 4\mu' \neq 0$. Then $Z(\omega, s)$ is analytic for $\Re(s) > 1/2$, unless $\omega\chi_{\nu'^2-4\mu'}$ is trivial and $\mathfrak{I}|\nu'^2 - 4\mu'$ (that is, for each finite valuation, $v(\nu'^2 - 4\mu') \geq v(\mathfrak{I})$). In that case, it has just one pole for $\Re(s) > 1/2$, at $s = 1$, and that is a simple pole with residue:

$$\mathrm{Res}_{s=1} Z(\omega, s) = C \equiv \frac{\mathrm{res}_{s=1}\zeta(F, s)}{\zeta(F, 2)\sqrt{\mathrm{Norm}(\mathfrak{I})}\prod_{\mathfrak{q}|\mathfrak{I}}(1 + \mathrm{Norm}(\mathfrak{q})^{-1})} \tag{4.29}$$

Suppose $\nu'^2 - 4\mu' = 0$. Then $Z(\omega, s)$ is analytic for $\Re(s) > 1/2$, unless ω is quadratic. In that case, it has just one pole for $\Re(s) > 1/2$, which is at $s = 1$, and it is a simple pole with residue $\frac{1}{2}C$, where C is as above.

In all cases, the function $Z(s)$ has slow growth along vertical strips for $\Re(s) > 1/2$: if $\sigma > 1/2$, one has an estimate: $|Z(\sigma + it)| \ll (1 + ||\nu'||_{\mathbb{A}_F})^{A(\sigma)}(1 + |t|)^{B(\sigma)}$, for some appropriate $A(\sigma), B(\sigma)$.

Proof. (Borrowed from upcoming section): contained in Corollary 3 and Equation 4.38. □

Using Equation 4.28 and the relation between $\mathcal{A}_{\mathfrak{a}}(\nu; \mathfrak{c}, \mu)$ and the adelic integral I, we obtain:

Corollary 2. *The sum $\sum_{\mathfrak{I}|\mathfrak{c},\mathfrak{c}\sim\mathfrak{d}^{-1}} \mathcal{A}_{\mathfrak{a}}(\nu; \mathfrak{c}, \mu)\mathrm{Norm}(\mathfrak{c})^{-1-s}$ is meromorphic, with at most a pole at $s = 1$ in the half-plane $\Re(s) > 1/2$, and it has a pole at $s = 1$ only if $\nu^2 - 4\mu \in S$ and $\nu^2 - 4\mu \in \mathfrak{I}\mathfrak{a}^{-2}$. (Here S is as in Subsection 4.3.1.) If C is the constant of Equation 4.29, the residue at $s = 1$ is given by:*

$$\begin{cases} h_F^{-1}C, & \nu^2 - 4\mu \neq 0 \\ \frac{|\widehat{C_F}(2)|}{2} h_F^{-1}C, & \nu^2 - 4\mu = 0 \end{cases}$$

Finally, it grows slowly along vertical lines, with the implicit constant a polynomial in ν*, that is, fixing a norm* $||\cdot||$ *on* F_∞*, the value of this zeta function at* $\sigma + it$ *is bounded by* $(1 + ||\nu||)^{A(\sigma)}(1 + |t|)^{B(\sigma)}$*.*

In particular, from the Proposition and its Corollary, one can deduce Proposition 11, by the usual techniques of expressing the sum as the integral of a Mellin transform against a zeta series, and then shifting contours.

4.4.3 Local factors of Zeta functions

We will evaluate the local factors that occur in the "zeta function" of Proposition 12. There is some overlap with Subsection 2.3.3 of Chapter 2. Our treatment here is better adapted to bad places, however.

We continue with notation from the previous section. Let v be a place of our base field F, $\mathfrak{o}_v$ the ring of integers in F_v. Let π_v be a local uniformizer. ν'_v and μ'_v denote the components of the adeles ν', μ' at v. Recall that ω is an unramified twist of $\chi = \chi_{K_0}$. The characters ω and χ restrict to characters ω_v, χ_v of F_v. Haar measure is fixed so as to assign to $\mathfrak{o}_v$ and $\mathfrak{o}_v^\times$ the masses 1 and $1 - q_v^{-1}$ respectively. Note that since ω and χ are twists of each other by an unramified character, they ramify at the same places with the same conductors; therefore we can speak of either being ramified interchangeably. Let $\mathfrak{f}_\chi$ be the conductor of χ; note that $\mathfrak{f}_\chi = \mathfrak{D}_{K/F}$; It is also the conductor of ω, but we will only ever use $\mathfrak{f}_\chi$, so as to emphasize that it is independent of choice of ω. All integrals in this section will be local - over F_v or a subset thereof.

Recall (Subsection 4.3.3) that $\mathfrak{I} = \mathfrak{f}_\chi \mathfrak{f}^2$; in particular, the parities of $v(\mathfrak{I})$ and $v(\mathfrak{f}_\chi)$ are the same for any finite place v.

Finally, we set $f_v = v(\mathfrak{I})$ and $f_{\chi,v} = v(\mathfrak{f}_\chi)$; thus $f_v = f_{\chi,v} + 2v(\mathfrak{f})$. (The proliferation of fs is unfortunate; they all refer to conductors of one type or the other.)

The local factor of the zeta function that we are interested in has been defined in Equation 4.27.

Evaluation at good places

We begin by evaluating at "good" places. Recall that $\chi_{\nu'^2-4\mu'}$ is the quadratic character associated with the extension $F(\sqrt{\nu'^2-4\mu'})$ – by definition, we take it to be trivial if ${\nu'}^2-4\mu'$ is a square.

Lemma 9. *For almost all places v,*

$$Z_v(\omega,s)=\begin{cases}1+\omega_v(\pi_v)\chi_{\nu'^2-4\mu'}(\pi_v)q_v^{-s}=\frac{L_v(s,\omega\chi_{\nu'^2-4\mu'})}{L_v(2s,\omega_v^2)},\ {\nu'}^2-4\mu'\neq 0\\ \frac{L_v(2s-1,\omega_v^2)}{L_v(2s,\omega_v^2)},\ {\nu'}^2-4\mu'=0\end{cases}\tag{4.30}$$

Here L_v is the factor at v of the Hecke-Tate L-function.

Proof. This follows as in Chapter 2, Subsection 2.3.3. □

As a Corollary we obtain part of Proposition 12:

Corollary 3. *$Z(\omega,s)$ extends to a meromorphic function of s, analytic in $\Re(s)>1/2$. It does not have a pole at $s=1$ unless either:*

1. *${\nu'}^2-4\mu'\neq 0$ and $\omega\chi_{\nu'^2-4\mu'}$ is the trivial character, or:*

2. *${\nu'}^2-4\mu'=0$ and ω is quadratic.*

Finally, it has a growth bound of the form $|Z(\sigma+it,\omega)|\ll(1+||\nu||)^{A(\sigma)}(1+|t|)^{B(\sigma)}$.

Proof. (Sketch) The conclusion is clearly true for the product over almost all v, and one must check, as in Chapter 2, that the finite number of bad factors do not jeopardize the conclusion. The only difficulty is at primes above 2, and even then it is easily verified by the methods developed in the rest of this Subsection. □

Local factors at bad places

After Corollary 3, we are only interested in evaluating the residue at $s = 1$ of $Z(\omega, s)$. Therefore, we now restrict ourself to the case where $\omega = \chi_{\nu'^2 - 4\mu'}$ or where $\nu'^2 - 4\mu' = 0$ and ω is quadratic. It will suffice to evaluate $Z_v(1)$ for all v, including "bad" places. The assumption on ω shows that $\omega_v(m^2 - m\nu'_v + \mu'_v) = 1$, because (if $\nu'^2 - 4\mu' \neq 0$) $m^2 - m\nu'_v + \mu'_v$ is a norm from the local extension $F_v(\sqrt{\nu'^2 - 4\mu'})$ and (if $\nu'^2 - 4\mu' = 0$) ω is quadratic and $m^2 - m\nu'_v + \mu'_v$ is a square. In both cases ω is quadratic so $\omega(x^{-1}) = \omega(x)$. Since we are working from the outset only with ω that are unramified twists of χ, it follows that ω must in all cases be trivial or associated to a quadratic extension $K' \in S$. Denoting $Q_v(m) = m^2 - m\nu'_v + \mu'_v$, the local factors become, for $r \geq 1$,

$$Z_{v,r} = \int_{v(x) = -r - v(\mathfrak{d}_v)} \int_{m \in \mathfrak{o}_v} \psi_v(xQ_v(m))\omega_v(xQ_v(m))d^\times x dm \tag{4.31}$$

This integral, in particular, can be computed purely from a knowledge of the measures, for each $s \in \mathbb{Z}$:

$$M_s = \text{Measure}\{m \in \mathfrak{o}_v : v(Q_v(m)) = s\} \tag{4.32}$$

We set also $M_{\geq s} = \sum_{t \geq s} M_t$. The strategy of the evaluation is now to evaluate M_s or equivalently $M_{\geq s}$, for each $s \in \mathbb{Z}$.

Lemma 10. *Let ψ_v be a character of F_v trivial on $\mathfrak{o}_v$ and let $\mathfrak{d}_v$ be the local different, so that ψ_v is trivial on $\mathfrak{d}_v^{-1}$ but on no larger ideal. Then $\int_{v(x)=r} \psi_v(x)d^\times x$ equals $(1 - q^{-1})$ for $r \geq -v(\mathfrak{d}_v)$ and $-q^{-1}$ for $r = -v(\mathfrak{d}_v) - 1$.*

Lemma 11. *Let ψ_v be as above and let ω_v be a character of $F_v^\times$ with conductor s, meaning that it is trivial on $1+(\pi_v)^s$ but not on $1+(\pi_v)^{s-1}$. Then $\int_{v(x)=r} \psi_v(x)\omega_v(x)d^\times x$ vanishes unless $r + s + v(\mathfrak{d}_v) = 0$.*

Proof. This is standard; see, for example, explicit computation of ϵ-factors as in Tate's thesis. $\square$

With ω_v, ψ_v as in Lemma 11, we denote the value of $\int_{v(x)=-s-v(\mathfrak{d}_v)} \psi_v(x)\omega_v(x)d^\times x$ by $g(\omega_v, \psi_v)$. It has absolute value $q^{s/2}$.

Definition 7. *Suppose ω_v is ramified at v. We denote by $\epsilon(\omega_v, \psi_v)$ the "sign" $\frac{g(\omega_v,\psi_v)}{|g(\omega_v,\psi_v)|}$.*

If ψ_v is ramified at v, i.e. $v(\mathfrak{d}) \neq 0$, but ω_v is not, we let $\epsilon(\omega_v, \psi_v) = \omega_v(\mathfrak{d}_v)$. Finally, if neither ψ_v or ω_v are ramified, we set $\epsilon(\omega_v, \psi_v) = 1$.

Lemma 12. *The product $\prod_{v \,\text{finite}} \epsilon(\omega_v, \psi_v)$ equals 1.*

Proof. ω is, as remarked, associated to a quadratic field $K' \in S$. The Lemma follows by computing the root numbers for the L-functions $L(K', s)$ and $L(F, s)$; the ϵ_v are essentially the local root numbers for their quotient. Standard computations and the fact that K' is unramified at ∞, that is to say, totally real, give the result. $\square$

It follows from the definitions that, if we set $\omega_v(\mathfrak{d}_v) = 0$ when ω_v is ramified:

$$Z_v(1) = \begin{cases} \sum_{r \geq v(\mathfrak{I})} Z_{v,r}, & v(\mathfrak{I}) \geq 1 \\ \omega(\mathfrak{d}_v) + \sum_{r \geq 1} Z_{v,r}, & v(\mathfrak{I}) = 0 \end{cases}$$

Let $k \geq 1$ be an integer. We see that, with M_s as in Equation 4.32:

$$\sum_{r \geq k} Z_{v,r} = \sum_{\substack{t \in \mathbb{Z} \\ r \leq -k-v(\mathfrak{d}_v)}} M_t \int_{v(x)=r+t} \psi_v(x)\omega_v(x)d^\times(x) \tag{4.33}$$

$$= \sum_{s \in \mathbb{Z}} \int_{v(x)=s} \psi_v(x)\omega_v(x) \sum_{t \geq s+k+v(\mathfrak{d}_v)} M_t \tag{4.34}$$

Recall the definitions $v(\mathfrak{I}) = f_v$, $v(\mathfrak{f}_\chi) = f_{\chi,v}$. We distinguish now three cases:

1. If ω_v is ramified with conductor $\mathfrak{f}_\chi$, the only contribution to the above integral

is when $s + f_{\chi,v} + v(\mathfrak{d}_v) = 0$. It follows that, by Lemma 11,

$$Z_v(1) = g(\omega_v, \psi_v) \sum_{t \geq f_v - f_{\chi,v}} M_t = g(\omega_v, \psi_v) M_{\geq f_v - f_{\chi,v}} \tag{4.35}$$

2. Suppose ω_v is unramified and nontrivial. Then by Lemma 10,

$$Z_v(1) = \begin{cases} (-1)^{v(\mathfrak{d}_v)} \left(q_v^{-1} M_{\geq f_v - 1} + (1 - q_v^{-1}) \sum_{s \geq f_v} (-1)^{s - f_v} M_{\geq s} \right), & f_v \geq 1 \\ (-1)^{v(\mathfrak{d}_v)} \left(1 + q_v^{-1} M_{\geq 0} + (1 - q_v^{-1}) \sum_{s \geq 1} M_{\geq s} \right), & f_v = 0 \end{cases} \tag{4.36}$$

3. If ω_v is trivial, then

$$\begin{cases} Z_v(1) = -q^{-1} M_{\geq f_v - 1} + (1 - q^{-1}) \sum_{s \geq f_v} M_{\geq s}, & f_v \geq 1 \\ Z_v(1) = 1 - q^{-1} M_{\geq 0} + (1 - q^{-1}) \sum_{s \geq 1} M_{\geq s} & f_v = 0 \end{cases} \tag{4.37}$$

4.4.4 Evaluation of measures and of zeta functions

Let, as before, v be a finite place of F, and Q_v as defined prior to Equation 4.31. The aim of this section is to compute the measures $M_s = \text{Measure}\{m \in \mathfrak{o}_v : v(Q_v(m)) = s\}$, or equivalently $M_{\geq s} = \sum_{t \geq s} M_t$, for each t; this will then complete our evaluation of local factor $Z_v(1)$ by the formulas 4.35, 4.36 and 4.37 of the last section. Again, the answer depends on the splitting type of the quadratic form $Q_v(m)$. We will list the evaluation of the measures and also give, in each case, the parity of the constants $f_{\chi,v}$, f_v and $v(\nu'^2 - 4\mu')$; it will then be easy to deduce $Z_v(1)$. (Recall the definitions of $f_{\chi,v}$ and f_v from Subsection 4.4.3.)

If ν' is (locally) divisible by 2, we can complete the square: $m^2 - m\nu'_v + \mu'_v = (m - \nu'_v/2)^2 + (\mu'_v - {\nu'_v}^2/4)$. Therefore $Q_v(m)$ is equivalent to the form $q(x) = x^2 - \beta$, with $\beta = {\nu'_v}^2 - \mu'_v/4$. If the residue characteristic is 2, we must proceed slightly differently.

1. Non-split case, residue characteristic $\neq 2$: By "non-split" we mean that $\beta = {\nu'_v}^2 - {\mu'}_v/4$ is not a square; equivalently, the local extension $F_v(\sqrt{{\nu'_v}^2 - 4\mu'_v})$ is not split. Then, if $v(x) < v(\beta)/2$, then $v(q(x)) = 2v(x)$; if $v(x) \geq v(\beta)/2$, then $v(q(x)) = v(\beta)$. Also $v(\beta) = v(\nu'^2 - 4\mu')$, as the residue characteristic is not 2. Thus $M_{\geq s} = q_v^{-\lceil s/2 \rceil}$ for $s \leq v(\nu'^2 - 4\mu')$, and $M_{\geq s} = 0$ for $s > v(\nu'^2 - 4\mu')$.

   We divide into two further subcases:

   (a) Ramified: In this case $Z_v(1)$ is, by Equation 4.35, $g(\omega_v, \psi_v) M_{\geq f - \mathfrak{f}_\chi}$. In this case $f_{\chi,v} = 1$, $v(\nu'^2 - 4\mu')$ is odd, and $f_v = v(\mathfrak{I})$ is odd as $f_v \equiv f_{\chi,v} \bmod 2$.

   (b) Nonramified: In this case $Z_v(1)$ is given by Equation 4.36; $f_{\chi,v} = 0$, and both f_v and $v(\nu'^2 - 4\mu')$ are even.

2. Nonsplit case, residue characteristic $= 2$:

   Write $Q_v(m) = (m - \beta_1)(m - \beta_2)$. Here β_1, β_2 are conjugate elements of the quadratic extension defined by Q_v, and $(\beta_1 - \beta_2)^2 = {\nu'}_v^2 - 4\mu'_v$. Also $v(Q(m)) = 2v(m - \beta_1)$, for $m \in F_v$.

   Let K_Q be the quadratic extension of F_v defined by Q_v, that is, the field obtained by adjoining β_1. We denote $\mathfrak{o}_v$ by $\mathfrak{o}_{F,v}$ so as to clearly distinguish between F_v and K_Q. Extend the valuation to K_Q; it may now take non-integral values.

   Let $\pi_{F,v}$ and π_K be uniformizers for F_v and K_Q respectively. We can choose $x \in \mathfrak{o}_{K_Q}$ so that $\mathfrak{o}_{K_Q} = \mathfrak{o}_{F,v} \oplus x\mathfrak{o}_{F,v}$. The local discriminant is then given by $(x - x^\sigma)^2$, where σ is the Galois automorphism. If K_Q is unramified, then we can take $\pi_K = \pi_{F,v}$, and $\mathfrak{o}_{K_Q}/\pi_K^l \mathfrak{o}_{K_Q} = \mathfrak{o}_{F,v}/\pi_{F,v}^l \mathfrak{o}_{F,v} \oplus x\mathfrak{o}_{F,v}/\pi_{F,v}^l \mathfrak{o}_{F,v}$.

   If K_Q is ramified over F_v, then we still have $\mathfrak{o}_{K_Q}/\pi_K^{2l} \mathfrak{o}_{K_Q} = \mathfrak{o}_{F,v}/\pi_{F,v}^l \mathfrak{o}_{F,v} \oplus x\mathfrak{o}_{F,v}/\pi_{F,v}^l \mathfrak{o}_{F,v}$. If an element of $\mathfrak{o}_{K_Q}$ is congruent to an element of $\mathfrak{o}_{F,v}$ mod π_K^{2l}, it is automatically so modulo π_K^{2l+1}; this is essentially a consequence of the isomorphism $\mathfrak{o}_{F,v}/\pi_{F,v}\mathfrak{o}_{F,v} \to \mathfrak{o}_{K_Q}/\pi_K \mathfrak{o}_{K_Q}$.

There is a maximal l for which β_1 is congruent to an element of $\mathfrak{o}_{F,v}$ modulo $(\pi_F)^l \mathfrak{o}_{K_Q}$. It is given by $v(\beta_1 - \beta_2) - f_{\chi,v}/2$ (this one can check by computing the discriminant of the $\mathfrak{o}_{K,Q}$-subring generated by $\mathfrak{o}_{F,v}$ together with β_1).

From this, one deduces that $M_{\geq s} = q_v^{-\lceil s/2 \rceil}$ for $s \leq v(\nu'^2 - 4\mu') - f_{\chi,v}$, and $M_{\geq s} = 0$ otherwise.

3. Split case: In this case, $Q_v(m)$ factors as $(m - \beta_1)(m - \beta_2)$ with $\beta_1, \beta_2 \in \mathfrak{o}_{F,v}$, and $v(\nu'^2 - 4\mu') = 2v(\beta_1 - \beta_2)$. Also $f_{\chi,v} = 0$ and f_v is even.

   We see that $M_{\geq s} = q^{-\lceil s/2 \rceil}$ if $s \leq v(\nu'^2 - 4\mu')$, and $M_{\geq s} = 2q^{-(s - v(\nu'^2 - 4\mu')/2)}$ if $s > v(\nu'^2 - 4\mu') = 2v(\beta_1 - \beta_2)$.

Now, we can compute $Z_v(1)$ case-by-case. We obtain:

Proposition 13. *$Z_v(1)$ vanishes unless $v(\nu'^2 - 4\mu') \geq v(\mathfrak{I})$. In that case,*

$$Z_v(1) = \begin{cases} 1 + q_v^{-1}, & f_v = v(\mathfrak{I}) = 0 \\ \epsilon(\omega_v, \psi_v) q_v^{-f_v/2}, & f_v > 0 \end{cases}$$

Now taking into account Lemma 9, Lemma 12 and the previous Proposition, we prove the last part of Proposition 12, namely:

Corollary 4. *Let ω be an unramified twist of $\chi = \chi_{K_0}$. Suppose that $\nu'^2 - 4\mu' \neq 0$ and $\omega = \chi_{\nu'^2 - 4\mu'}$, or $\nu'^2 - 4\mu' = 0$ and ω is quadratic. We say that $\mathfrak{I} | \nu'^2 - 4\mu'$ if $v(\nu'^2 - 4\mu') \geq v(\mathfrak{I})$ for all finite places v. Let $C = \frac{\mathrm{Res}_{s=1}\zeta_F(s)}{\zeta_F(2)} \frac{1}{\sqrt{\mathrm{Norm}(\mathfrak{I})} \prod_{v|\mathfrak{I}} (1 + q_v^{-1})}$. Then:*

$$\mathrm{Res}_{s=1} Z(\omega, s) = \begin{cases} C, \nu'^2 - 4\mu' \neq 0, & \mathfrak{I} | (\nu'^2 - 4\mu') \\ \frac{1}{2} C, & \nu'^2 - 4\mu' = 0 \\ 0, & \text{else} \end{cases} \tag{4.38}$$

4.5 Expected answer

Here we sketch the *expected* answer, that is, the answer we expect when we assume that dihedral forms arise from Grossencharacters of quadratic field extensions K. As in Chapter 2, the matching of this expected answer with what we derived (in Equation 4.23 or Equation 4.24) leads formally to the classification of dihedral forms.

One expects:

Expectation 1. *Automorphic forms π on $\mathrm{GL}(2)$ over F, such that $L(s, \pi, \mathrm{Sym}^2)$ has a pole, correspond to Grössencharacters ω of quadratic K/F so that the restriction of ω to $\mathbb{A}_F$ is trivial; the correspondence is specified by the matching of L-functions. We will denote by $\pi(\omega)$ the $\mathrm{GL}(2)$-form on F associated to the K-Grössencharacter ω. The conductor of $\pi(\omega)$ is given by $\mathfrak{D}_{K/F}\mathrm{Norm}(\mathfrak{f}_\omega)$, where $\mathfrak{f}_\omega$ is the conductor of ω.*

In particular, if $\mathfrak{J}$ is a prime ideal of F prime to any ramification of K or ω, then the $\mathfrak{J}$th Hecke coefficient of $\pi(\omega)$ is given by:

$$\lambda_{\mathfrak{J}}(\pi(\omega)) = \sum_{\mathfrak{B}:\mathrm{Norm}(\mathfrak{B})=\mathfrak{J}} \omega(\mathfrak{B}) \tag{4.39}$$

As before, we have fixed a totally real quadratic extension K of F, and an integral ideal $\mathfrak{J} = \mathfrak{D}_{K/F}\mathfrak{f}^2$ of F. It determines an order $\mathfrak{o}_{K,\mathfrak{J}}$ of K by $\mathfrak{o}_F + \mathfrak{f}\mathfrak{o}_K$; this is a subring of $\mathfrak{o}_K$. For each finite valuation v of K, we set $U_{\mathfrak{J},v}$ to be the units (elements of valuation 0) in the closure of $\mathfrak{o}_{K,\mathfrak{J}}$ in K_v. We set $U_{\mathfrak{J}} = \prod_v U_{\mathfrak{J},v}$; it is an open compact subgroup of $\mathbb{A}_{K,f}^\times$ (the ring of finite ideles of K.) Since $\mathfrak{J}$ is fixed, we wil often refer to $U_{\mathfrak{J}}$ simply as U.

One sees that:

Lemma 13. *Let ω be a character of $\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times$. Then the conductor of ω is the extension of an F-ideal; and it has conductor dividing $(\mathfrak{f})_K$ (the extension of $\mathfrak{f}$ to K) if and only if it is trivial on $U_{\mathfrak{J}}$. In particular, one expects π such that $L(s, \pi, \mathrm{Sym}^2)$*

has a pole and Conductor$(\pi)|\mathfrak{I}$ *to correspond to characters of* $\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times U_{\mathfrak{I}}$.

We are therefore interested in making explicit the structure of $\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times U$. It is convenient, as it turns out, to work out the structure of $\mathbb{A}_K^\times/K^\times F_\infty^\times U$ and then "take the quotient by $\mathbb{A}_F^\times$ at the end." The notation is, unfortunately, rather unwieldy, and we introduce it all now just for a convenient reference.

Notation: Let Λ_U be the set of U-units of K, that is to say, elements of K that are in U at every finite place; equivalently, this is the set of units in the order $\mathfrak{o}_{K,\mathfrak{I}}$. We also define Λ'_U to be the intersection $K \cap (U \cdot \mathbb{A}_F^\times)$; equivalently, this is the set of elements $\lambda \in K$ that generate an $\mathfrak{o}_{K,\mathfrak{I}}$-ideal that is the extension of an ideal of F.

We also introduce notation for the class groups: $C_{K,\mathfrak{I}} = \mathbb{A}_K^\times/(K^\times K_\infty^\times U)$ will be the class group of the order $\mathfrak{o}_{K,\mathfrak{I}}$ (although we never need the interpretation by $\mathfrak{o}_{K,\mathfrak{I}}$-ideals.) Define $C_{K,\mathfrak{I};F}$ to be the relative class group : the quotient of $C_{K,\mathfrak{I}}$ by the image of $\mathbb{A}_F$.

Let Pl_∞ be the set of infinite places of F. Let σ_K be the Galois automorphism of K over F, and let $\{\sigma_v\}_{v\in\mathrm{Pl}_\infty}$ be a set of representatives, modulo the action of σ_K, for embeddings of K into $\mathbb{R}$. This choice induces an isomorphism of $K_\infty^\times/F_\infty^\times$ with $(\mathbb{R}^*)^{\mathrm{Pl}_\infty}$.

Let $\Delta = K_\infty^\times/K_\infty^+F_\infty^\times\Lambda_U$ and $\Delta' = K_\infty^\times/K_\infty^+F_\infty^\times\Lambda'_U$. They are both finite groups, in fact, products of $(\mathbb{Z}/2\mathbb{Z})$s. We define sublattices $\Lambda_{U,+}$ and $\Lambda'_{U,+}$ of $K_{\infty,+}^\times/F_{\infty,+}^\times$ as:

$$\Lambda_{U,+} = (K_{\infty,+}^\times \cap F_\infty^\times\Lambda_U)/F_{\infty,+}^\times, \quad \Lambda'_{U,+} = (K_{\infty,+}^\times \cap F_\infty^\times\Lambda'_U)/F_{\infty,+}^\times$$

$\Lambda_{U,+}$ can be relatively easily described: let $\mathfrak{o}'_{K,\mathfrak{I}}$ be the subgroup of elements in $\mathfrak{o}_{K,\mathfrak{I}}$ whose norm is a *totally positive* unit of $\mathfrak{o}_F^*$. $\mathfrak{o}'_{K,\mathfrak{I}}$ is a sublattice of Λ_U. Then $\Lambda_{U,+}$ is isomorphic to $\mathfrak{o}'_{K,\mathfrak{I}}/\mathfrak{o}_F^*$. To describe $\Lambda'_{U,+}$ is more complicated: it is the group of elements of K that generate an $\mathfrak{o}_{K,\mathfrak{I}}$-ideal that is the extension of an F-ideal, modulo $F^\times$.

Once this notation has been introduced, it is easy to describe the structure of $\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times U$. Its connected component is a torus, and "most" of the disconnectedness comes from the class group, with some coming from the ∞ components also. We have exact sequences:

$$K_\infty^\times/F_\infty^\times\Lambda_U \rightarrowtail \mathbb{A}_K^\times/(K^\times F_\infty^\times U) \twoheadrightarrow C_{F,\mathfrak{I}} \tag{4.40}$$

$$K_\infty^\times/F_\infty^\times\Lambda_U' \rightarrowtail \mathbb{A}_K^\times/(K^\times \mathbb{A}_F^\times U) \twoheadrightarrow C_{F,\mathfrak{I};K} \tag{4.41}$$

and, to understand the initial groups, we have further sequences:

$$\frac{K_{\infty,+}^\times/F_{\infty,+}^\times}{\Lambda_{U,+}} \rightarrowtail K_\infty^\times/F_\infty^\times\Lambda_U \twoheadrightarrow \Delta \tag{4.42}$$

$$\frac{K_{\infty,+}^\times/F_{\infty,+}^\times}{\Lambda_{U,+}} \rightarrowtail K_\infty^\times/F_\infty^\times\Lambda_{U'} \twoheadrightarrow \Delta' \tag{4.43}$$

We are now in a position to make explicit computations.

4.5.1 Fourier coefficients of the corresponding forms

We continue with the notations of the previous section; in particular, we regard as fixed an ideal $\mathfrak{I} = \mathfrak{D}_{K/F}\mathfrak{f}^2$ of F. Let $\mathfrak{J}$ be an ideal of K coprime to $\mathfrak{I}$. (A warning about the typesetting: note that $\mathfrak{I}$ and $\mathfrak{J}$ are not the same symbol!) Fix a character ω_∞ of $K_{\infty,+}^\times/F_{\infty,+}^\times\Lambda_{U,+}$.

More notation: Recall (**Notations** in the previous section) that Pl_∞ was a set of representatives for embeddings of K into $\mathbb{R}$ modulo the action of $\mathrm{Gal}(K/F)$; let $\mathbb{R}^{\mathrm{Pl}_\infty} = \prod_{v\in\mathrm{Pl}_\infty}\mathbb{R}$. We then define an homomorphism $\mathbf{log} : K_\infty^\times/F_\infty^\times \to \mathbb{R}^{\mathrm{Pl}_\infty}$ via $x \to (\log|\frac{\sigma_v(x)}{\sigma_K\sigma_v(x)}|)$; this is essentially compatible with our earlier use of $\mathbf{log}$. It induces an isomorphism of $K_{\infty,+}^\times/F_{\infty,+}^\times$ with $\mathbb{R}^{\mathrm{Pl}_\infty}$.

Fix the standard character of $\mathbb{R}$, namely $x \to e(x) = e^{2\pi i x}$. We also fix the measure on $F_\infty = \mathbb{R}^{\mathrm{Pl}_\infty}$ to be the product Lebesgue measure.

We can identify the dual space $(\mathbb{R}^{\mathrm{Pl}_\infty})^*$ with the space of characters of $K^\times_{\infty,+}/F^\times_{\infty,+}$: $\lambda \in (\mathbb{R}^{\mathrm{Pl}_\infty})^*$ gives rise to the character $x \mapsto e(\langle \lambda, \mathbf{log}(x)\rangle)$. Given a Schwarz function h on $(\mathbb{R}^{\mathrm{Pl}_\infty})^*$ – one may use this isomorphism to identify h with a function on $\widehat{K^\times_{\infty,+}/F^\times_{\infty,+}}$, the character group of $K^\times_{\infty,+}/F^\times_{\infty,+}$.

Define $\mathbf{t}_\omega \in (\mathbb{R}^{\mathrm{Pl}_\infty})^*$ to be the parameter of the character ω_∞, so that $\omega_\infty(x) = e(\langle \mathbf{t}_\omega, \log(x)\rangle)$. Then, under the identification of the previous paragraph, $h(\omega_\infty) = h(\mathbf{t}_\omega)$. We also define the Fourier transform of h to be the function $\hat{h}$ on $\mathbb{R}^{\mathrm{Pl}_\infty}$ given by $\hat{h}(v) = \int_{w\in(\mathbb{R}^{\mathrm{Pl}_\infty})^*} h(w)e(\langle v, w\rangle)dw$.

Further, $\Lambda_{U,+}$ becomes a lattice in $\mathbb{R}^{\mathrm{Pl}_\infty}$. With measures as above, the covolume of $\Lambda_{U,+}$ becomes $\frac{R_{K,\mathfrak{J}}}{|\Delta| R_F}$. Here R_K and R_F are the regulators of K and F respectively, and $R_{K,\mathfrak{J}}$ is the regulator of the unit lattice in $\mathfrak{o}_{K,\mathfrak{J}}$, a sublattice of $\mathfrak{o}_K^*$; in particular $R_{K,\mathfrak{J}} = R_K[\mathfrak{o}_K^* : \Lambda_U]$. (These statements would become more complicated in the presence of roots of unity, which are not present since the fields are totally real.)

Given a character ω of $\mathbb{A}_K^\times/(K^\times F_\infty^\times U)$, we say $\omega \sim \omega_\infty$ if the restriction of χ to $K_{\infty,+}$ coincides with ω_∞. The possible ω_∞ that can occur in this way are identified, under the isomorphisms given above, with characters of $\mathbb{R}^{\mathrm{Pl}_\infty}$ trivial on $\Lambda_{U,+}$.

Remark 14. Suppose $\omega \in \widehat{\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times U}$. Then the $\mathrm{GL}_2(F_\infty)$ representation that is the ∞ component of $\pi(\omega)$ is, for an appropriate of $\mathbf{sgn} \in \{0,1\}^{\mathrm{Pl}_\infty}$, the representation $\pi((2\pi)\mathbf{t}_\omega, \mathbf{sgn})$. (Here the notation is bad – the outer π means "representation associated to the data $(\mathbf{t}, \mathbf{sgn})$" in the sense of Section 4.3.2, and the inner π is 3.1415....)

Let h be a Schwarz function on $(\mathbb{R}^{\mathrm{Pl}_\infty})^*$, which is even in each coordinate. We aim to evaluate, a generalization of Equation 2.27 of Chapter 2, the "expected answer"

$$EA = \sum_{\omega \in (\widehat{\mathbb{A}_K^\times/K^\times F_\infty^\times U})} h(\omega_\infty)\lambda_{\mathfrak{J}}(\pi(\omega)) \tag{4.44}$$

and we will deduce the value of the corresponding sum when ω ranges only over characters of $\mathbb{A}_K^\times/(K^\times\mathbb{A}_F^\times U)$. Using Expectation 1 and Equation 4.44, we see that:

$$EA = \sum_{\omega_\infty} h(\omega_\infty)\left(\sum_{\omega\sim\omega_\infty}\sum_{\mathfrak{B}:\mathrm{Norm}(\mathfrak{B})=\mathfrak{J}} \omega(\mathfrak{B})\right)$$

Choose an element $x_{\mathfrak{B}} \in \mathbb{A}_{K,f}^\times$ that corresponds to the ideal $\mathfrak{B}$, and we keep in mind that $\mathfrak{B}$ is prime to $\mathfrak{J}$. Then, by definition, $\omega(\mathfrak{B}) = \omega(x_{\mathfrak{B}})$. The inner sum, then, will *vanish* unless $x_{\mathfrak{B}} \in K_{\infty,+}F_\infty^\times K^\times U$; this is equivalent to requiring that $\mathfrak{B}$ be principal and generated by a element $x_0 \in \mathfrak{o}_{K,\mathfrak{J}}$ with *totally positive norm*. Further, if two different such elements $x_0, x_0' \in \mathfrak{o}_{K,\mathfrak{J}}$ generate $\mathfrak{B}$, then they differ by an element of $\mathfrak{o}'_{K,\mathfrak{J}}$ – (the subset of $\mathfrak{o}_{K,\mathfrak{J}}$ consisting of elements whose norms are totally positive units of F); this follows since one sees that $x_0'x_0^{-1}$ must lie in $U_{\mathfrak{J},v}$ for all v, and also must lie in $K_{\infty,+}F_\infty$.

Let $X_{\mathfrak{J}}$ be a set of representatives, *modulo* the action of $\mathfrak{o}'_{K,\mathfrak{J}}$, for elements $x_0 \in \mathfrak{o}_{K,\mathfrak{J}}$ such that $\mathrm{Norm}_{K/F}(x_0)$ is totally positive and $(\mathrm{Norm}_{K/F}(x_0)) = \mathfrak{J}$. The exact sequences Equation 4.40 and Equation 4.42 show that there are $|C_{K,\mathfrak{J}}||\Delta|$ characters ω so that $\omega \sim \omega_\infty$. Using these observations, our sum becomes:

$$EA = |C_{K,\mathfrak{J}}||\Delta| \sum_{x\in X_{\mathfrak{J}}}\sum_{\omega_\infty} h(\omega_\infty)\omega_\infty(x) \tag{4.45}$$

As remarked, the set of characters ω_∞ that we sum over are, when one identifies $K_{\infty,+}/F_{\infty,+}$ with $\mathbb{R}^{\mathrm{Pl}_\infty}$, just those trivial on $\Lambda_{U,+}$.

One carries through the same procedure for $\mathbb{A}_K^\times/\widehat{(K^\times\mathbb{A}_F^\times U)}$, by using elements of C_F to "pick out" characters that are trivial on $\mathbb{A}_F^\times$. Let A be, as before, a set of representatives for F-ideal classes. For $\mathfrak{a} \in A$, let $X_{\mathfrak{a},\mathfrak{J}}$ be a set of representatives, modulo the multiplicative action of $\mathfrak{o}'_{K,\mathfrak{J}}$, for elements $x_0 \in \mathfrak{a}^{-1}\mathfrak{o}_{K,\mathfrak{J}}$ with totally positive norm $\mathrm{Norm}_{K/F}(x_0)$ and so that $\mathrm{Norm}_{K/F}(x_0)$ generates $\mathfrak{J}\mathfrak{a}^{-2}$. As before, let

$\pi_{\mathfrak{a}}$ be an idele corresponding to $\mathfrak{a} \in A$. One obtains:

$$\sum_{\omega \in (\widehat{\mathbb{A}_K^\times / K^\times \mathbb{A}_F^\times U})} h(\omega_\infty)\lambda_{\mathfrak{J}}(\pi(\omega)) = \sum_{\omega \in \widehat{\mathbb{A}_K^\times / K^\times F_\infty^\times U}} \left(\frac{1}{|C_F|} \sum_{\mathfrak{a} \in C_F} \omega(\pi_{\mathfrak{a}}) \right) h(\omega_\infty)\lambda_{\mathfrak{J}}(\pi(\omega))$$

$$= \frac{1}{|C_F|} \sum_{\mathfrak{a} \in C_F} \sum_{\omega \in \widehat{\mathbb{A}_K^\times / K^\times F_\infty^\times U}} h(\omega_\infty)\omega(\pi_{\mathfrak{a}})\lambda_{\mathfrak{J}}(\pi(\omega)) = \frac{|C_{K,\mathfrak{J}}|}{|C_F|} |\Delta| \sum_{\mathfrak{a} \in A} \sum_{x \in X_{\mathfrak{a},\mathfrak{J}}} \sum_{\omega_\infty} h(\omega_\infty)\omega_\infty(x) \tag{4.46}$$

where the final equality is derived in an identical fashion to Equation 4.45.

Finally, we apply Poisson sum to Equation 4.46. Recall (see discussion preceding Remark 14) that ω_∞ ranges over the lattice dual to $\Lambda_{U,+}$ (it would be the same to restrict it to the lattice dual to $\Lambda'_{U,+}$ – the other ω contribute 0 – but this is more convenient), and we can identify $\Lambda_{U,+}$ with the quotient $\mathfrak{o}'_{K,\mathfrak{J}}/\mathfrak{o}_F^*$; also, the covolume of $\Lambda_{U,+}$ is $\frac{R_{K,\mathfrak{J}}}{|\Delta| R_L}$. Also, our identifications are such that $h(\omega_\infty) = h(\mathbf{t}_\omega)$. We obtain, with $\hat{h}$ the Fourier transform of h:

$$\sum_{\omega \in (\widehat{\mathbb{A}_K^\times / K^\times \mathbb{A}_F^\times U})} h(\mathbf{t}_\omega)\lambda_{\mathfrak{J}}(\pi(\omega)) = \frac{|C_{K,\mathfrak{J}}||\Delta|}{|C_F|} \cdot \frac{R_{K,\mathfrak{J}}}{|\Delta| R_F} \sum_{\mathfrak{a} \in A} \sum_{x \in X_{\mathfrak{a},\mathfrak{J}}} \sum_{\epsilon' \in \mathfrak{o}'_{K,\mathfrak{J}}/\mathfrak{o}_F} \hat{h}(\mathbf{log}(x\epsilon')) \tag{4.47}$$

We can finally make the comparison between this and Equation 4.24. Taking into account Remark 14, we see that they do indeed "agree," up to a constant; as in Chapter 2, one can translate this to the classification theorem for π such that $m(\pi, \mathrm{Sym}^2) = 1$. (Actually, there is one added detail, which is to sift apart the contributions of the different fields $K \in S$. This requires some further effort.)

This completes our treatment of dihedral forms over F.

4.6 Derivation of Petersson Formula (after Cogdell, Piatetski-Shapiro)

The aim of this section is to derive a formula of Petersson-Kuznetsov type. Such a formula can be found in the article of Bruggeman and Miatello ([2]), and, indeed, their formula is sufficient for the application of this Chapter and circumvents entirely the need to invoke any of the "approximation" results from the Appendix. However, we include this section for (partial) completeness, and also because the method of Cogdell and Piatetski-Shapiro does not seem to have been widely applied, and provides an elegant derivation of the formula.

For our application, which is essentially a statement about the whole spectrum, it is convenient to write down a formula that includes both "Maass" and "holomorphic" forms: that is, it combines the contributions of forms with different representation types at ∞. Such a formula is, of course, slightly less precise than its sibling that sifts apart the ∞-types, but we have no need for the latter, and the derivation of a formula of the desired type follows relatively elegantly from representation-theoretic considerations; it also "explains" the disappearance of the diagonal term in the combined formula. It has the defect, however, that the "geometric test function" φ is compactly supported; actually, the two facts are related.

The reader can either take on faith that the conclusions of this Section can be extended easily to φ that are not of compact support, by the method of Bruggeman and Miatello; or, if so wished, we can use only compactly supported φ in our arguments, in combination with the approximation Theorem 7 of the appendix.

We shall work relatively classically, and will closely follow Cogdell and Piateski-Shapiro in [4]; the modifications are that we have dealt with GL_2 over a number field and not PGL_2 over $\mathbb{Q}$, but this does not introduce any essentially new complications. In short, the method is to compute the Fourier coefficient of a Poincaré series in two ways. One way is directly, which leads to a "geometric" formula involving Kloosterman sums. The other way is to spectrally expand the Poincaré series. This leads to the "spectral" side of the formula.

Cogdell and Piatetski-Shapiro, at least in the $\mathbb{Q}$ case, treat convergence carefully. We only reproduce the central point here, namely, the absolute convergence and rapid decay of the Poincaré series. (Indeed, I could not follow some points of this particular proof in [4].)

Notation: Let $G = GL_2(F_\infty)$; it is isomorphic to a product of $\mathrm{GL}_2(\mathbb{R})$ factors. Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ be the Borel subgroup of upper triangular matrices, $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ its unipotent radical, and $\bar{N} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ its opposite, A the subgroup of diagonal matrices. (Here, an asterisk $*$ denotes an arbitrary element of F_∞). Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be a representative for the nontrivial element of the Weyl group. We denote by $n(x)$ the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and by $a(y_1, y_2)$ the diagonal matrix $\mathrm{diag}(y_1, y_2) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$. Finally, Z is the center of G.

All representations mentioned or used in this section will be unitary.

Normalization of Measures: We fix the standard measure dx on $\mathbb{R}$ and therefore a measure on F_∞, via the identification $F_\infty = \prod_{\infty|v} F_v = \mathbb{R}^{[F:\mathbb{Q}]}$. This gives a measure on N (by means of the homeomorphism $x \to n(x)$). The measure on F_∞ also gives a Haar measure on $F_\infty^\times$, via $dx/\mathrm{Norm}(x)$; this then also gives a measure on the subgroup A of diagonal matrices and on the centre Z. We fix Haar measure on the maximal compact so it has mass 1. Finally, the Iwasawa decomposition for $\mathrm{GL}_2(F_\infty)$ now specifies the measure on $\mathrm{GL}_2(F_\infty)$. (One can also use the Bruhat decomposition for the big cell $NwAN$, and, using product measure, one obtains a Haar measure μ_B on G. If one denotes the "Iwasawa" Haar measure by μ_I, the two are related by $\mu_I = (\frac{1}{4\pi})^{[F:\mathbb{Q}]}\mu_B$. By default, we use μ_I, but it will be convenient to work in the Bruhat decomposition at a certain point, so we will require this.)

Let Z be the centre of G. Let $\Gamma = \Gamma_0(\mathfrak{I};\mathfrak{a})$ as defined earlier. Fixing a unitary adelic character χ with conductor (dividing) $\mathfrak{I}$, we let χ_f be the character of $(\mathfrak{o}_F/\mathfrak{I})^\times$ induced by χ and χ_∞ the character of $F_\infty^\times$. (To be precise, we can regard χ_f as being the restriction of χ to $\prod_v \mathfrak{o}_v^\times$, or, what is the same, the restriction of χ to $\prod_v \mathfrak{o}_v^\times$, where one takes the product only over ramified v.)

Together, (χ_f, χ_∞) determine χ up to twists by unramified characters. We will often significantly abuse notation and write χ instead of χ_f. This never causes ambiguity, but it is worth remembering that, in contrast to the case over $\mathbb{Q}$, χ_f and χ_∞ together do not determine χ.

In any case, χ_f determines a character of $\Gamma_0(\mathfrak{I};\mathfrak{a})$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \chi_f(a)$.

Let $L^2_\chi(\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty))$ denote the space of functions f on $\mathrm{GL}_2(F_\infty)$ that transform under the center $\mathrm{diag}(z,z)$ via $\chi_\infty(z)$ and transform on the left by χ_f^{-1} under $\Gamma_0(\mathfrak{I};\mathfrak{a})$, and so that the integral

$$||f||_2^2 = \int_{\Gamma_0(\mathfrak{I};\mathfrak{a})Z\backslash \mathrm{GL}_2(F_\infty)} |f(g)|^2 dg < \infty$$

(Recall that the inverse is for appropriate compatibility between adelic and classical: when one unwinds an adelic form with character χ, one obtains classically a form transforming under χ_f^{-1}.)

Let ψ_1 and ψ_2 be two additive characters of $F \otimes \mathbb{R}$ trivial on $\mathfrak{I}\mathfrak{a}^{-1}$; they are identified with characters of N trivial on $\Gamma_N \equiv \Gamma \cap N$.

It is convenient to note now the Γ_N-double coset decomposition of $\Gamma_0(\mathfrak{I};\mathfrak{a})$. The proof is elementary matrix manipulation and we omit it.

Proposition 14. *There is a natural map* $\Gamma_N\backslash\Gamma/\Gamma_N$ *to* $\{c : c \in \mathfrak{I}\mathfrak{a}^{-1}\}$*, the map being that which sends a matrix to its lower left-hand entry.*

The fibre Σ_c *above* $c \in \mathfrak{I}\mathfrak{a}^{-1}$ *can be identified with pairs of elements* (x, ϵ)*, where* $\epsilon \in \mathfrak{o}_F^*$ *and* $x \in (\mathfrak{o}_F/c\mathfrak{a})^\times$*; a representative for the fibre corresponding to* (x, ϵ) *is* $\begin{pmatrix} [x]_{c\mathfrak{a}} & ? \\ c & [\epsilon x^{-1}]_{c\mathfrak{a}} \end{pmatrix}$*, where, for example,* $[x]_{c\mathfrak{a}}$ *denotes any element of* $\mathfrak{o}_F$ *that reduces to* x *mod* $c\mathfrak{a}$*.*

Given η a Schwarz function on $F \otimes \mathbb{R}$ (the notion of Schwarz function is the usual one, as $F \otimes \mathbb{R}$ is a real vector space), and ν a compactly supported function on $(F \otimes \mathbb{R})^\times$, we define the function $f_{\eta,\nu}$ on $G(\mathbb{R})$ via

$$f_{\eta,\nu}(n(x_1)wn(x_2)a(y_1, y_2)) = \psi_1(x_1)\eta(x_2)\nu(y_1y_2^{-1})\chi_\infty(y_2)$$

and we define $f_{\eta,\nu}$ to vanish identically on the complement of the large Bruhat cell. This function $f_{\eta,\nu}$ transforms on the left under N according to ψ_1 and under the center according to χ_∞.

Note that $n(x_1)wn(x_2)a(y_1, y_2)$ is the matrix $\begin{pmatrix} y_1x_1 & y_2(x_1x_2 - 1) \\ y_1 & y_2x_2 \end{pmatrix}$. Therefore, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:

$$f_{\eta,\nu}(g) = \begin{cases} \psi_1(\frac{a}{c})\eta(cd\det(g)^{-1})\nu(c^2\det(g)^{-1})\chi_\infty(\det(g)c^{-1}),\ c \in F_\infty^\times \\ 0, \text{ else} \end{cases}$$

One constructs a Poincaré series $P_{\eta,\nu}$ associated with f, simply by averaging over Γ. (Note that, since ψ_1 is trivial on $\Gamma \cap N$, it follows that f is Γ_N invariant.) Define $Z_\Gamma = Z \cap \Gamma$. We then define:

$$P_{\eta,\nu}(g) = \sum_{\Gamma_N Z_\Gamma \backslash \Gamma} \chi_f(\gamma) f_{\eta,\nu}(\gamma g)$$

This transforms under Γ on the left by the factor $\chi_f(\gamma)^{-1}$.

Lemma 14. *The series defining $P_{\eta,\nu}(g)$ converges absolutely and locally uniformly, defining a function of rapid decay on $\Gamma\backslash \mathrm{GL}_2(F_\infty)$.*

We give a proof via the theory of Eisenstein series. We first note the following Lemma, controlling sums over the units of $\mathfrak{o}_F$. It is contained in [2] and it will be of use both here and in the final chapter.

Lemma 15. *Suppose one is given a function f on F_∞ and non-negative a, b with $a+b>0$ so that f satisfies, for $y \in F_\infty = \prod_{v|\infty} F_v$, $f(y) \ll \prod_{v|\infty} \min(|y|_v^a, |y|_v^{-b})$. Then, for $y \in F_\infty$,*

$$\sum_{\alpha \in \mathfrak{o}_F^*} f(\alpha y) \ll_{\epsilon,F} \min(\mathrm{Norm}(y)^{a-\epsilon}, \mathrm{Norm}(y)^{-b+\epsilon})$$

Proof. Approximate the sum by an integral; see [2] for details. □

Proof. (Of Lemma 14) We may write, with $\Gamma_P = \Gamma \cap B$,

$$P_{\eta,\nu}(g) = \sum_{\Gamma_P \backslash \Gamma} \sum_{Z_\Gamma \Gamma_N \backslash \Gamma_P} \chi_f(\gamma) f_{\eta,\nu}(\gamma g)$$

Now the function $F(g)$ defined by the inner sum:

$$F(g) = \sum_{\gamma \in Z_\Gamma \Gamma_N \backslash \Gamma_P} \chi_f(\gamma) f_{\eta,\nu}(\gamma g) \tag{4.48}$$

is a function which transforms by ψ_1 under N and is, in absolute value, invariant additionally under $a(\epsilon_1, \epsilon_2)$ for $\epsilon_1, \epsilon_2 \in \mathfrak{o}_F^*$. The idea is to bound $|F|$ and, consequently, bound $P_{\eta,\nu}$ by an appropriate Eisenstein series. Fix a norm $||\cdot||$ on F_∞.

First, one verifies the following bound on $f_{\eta,\nu}$: there is a function ψ_0 on $\mathbb{R}^+$ such that $\psi_0(x) \ll_k x^k$ for all $k \in \mathbb{Z}$, i.e. ψ_0 is rapidly decreasing both at 0 and ∞; and, if in the Iwasawa decomposition $g = n_g a_g k_g$ with $n_g \in N$ and $a_g = \mathrm{diag}(a_1, a_2) \in A$, then $|f_{\eta,\nu}(n_g a_g k_g)| \ll \prod_{v|\infty} \psi_0(|a_{1,v} a_{2,v}^{-1}|)$.

Suppose $a_g = a(y_1, y_2)$. The sum in Equation 4.48 can be taken over the set of representatives $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$ for $Z_\Gamma \Gamma_N \backslash \Gamma_P$, where $\epsilon \in \mathfrak{o}_F^*$. Then, applying Lemma 15, we see that $F(g) \ll_r \mathrm{Norm}(y_1 y_2^{-1})^r$, for all r, both positive and negative. It follows that $F(g)$ is a function of rapid decay on $\Gamma_P Z_\Gamma \backslash G$.

The result now follows from well-known properties of Eisenstein series (the Eisenstein series associated to a function of rapid decay converges absolutely and also has rapid decay.) □

We will derive a form of the Kuznetsov formula by computing the Fourier coefficients of $P_{\eta,\nu}$ in two ways: "geometrically", directly from the definition; or "spectrally," using the expansion of $P_{\eta,\nu}$ in terms of the spectrum of $\Gamma_0(\mathfrak{I}; \mathfrak{a}) \backslash \mathrm{GL}_2(F_\infty)$.

4.6.1 Geometric Computation of Fourier Coefficients

It is simple to compute the ψ_2-Fourier coefficient of $P_{\eta,\nu}$, which we will denote by $(P_{\eta,\nu})_{\psi_2}$ and will evaluate at the identity:

$$(P_{\eta,\nu})_{\psi_2}(1) \equiv \int_{\Gamma_N \backslash N} \psi_2(n)^{-1} P_{\eta,\nu}(n) dn = \int_{\Gamma_N \backslash N} \psi_2(n)^{-1} dn \sum_{\gamma \in Z_\Gamma \Gamma_N \backslash \Gamma} \chi_f(\gamma) f_{\eta,\nu}(\gamma n)$$

Note that γ in the complement of the large Bruhat cell do not contribute to this sum, since $f_{\eta,\nu}$ is supported on the large Bruhat cell. If γ is not in the large Bruhat cell, the Bruhat decomposition allows one to write $\gamma = n_{1,\gamma} w a_\gamma n_{2,\gamma}$, with

$n_{1,\gamma}, n_{2,\gamma} \in N$ and $a_\gamma \in A$. The integral then equals:

$$\int_{\Gamma_N \backslash N} \psi_2(n)^{-1} dn \sum_{\gamma \in Z_\Gamma \Gamma_N \backslash \Gamma} \psi_1(n_{1,\gamma}) \chi_f(\gamma) f_{\eta,\nu}(w a_\gamma n_{2,\gamma} n)$$

By expanding $\Gamma_N \backslash \Gamma$ in terms of Γ_N-double cosets – the value of χ_f only depends on the double coset – one obtains:

$$(P_{\eta,\nu})_{\psi_2}(1) = \int_N \psi_2(n)^{-1} \psi_1(n_{1,\gamma}) \psi_2(n_{2,\gamma}) dn \sum_{\gamma \in Z_\Gamma \Gamma_N \backslash \Gamma / \Gamma_N} \chi_f(\gamma) f_{\eta,\nu}(w a_\gamma n) \qquad (4.49)$$

Each a_γ for $\gamma \in \Gamma_N \backslash \Gamma / \Gamma_N$ can be written as $a(\omega_1, \omega_2)$ for some $\omega_1, \omega_2 \in F_\infty$. Let Ω be the set of these pairs (ω_1, ω_2) counted *without* multiplicity. There is a natural map from $\Gamma_N \backslash \Gamma / \Gamma_N$ to Ω; also Z_Γ acts in a natural way on the set Ω, in such a way that the map $\Gamma_N \backslash \Gamma / \Gamma_N \to \Omega$ is compatible with the Z_Γ actions on both sides. One verifies:

Lemma 16. *For $\Gamma = \Gamma_0(\mathfrak{I}; \mathfrak{a})$, the set Ω consists of pairs (ω_1, ω_2) with $\omega_1 \in \mathfrak{I}\mathfrak{a}^{-1}$ and $\omega_1 \omega_2 \in \mathfrak{o}_F^*$. The Γ_N-double coset parametrized (notation of Proposition 14) by (c, x, ϵ) maps to the element of Ω given by $(c, \epsilon c^{-1})$.*

For $\omega \in \Omega$, let $S(\omega)$ be a set of representatives for $\Gamma_N \backslash \{\gamma \in \Gamma : a_\gamma = \omega\} / \Gamma_N$. We may define, for each $\omega \in \Omega$, the sum $KS_\chi(\omega) = \sum_{\gamma \in S(\omega)} \psi_1(n_1(\gamma)) \chi_f(\gamma) \psi_2(n_2(\gamma))$. This is a first version of the Kloosterman sum – we will eventually introduce a slightly modified sum to account for χ_∞ better. For $\Gamma = \Gamma_0(\mathfrak{I}; \mathfrak{a})$, if $\omega = (\omega_1, \omega_2) \in \Omega$, then the Kloosterman sum $KS_\chi(\omega)$ equals

$$\sum_{xy = \omega_1 \omega_2 (\mathrm{mod}\, \omega_1 \mathfrak{a})} \chi_f(x) \psi_1(\frac{x}{\omega_1}) \psi_2(\frac{y}{\omega_1}) = \sum_{x \in (\mathfrak{o}_F / \omega_1 \mathfrak{a})^\times} \chi_f(x) \psi_1(\frac{x}{\omega_1}) \psi_2(x^{-1} \omega_2)$$

In terms of these sums, we may rewrite Equation 4.49:

$$(P_{\eta,\nu})_{\psi_2}(1) = \int_{F_\infty} dx\, \psi_2^{-1}(x) \sum_{\omega=(\omega_1,\omega_2)\in Z_\Gamma\backslash\Omega} KS_\chi(\omega) f_{\eta,\nu}(wn(\omega_1\omega_2^{-1}x)a(\omega_1,\omega_2))$$

$$= \sum_{\omega\in Z_\Gamma\backslash\Omega} KS_\chi(\omega) \int_{x\in F_\infty} dx\, \psi_2^{-1}(x)\eta(\omega_1\omega_2^{-1}x)\chi_\infty(\omega_2)\nu(\omega_1\omega_2^{-1})$$

$$= \sum_{\omega\in Z_\Gamma\backslash\Omega} KS_\chi(\omega)\hat{\eta}_{\psi_2{}^{-1}}((\frac{\omega_1}{\omega_2})^{-1})\nu(\omega_1/\omega_2)\frac{1}{|\mathrm{Norm}(\omega_1/\omega_2)|}\chi_\infty(\omega_2)$$

Here $\hat{\eta}_{\psi_2^{-1}}(k) = \int_{F_\infty} \psi_2^{-1}(kx)\eta(x)dx$, the Fourier transform of η with respect to ψ_2^{-1}.

We make two additional definitions to simplify the expression for $(P_{\eta,\nu})_{\psi_2}$: we define a function φ on $F_\infty^\times$ via $\varphi(x) = \nu(1/x)\hat{\eta}_{\psi_2^{-1}}(x)$, and we define a modified Kloosterman sum via, for $\epsilon \in \mathfrak{o}_F^*$ and $c \in \mathfrak{I}\mathfrak{a}^{-1}$,

$$KS_\chi(\psi_1, \psi_2, c, \epsilon) = \chi_\infty(\epsilon/c) \sum_{x\in(\mathfrak{o}_F/c\mathfrak{a})^\times} \psi_1(\frac{x}{c})\psi_2(\frac{\epsilon x^{-1}}{c})\chi_f(x) \tag{4.50}$$

Note that, by choosing η, ν appropriately one can achieve any φ that is smooth and compactly supported on $F_\infty^\times$.

We also note (see Lemma 16) that the map $(\omega_1, \omega_2) \to (\omega_1, \omega_1\omega_2)$ identifies $Z_\Gamma\backslash\Omega$ with the set of pairs (c, ϵ), where $\epsilon \in \mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2$ and $c \in \mathfrak{I}\mathfrak{a}^{-1}$ is defined up to sign – or, rather, this is a set of representatives for $Z_\Gamma\backslash\Omega$. Note that $KS_\chi(\psi_1, \psi_2, c, \epsilon)$ is unchanged under $c \mapsto -c$. We can now phrase the formula for $P_{\eta,\nu}$ quite compactly, and we state it as a Lemma:

Lemma 17.

$$(P_{\eta,\nu})_{\psi_2}(1) = \frac{1}{2} \sum_{\substack{c\in\mathfrak{I}\mathfrak{a}^{-1}-\{0\} \\ \epsilon\in\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2}} \frac{1}{\mathrm{Norm}(c)^2}\varphi(\epsilon/c^2)KS_\chi(\psi_1, \psi_2, c, \epsilon)$$

Here the factor of $\frac{1}{2}$ arises from c being defined only up to sign.

4.6.2 Spectral Computation of Fourier Coefficient

The L^2-spectral expansion of $P_{\eta,\nu}$ looks like

$$P_{\eta,\nu} = \int_\pi \mathrm{Proj}_\pi(P_{\eta,\nu}) d\pi$$

where π ranges over those representations occurring in the spectral decomposition of $L^2_\chi(\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty))$, and Proj_π denotes the projection onto the space corresponding to π; note that π is unitary. $d\pi$ is an appropriate spectral measure. Now, we will compute only the terms corresponding to the discrete spectrum; for the computations on the continuous spectrum we refer to [4], at least over $\mathbb{Q}$. The formal reasoning is identical in general.

To be precise, we will compute, for each π in the discrete spectrum, the Fourier coefficients of the projection $\mathrm{Proj}_\pi P_{\eta,\nu}$. Putting these together, for various π, gives the "spectral" computation of the Fourier coefficients. We will also not address the issue of L^2 against pointwise convergence; it is treated in [4].

Fix, then, a $\mathrm{GL}_2(F_\infty)$ representation π that occurs in the discrete part of the spectrum:

$$\pi \subset L^2_{\chi,\mathrm{Disc}}(\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty))$$

π has a ψ_1-Whittaker model: a unique model acting by right translation on a space of functions $\mathcal{W}_{\psi_1}$ on $\mathrm{GL}_2(F_\infty)$ such that, for each $W \in \mathcal{W}_{\psi_\infty}$, $W(ng) = \psi_1(n)W(g)$. For each $\phi \in \pi$, we denote by W_{ϕ,ψ_1} the "ψ_1-Fourier coefficient": $\int_{\Gamma_N\backslash N} \psi_1(n)^{-1}\phi(ng)dn$; the map $\phi \mapsto W_{\phi,\psi_1}$ gives a nonzero intertwiner between π and $\mathcal{W}_{\psi_1}$. Of course, one can replace ψ_1 with any other character; in that case, we change the subscripts accordingly.

For any ϕ belonging to the space of π, one computes by unfolding the inner product:

$$\langle P_{\eta,\nu}, \phi \rangle_{\Gamma\backslash G} = \int_{Z\Gamma\backslash G} P_{\eta,\nu}(g)\overline{\phi(g)}dg = \int_{Z\Gamma_N\backslash G} f_{\eta,\nu}(g)\overline{\phi(g)}dg$$

which equals

$$\int_{ZN\backslash G} f_{\eta,\nu}(g)\overline{W_{\phi,\psi_1}}(g)dg$$

One may define an inner product on $\mathcal{W}_{\psi_1}$ by:

$$\langle W_1, W_2 \rangle_W = \int_{F_\infty} W_1(a(x,1))\overline{W_2(a(x,1))}dx$$

Indeed, one verifies that this is convergent and that it is GL_2-invariant. (It defines an invariant inner product for the "mirabolic" subgroup of matrices $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$; since, by a result of Kirillov, the representation restricted to the mirabolic subgroup is irreducible, it must be the inner product for the entire group.)

Let $c(\pi, \psi_1)$ be the positive real constant such that, for ϕ_1, ϕ_2 in the space of π, we have $\langle W_{\phi_1,\psi_1}, W_{\phi_2,\psi_1} \rangle_W = c(\pi,\psi_1)\langle \phi_1, \phi_2 \rangle_{\Gamma\backslash G}$; such a constant exists, since a GL_2-invariant inner product on an irreducible representation is unique up to scaling. It will be evaluated in Lemma 18.

Now, having fixed the inner product $\langle \cdot, \cdot \rangle_W$ on the Whittaker model in this fashion, there exists $F_\pi(f_{\eta,\nu}) \in \mathcal{W}_{\psi_1}$ so that, for all ϕ in the space of $\pi \subset L^2_\chi$, we have:

$$\langle \mathrm{Proj}_\pi P_{\eta,\nu}, \phi \rangle \left(= \langle P_{\eta,\nu}, \phi \rangle = \int_{ZN\backslash G} f_{\eta,\nu}(g)\overline{W_\phi}(g) \right) = \langle F_\pi(f_{\eta,\nu}), W_\phi \rangle_W \tag{4.51}$$

The existence of such an F_π follows from the continuity of the functional $\phi \to \langle P_{\eta,\nu}, \phi \rangle$. It follows from the equality above that $c(\pi,\psi_1)F_\pi(f_{\eta,\nu}) = W_{\mathrm{Proj}_\pi P_{\eta,\nu},\psi_1} = (\mathrm{Proj}_\pi P_{\eta,\nu})_{\psi_1}$, the ψ_1-Fourier coefficient of $\mathrm{Proj}_\pi P_{\eta,\nu}$. It will not be difficult to deduce from this the ψ_2-Fourier coefficient.

It remains to compute $F_\pi(f_{\eta,\nu})$ relatively explicitly and to evaluate it at 1. We have:

$$\langle P_{\eta,\nu}, \phi \rangle = \int_{ZN\backslash G} f_{\eta,\nu}(g)\overline{W_{\phi,\psi_1}(g)}dg$$

This integral can be evaluated conveniently in the Bruhat decomposition. Recall the remark on normalization of measures in the Bruhat decomposition. We obtain:

$$\langle P_{\eta,\nu}, \phi \rangle = (\frac{1}{4\pi})^{[F:\mathbb{Q}]} \int_{x\in N, a\in Z\backslash A} f_{\eta,\nu}(wn(x)a(y_1,y_2))\overline{W_{\phi,\psi_1}(wn(x)a(y_1,y_2))}dnda$$

$$= (\frac{1}{4\pi})^{[F:\mathbb{Q}]} \int_{Z\backslash N\times A} \eta(x)\nu(y_1y_2^{-1})\chi_\infty(y_2)\overline{W_{\phi,\psi_1}(wn(x)a(y_1,y_2))}dnda$$

There is an explicit expression for the Whittaker function that occurs. Namely, there is a function J_{π,ψ_1} on $F_\infty^\times$ – the "Bessel function" of π in the nomenclature of Cogdell and Piatetski-Shapiro – so that the action of $\pi(w)$ on the Kirillov model is expressed by multiplicative correlation with K, i.e.:

$$\pi(w)K(x) = \int_{F_\infty^\times} J_{\pi,\psi_1}(ux)\chi_\infty(u)^{-1}K(u)d^\times u \tag{4.52}$$

We will not define this more carefully, referring to [4], as we only need the consequence, namely, it allows us to compute $W_{\phi,\psi_1}(wn(x)a(y_1,y_2))$. We will suppress the dependence on the additive character for now and refer to J_{π,ψ_1} only as J_π. We see:

$$W_{\phi,\psi_1}(wn(x)a(y_1,y_2)) = W_{\pi(n(x)a(y_1,y_2))\phi}(w)$$

$$= \int_{F_\infty^\times} \chi_\infty(u)^{-1}J_\pi(u)\psi_1(ux)W_{\phi,\psi_1}(a(y_1u,y_2))d^\times u$$

$$= \chi_\infty(y_2)\int_{F_\infty^\times} \chi_\infty(y_2y_1^{-1}u)^{-1}\psi_1(\frac{ux}{y_1y_2^{-1}})J_\pi(\frac{u}{y_1y_2^{-1}})W_{\phi,\psi_1}(a(u,1))d^\times u$$

where, for the last simplification, we have replaced u by $uy_2y_1^{-1}$. Noting this, the integral for $\langle P_{\eta,\nu}, \phi\rangle$ becomes:

$$\langle P_{\eta,\nu}, \phi\rangle = (\frac{1}{4\pi})^{[F:\mathbb{Q}]} \int_{\substack{n(x)\in N \\ a(y_1,y_2)\in Z\backslash A}} \eta(x)\chi_\infty(y_2y_1^{-1})\nu(y_1y_2^{-1})$$
$$\left(\int_{F_\infty^\times} \overline{\psi_1(\frac{ux}{y_1y_2^{-1}})J_\pi(\frac{u}{y_1y_2^{-1}})\chi_\infty(u)^{-1}W_{\phi,\psi_1}(a(u,1))d^\times u}\right) dxd^\times a$$

Replacing $y_1y_2^{-1}$ by a single variable y, and using the identity $\overline{J_\pi(u)} = \chi_\infty(-1)J_\pi(u)$, this becomes

$$\frac{\chi_\infty(-1)}{(4\pi)^{[F:\mathbb{Q}]}} \int_{F_\infty^\times} \overline{W_{\phi,\psi_1}(a(u,1))}d^\times u \int_{y\in F_\infty^\times} \chi_\infty(y)^{-1}\nu(y)\hat{\eta}_{\psi_1^{-1}}(\frac{u}{y})J_\pi(\frac{u}{y})\chi_\infty(u)d^\times y$$

It follows that $F_\pi(f_{\eta,\nu})$, defined by Equation 4.51, is given by

$$F_\pi(f_{\eta,\nu})(\text{diag}(u,1)) = \frac{1}{(4\pi)^{[F:\mathbb{Q}]}} \int_{F_\infty^\times} \nu(y)\hat{\eta}_{\psi_1^{-1}}(\frac{u}{y})J_\pi(\frac{u}{y})\chi_\infty(-\frac{u}{y})d^\times y \tag{4.53}$$

Definition 8. *(Change of character) Let ψ_1, ψ_2 be characters of F_∞, identified with characters of N, and let $\beta \in F_\infty^\times$ be such that $\psi_2(x) = \psi_1(\beta x)$. (Such a β* a priori *need not exist, but will in our applications; the Fourier expansion is supported on nondegenerate characters.) We define the constant $c(\pi : \psi_1 \to \psi_2)$ so that, for all ϕ in the space of π,*

$$W_{\phi,\psi_2}(g) = c(\pi : \psi_1 \to \psi_2)W_{\phi,\psi_1}(\text{diag}(\beta,1)g)$$

(The existence of such a constant follows from the uniqueness of Whittaker models.)

It is a consequence of this definition that, fixing β as in Definition 8, we have:

Corollary 5. $(\text{Proj}_\pi P_{\eta,\nu})_{\psi_2}(1) = c(\pi : \psi_1 \to \psi_2)(\text{Proj}_\pi P_{\eta,\nu})_{\psi_1}(\text{diag}(\beta,1))$.

Now, recalling the definition $\varphi(y) = \nu(1/y)\hat{\eta}_{\psi_2^{-1}}(y)$, we see from Equation 4.53

that

$$F_\pi(f_{\eta,\nu})(\mathrm{diag}(\beta,1)) = \frac{1}{(4\pi)^{[F:\mathbb{Q}]}} \int_{F_\infty^\times} \varphi(y) J_\pi(\beta y)\chi_\infty(-\beta y) d^\times y \tag{4.54}$$

Putting together Equation 4.51 and the remarks that follow it, Equation 4.54, and Corollary 5, we have thus proven:

$$(P_{\eta,\nu})_{\psi_2}(1) = \sum_\pi c(\pi,\psi_1)c(\pi:\psi_1\to\psi_2)F_\pi(f)(\mathrm{diag}(\beta,1)) + \text{C.S.C.} \tag{4.55}$$

$$= \sum_\pi \frac{c(\pi,\psi_1)c(\pi:\psi_1\to\psi_2)}{(4\pi)^{[F:\mathbb{Q}]}} \int_{F_\infty^\times} \varphi(u) J_\pi(\beta u)\chi_\infty(-\beta u) d^\times u + \text{C.S.C.} \tag{4.56}$$

4.6.3 Normalization of Fourier Coefficients (and Comparisons)

In this section we define precisely the normalization of Fourier coefficients that will be used. The definition given below is somewhat unenlightening, since most of the details are buried, so we add a brief section comparing this to some other normalizations in the literature. Actually, it turns out that our normalization will agree with theirs, although phrased in slightly different terms.

Recall we had fixed the basic character ψ of F_∞: $\alpha \mapsto e(\mathrm{Tr}_{\mathbb{Q}}^F \alpha)$. ψ induces a character of N. For each representation π of $\mathrm{GL}_2(F_\infty)$, we choose once and for all a fixed Whittaker function W_π^0 which is in the Whittaker model transforming under ψ and is of minimal SO(2)-weight for π, and so that:

$$\int_{x\in F_\infty^\times} \left| W_\pi^0 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right|^2 d^\times x = 1$$

It is unique up to scaling by a complex number of absolute value 1, and we choose a specific representative. The definition of "Fourier coefficient" will depend, up to a scalar of magnitude 1, on this choice of representative, but the Petersson-Kuznetsov formula will not. (Earlier in the chapter, we chose a specific choice of representative for concreteness; this is entirely peripheral.)

Definition 9. *Suppose π occurs discretely in $L^2_\chi(\Gamma\backslash \mathrm{GL}_2(F_\infty))$. Let f be a vector in the space of π that is of the same* SO(2)*-weight as W^0_π. Expand f in terms of Whittaker functions, normalized as above:*

$$f(g) = \frac{1}{\mathrm{vol}(\Gamma_N\backslash N)} \sum_{\alpha\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}} a_f^{un}(\alpha) W^0_\pi(\mathrm{diag}(\alpha, 1)g)$$

The $a_f^{un}(\alpha)$ are the "unnormalized Fourier coefficients" of f.

We now return to the formula Equation 4.55. We can find $\alpha_1, \alpha_2 \in F_\infty$ so that $\psi_1(x) = \psi(\alpha_1 x), \psi_2(x) = \psi(\alpha_2 x)$. Then $\beta = \alpha_2/\alpha_1$.

At this point, the constants $c(\pi, \psi_1)$ and $c(\pi; \psi_1 \to \psi_2)$ are easily computable, and we state the result as a Lemma.

Lemma 18.

$$c(\pi, \psi_1) = |a_f^{un}(\alpha_1)|^2, \quad c(\pi; \psi_1 \to \psi_2) = a_f^{un}(\alpha_2)/a_f^{un}(\alpha_1)$$

Proof. Immediate from the definitions. □

Recall also that $\varphi(x) = \nu(1/x)\hat{\eta}_{\psi_2^{-1}}(x)$, and the Bessel functions J_π were computed with respect to ψ_1-Whittaker models. Then one may check that $J_{\pi,\psi_1}(x) = J_{\pi,\psi}(\alpha_1^2 x)$. Substituting this into Equation 4.56, we find that:

$$(P_{\eta,\nu})_{\psi_2}(1) = (\frac{1}{4\pi})^{[F:\mathbb{Q}]} \sum_f \overline{a_f^{un}(\alpha_1)} a_f^{un}(\alpha_2) \int_u J_{\pi,\psi}(\alpha_1\alpha_2 u)\varphi(u)\chi_\infty(-\alpha_1\alpha_2 u) d^\times u \tag{4.57}$$

Definition 10. *We define normalized coefficients by $a_f^{nm}(\alpha) = \frac{\sqrt{\mathrm{Norm}(\alpha)}}{\sqrt{\mathrm{Norm}(\mathfrak{a})}} a_f^{un}(\alpha)$.*

Hypothesis 3. We now assume that φ is supported in the positive multiquadrant of totally positive elements, and $\chi_\infty = 1$.

Under this hypothesis, we combine Equation 4.57 with Lemma 17, and after some simple manipulations (which involve changing the variable inside φ) we obtain:

Proposition 15. *(Petersson-Kuznetsov Formula) Suppose $\alpha_1, \alpha_2 \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}$, and let φ be a compactly supported function on $F_{\infty,+}$. Then:*

$$\sum_{\substack{\epsilon \in \mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2 \\ c \in \mathfrak{I}\mathfrak{a}^{-1} - \{0\}}} \frac{1}{\mathrm{Norm}(c\mathfrak{a})} \varphi(\frac{\sqrt{\epsilon\alpha_1\alpha_2}}{c}) KS_\chi(\alpha_1, \alpha_2, \epsilon, c) = \tag{4.58}$$

$$\sum_f a_f^{nm}(\alpha_1) a_f^{nm}(\alpha_2) h_f(\varphi) + \textit{Continuous Spectrum Contribution} \tag{4.59}$$

Here the sum is taken over a set of representatives f for each of the representations π that occur discretely in $L^2_\chi(\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty))$, and where, if f lies in π, then

$$h_f(\varphi) = 2(\frac{1}{2\pi})^{[F:\mathbb{Q}]} \int_{F_\infty^\times} \varphi(u) \frac{1}{\mathrm{Norm}(u)} J_{\pi,\psi}(u^2) d^\times u$$

4.6.4 Comparison of normalizations of Fourier Coefficients

There are several different normalizations possible for the Fourier coefficients. Ours will agree, as regards dependence on D_F, with that of Shimura in his work on Hilbert modular forms, and also Bruggeman-Miatello [2].

Let f be a function belonging to a discrete subrepresentation π, so again

$$\pi \subset L^2_{\chi,\mathrm{Disc}}(\Gamma_0(\mathfrak{I};\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty))$$

Noting that $\mathrm{vol}(\Gamma_N \backslash N) = D_F^{1/2}\mathrm{Norm}(\mathfrak{a})$, we have defined normalized Fourier coefficients $a_f^{nm}(\alpha)$ in such a way that:

$$f(g) = \frac{1}{D_F^{1/2}\mathrm{Norm}(\mathfrak{a})^{1/2}} \sum_\alpha a_f^{nm}(\alpha) \frac{W_\pi^0(\mathrm{diag}(\alpha,1)g)}{\sqrt{\mathrm{Norm}(\alpha)}}$$

This agrees, as far as factors of D_F go, with two other normalizations. The first is that of Bruggeman-Miatello in [2]; they explicitly write out $W_\pi^0(\mathrm{diag}(\alpha,1)g)$, but when unwound their definition of Fourier coefficient is as above.

The other normalization is to expand, naively,

$$f(g) = \sum_{\alpha \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}} b_f(\alpha) W_\pi^0(\mathrm{diag}(\alpha,1)g)$$

and define the normalized coefficients via $b_f^{nm}(\alpha) = b_f(\alpha)\sqrt{\mathrm{Norm}(\mathfrak{d}\mathfrak{a}\alpha)}$. Here we note that $\mathfrak{d}\mathfrak{a}\alpha$ is an *integral* ideal. This is essentially the normalization used by Shimura, and we see that $b_f^{nm}(\alpha) = a_f^{nm}(\alpha)$.

4.6.5 A transformation formula

We state a transformation formula over $\mathbb{R}$; it then implies formula 4.8.

Let $\pi(t)$ be the representation of $\mathrm{GL}_2(\mathbb{R})$ that has trivial central character and corresponds to Maass forms of eigenvalue $1/4 + t^2$. In other words, it is that representation induced from the character $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto |xy^{-1}|^{it}$ of the torus. Let ψ be the character $x \to e(x)$ of $\mathbb{R}$.

The Bessel function $J_{\pi(t)}(u)$ is computed in [4], or, at least, the answer is quoted there. It is not difficult to derive directly. In any case:

$$J_{\pi(t)}(u^2) = -\pi u \frac{J_{2it}(4\pi u) - J_{-2it}(4\pi u)}{\sin(i\pi t)}$$

Let φ be a compactly supported function on $(0, \infty)$. If we define $h_\varphi(t)$ by:

$$h_\varphi(t) \equiv \frac{1}{2\pi} \int_{u \in \mathbb{R}} \varphi(u) \frac{1}{u} J_{\pi(t)}(u^2) d^\times u = \int_u B_{2it}(4\pi u) \varphi(u) u^{-1} du$$

where $B_\nu(x) = (2 \sin(\pi\nu/2))^{-1}(J_{-\nu}(x) - J_\nu(x))$, then one checks that:

$$\hat{h}_\varphi(k) = \frac{1}{2} \Delta(2 \cosh(\pi k))$$

where $\hat{h_\varphi}(k) = \int_t h(t) e^{2\pi i k t} dt$ and $\Delta(k) = \int_x \varphi(|x|) x^{-1} e^{2\pi i k x} dx$. This may be deduced, for instance, from the results in the Appendix, Subsection 6.3.1. This, accounting for the appropriate normalization factors, implies Equation 4.8.

Chapter 5

Automorphic forms of Galois type

5.1 Introduction

Of course, one would like to extend the work of the previous chapters to cover classification of π such that $L(s, \mathrm{Sym}^r\pi)$ has a pole, for some $r \geq 3$. One expects, if such a form is non-dihedral, that it corresponds to a Galois representation. For example, if π corresponds to an icosahedral Galois representation with image in $\mathrm{SL}_2(\mathbb{C})$, then $L(s, \mathrm{Sym}^{12}\pi)$ has a pole.

We shall briefly discuss in this chapter the reason that the techniques of this thesis *fail* to classify such forms in the strong sense that one is able to treat the dihedral case. This was already pointed out by Sarnak in his letter to Langlands, and analyzed in sufficient detail to compute the exact bounds needed on the exponential sums in order to make an improvement. In view of this, we will keep our discussion to a brief summary of the main obstacle.

We will then relax our aim somewhat: rather than obtaining a spectral formula that will precisely classify the forms, we use the same technique to *bound the number* of these "Galois type" automorphic forms.

As pointed out by Sarnak, it is essentially this idea that underlies the paper of

Duke [5] in which he bounds the dimension of weight 1 holomorphic forms for $\Gamma_0(q)$. (The trivial bound is that the dimension of this space is $o(q)$; Duke improved it to $O(q^{11/12+\epsilon})$, and we will improve it further to $O(q^{6/7+\epsilon})$. This improvement – by what is essentially exactly the same technique – was simultaneously obtained by P. Michel. I would like to thank him for interesting discussions on this work and possible extensions. We have jointly published this, over $\mathbb{Q}$, in [15].)

The precise result is contained in Theorem 6; it is a result that holds over a number field and over $\mathbb{Q}$ gives the result mentioned. It is also possible with some more care to achieve uniformity in the number field; this will save, I believe, a power of the discriminant. This extension has not been treated here (nor have I worked it out in detail, so the suggested conclusion should be regarded with a little caution.)

5.2 Higher Symmetric Powers

As remarked earlier, this material is contained in Sarnak, [17]. We shall sketch how one can set up a spectral sum that would isolate π for which a higher symmetric power has a pole, and the obstacle encountered in carrying through the procedure of the previous chapters.

Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$; for simplicity we shall assume that π is unramified at all finite places. At each prime p, let A_p be the Hecke matrix, so $\mathrm{tr}(A_p)$ is the pth coefficient in the L-series of π. To make matters even simpler, we shall assume the Ramanujan conjecture for all such forms (that is, for all p, A_p is conjugate to a unitary matrix). The computation that we are about to do makes sense without this assumption, but its interpretation is a little clearer with it. We denote by $\lambda_\pi(n)$ the nth coefficient in the L-series attached to π.

The finite part of the symmetric rth power L-function is given by:

$$L(s, \mathrm{Sym}^r \pi) = \prod_p \det(1 - \mathrm{Sym}^r(A_p)p^{-s})^{-1}$$

Now, for each p, $\det(1 - \mathrm{Sym}^r(A_p)p^{-s})^{-1} = \sum_{t\geq 0} \mathrm{tr}(\mathrm{Sym}^t(\mathrm{Sym}^r A_p))p^{-st} = 1 + \lambda_\pi(p^r)p^{-s} + O(p^{-2s})$ where, on account of the Ramanujan conjecture that we are assuming, the implicit constant in the O is independent of p. In particular, we may write

$$\det(1 - \mathrm{Sym}^r A_p p^{-s})^{-1} = (\sum_{t\geq 0} \lambda_\pi(p^{rt})p^{-ts})(1 + O(p^{-2s}))$$

where the implicit constant is still independent of p. Consequently – and possibly removing Euler factors associated to small prime factors so that the $O(p^{-2s})$ cause no problems –

$$L(s, \mathrm{Sym}^r \pi) = E(s) \sum_n \lambda_\pi(n^r)n^{-s}$$

where $E(s)$ is a function with analytic continuation to $\Re(s) > 1/2$, and $E(1) \neq 0$. In particular, the existence of a pole for $L(s, \mathrm{Sym}^r \pi)$ can be analyzed in terms of the limit:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X} \lambda_\pi(n^r)$$

which is nonzero if and only if the L-function has a pole. The analogue of our earlier analysis, then, is to carry out the spectral sum:

$$\sum_f \frac{1}{X} \sum_{n<X} a_{n^r}(f)\overline{a_m(f)}h(t_f)$$

where the sum is over, say, an orthonormal basis of Maass forms for $\mathrm{SL}_2(\mathbb{Z})$. This is done, again, via the Kuznetsov formula. Take, for example, the case $m = 1$. The geometric side of the formula will then look like (the sum of Chapter 1 now generalized to $r \neq 2$):

$$\sum_{c=1}^{\infty} \sum_{n<X} S(n^r, 1, c) \varphi(\frac{4\pi n^{r/2}}{c})$$

The Kloosterman sum $S(n^r, 1, c)$ is dependent only on the residue class of n mod c. The technique used in Chapter 2 was to break the n-sum into residue classes mod c. Since φ is compactly supported on $(0, \infty)$, the n-sum has length around $c^{2/r}$.

If $r = 2$, the n-sum has length c, and therefore one expects the n sum to include all c residue classes; it is therefore reasonable to expect that the exponential sums that occur are *complete* – summed over complete residue classes mod c. These exponential sums are just the $S(\nu; c, \alpha)$ that were analyzed in Chapters 2 and 4.

If $r > 2$, however, the n sum becomes much shorter: even if $r = 3$, the sum over n is only of length about $c^{2/3}$. The exponential sums that occur are therefore fundamentally *incomplete* and their analysis becomes progressively harder as r increases. Even the gap between $r = 2$ and $r = 3$ is enormous.

Therefore, an exact treatment of higher symmetric powers in this fashion is expected to be a rather difficult endeavor. The goal of the rest of this chapter is to show how one obtains interesting, though approximate, results, by relaxing our requirements: rather than trying to exactly characterize those f such that $m(f, \mathrm{Sym}^{12}) = 1$, we merely try to bound the *number* of such f.

This is interesting for other reasons: such forms are associated to Galois representations, and one can deduce interesting arithmetic information from analytic bounds; see [5] and [18].

Remark 15. In what follows, it is more convenient, rather than spectrally averaging the sum

$$\sum_{n<X} a_{n^r}(f)$$

to spectrally average a somewhat more convoluted sum involving sums over various powers of primes. This has the advantage of making the result *unconditional*; without it, we would require various hypotheses (regarding Siegel zeros) on the L-functions attached to Galois representations. On the other hand, the more convoluted approach that we follow does sacrifice some powers of log, but that is not particularly important.

5.3 Estimate of the number of automorphic forms of Galois type

Let F be a totally real number field. (Again, this assumption is imposed so that the Bessel transforms are familiar, but the reasoning below should hold for any field.) An automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is said to be of *Galois type* if there is a Galois representation $\rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\mathbb{C})$ that corresponds, under the local Langlands correspondence, to π at all places. For our purposes, it suffices to assume this at archimedean places and at all finite places prime to the conductor.

In this section, the notation and the normalization of Fourier coefficients will be as in Chapter 4 (see especially Definition 4); in particular, $\mathfrak{d}$ is the different of F, D_F the discriminant, $\mathfrak{o}_F$ the ring of integers and $\mathfrak{o}_F^*$ the group of units, and h_F the class number.

We say that a Galois type π is of *weight* 0 if it is *even*, that is to say, for every infinite place of F, the corresponding complex conjugation acts in ρ_π with determinant 1; equivalently, $\pi_\infty = \prod_{v|\infty} \pi_v$ where each π_v is a representation associated to weight 0, Maass forms of eigenvalue 1/4.

We will prove a bound for the number of weight 0, Galois type automorphic forms, on GL_2 over F, with specified central character and conductor $\mathfrak{q}$, as $\mathrm{Norm}(\mathfrak{q}) \to \infty$.

The assumption of weight 0 is not at all important; in particular, the argument and bound applies to weight 1 (at every place) holomorphic Hilbert modular forms over a totally real field. A uniform proof over $\mathbb{Q}$ is given in [15]. Indeed, the argument is essentially that of [15] generalized, and also shares many features with [5]. Consequently, we will only sketch certain points that are treated there in more detail; in each case, the relevant argument generalizes from $\mathbb{Q}$ to F.

Theorem 6. *Let $\mathfrak{q}$ be a prime ideal of $\mathfrak{o}_F$ (primality is not essential). Let χ be a Grössencharacter of conductor dividing $\mathfrak{q}$, and let $N_\chi(\mathfrak{q})$ be the number of automorphic forms on GL_2 over F of Galois type, of weight 0, and with central character χ.*

One then has the bound, for any $\epsilon > 0$,

$$N_\chi(\mathfrak{q}) \ll_{F,\epsilon} \mathrm{Norm}(\mathfrak{q})^{6/7+\epsilon} \tag{5.1}$$

Over $\mathbb{Q}$, this Theorem was proved by Duke, with a weaker exponent (11/12 instead of 6/7). It is also helpful to keep in mind that the "trivial bound" is $N_\chi(\mathfrak{q}) \ll \mathrm{Norm}(\mathfrak{q})$; indeed, the trivial bound is determined by the volume of the symmetric space, and the index of $K_0(\mathfrak{q})$ in $K_0(\mathfrak{o}_F)$ – notation of Subsection 4.2.1 – is $\mathrm{Norm}(\mathfrak{q})+1$.

One knows, by a theorem of Deligne-Serre, that all holomorphic forms of weight 1 (over $\mathbb{Q}$) are of Galois type; a similar result holds over totally real fields. It follows that Theorem 6 gives a bound on the number of automorphic forms of a specified infinity type, e.g. Theorem 2.

We need to use a slightly sharper form of the Kuznetsov formula than that used in Chapter 4; this sharper form will allow more versatility in the spectral test function. (Previously, we were carrying through a limit while fixing the level. If varying the level, one needs finer control.) This sharper form of the Kuznetsov formula may be found in Bruggeman-Miatello [2], and the Weil-type bounds for Kloosterman sums that we will use may be found in [1]. One must modify the formula for the influence of multiple ideal classes, but this is not difficult. Let $\chi, \mathfrak{q}$ be as in the statement of Theorem 6. Let χ_f, χ_∞ be defined as in Chapter 4; they are, respectively, the characters of the finite ideles and of $F_\infty^\times$ obtained by restriction from χ.

Let $\mathcal{B}$ be an L^2-basis for the discrete Laplacian spectrum on

$$L^2_\chi(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/F_\infty^\times K_0(\mathfrak{q})\prod_{v|\infty}\mathrm{SO}_2(F_v))$$

That is, the space of L^2_{discrete}-automorphic forms on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ that transform under $K_0(\mathfrak{q})$ by χ_f (as in Chapter 4), and are fixed by $\prod_{v|\infty}\mathrm{SO}_2(F_v)$ and the center $Z(F_\infty)$. Note that χ_∞ is trivial on account of the weight zero assumption, whence the assumption on the action of the center $Z(F_\infty)$.) In particular, this space includes the Galois type forms of weight 0 with central character χ.

The Kuznetsov formula we will use will be a sum over the set $\mathcal{B}$, which amounts to considering only "Hilbert Maass forms of weight 0." As in Chapter 4, Subsection 4.2.1, this set really includes all forms whose central character is an unramified twist of χ.

Recall from Chapter 4 that we fixed A, a set of representatives for ideal classes of F. For each $\mathfrak{a} \in A$ and $\mathbf{f} \in \mathcal{B}$, one has an associated set of Fourier coefficients $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ for $\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. With this notation

Proposition 16. *(Kuznetsov formula, after Bruggeman-Miatello) Let $\mathfrak{a}_1, \mathfrak{a}_2 \in A$ and $\alpha_i \in \mathfrak{d}^{-1}\mathfrak{a}_i^{-1}$. Then:*

$$\sum_{\mathbf{f}\in\mathcal{B}} a_{\mathbf{f}}(\mathfrak{a}_1,\alpha_1)\overline{a_{\mathbf{f}}(\mathfrak{a}_2,\alpha_2)}h_{\mathbf{f}}(\varphi) + \textit{Continuous Spectrum Contribution} = \delta_{\mathfrak{a}_1,\mathfrak{a}_2}\cdot$$

$$\left(h_0(\varphi)\delta'_{\alpha_1,\alpha_2}D_F^{1/2} + \sum_{c\in\mathfrak{q}\mathfrak{a}_1^{-1}-\{0\}}\ \sum_{\epsilon\in\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2}\frac{1}{\mathrm{Norm}(c\mathfrak{a}_1)}KS_\chi(\alpha,\epsilon\alpha';c)\varphi(\frac{\alpha\alpha'\epsilon}{c^2})\right)$$

where $\delta'(\alpha_1,\alpha_2) = 1$ if $\alpha_1\alpha_2^{-1}$ is a unit, and 0 otherwise. φ is a function on $F \otimes \mathbb{R} = \mathbb{R}^d$ that is a product of functions at each place. φ is assumed to be supported on $F_{\infty,+} = (\mathbb{R}^+)^d$, the multiquadrant of totally positive elements. $\mathbf{f} \mapsto h_{\mathbf{f}}(\varphi)$ is a spectral transform of φ. The Kloosterman sum is defined as $KS_\chi(A, B; C) =$

$\sum_{x\in(\mathfrak{o}_F/\mathfrak{a}_1 C)^\times} \chi_f(x) e(\mathrm{tr}(\frac{Ax+Bx^{-1}}{C}))$; *it strictly depends on* $\mathfrak{a}_1$*, but we suppress that dependence.*

Remark 16. The phrasing of this formula is a little different to that of Chapter 4; namely, we have replaced $\mathfrak{o}_F^{*,+}/(\mathfrak{o}_F^*)^2$ with $\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2$, and have replaced $\varphi(x)$ by $\varphi(x^2)$. There is no change in the underlying substance; this new phrasing just allows us to avoid the "total positivity" Remark 10 of Chapter 4, which would be a little annoying to have to deal with in what follows. Of course, this changes the transformation rule $\varphi \mapsto h_{\mathbf{f}}(\varphi)$, but we will have no need for the explicit formula, using only Lemma 19 below.

Lemma 19. *We may choose* φ *so that the spectral transform satisfies* $h_{\mathbf{f}}(\varphi) > 0$ *for all* $\mathbf{f}$*, and* φ *satisfies the following decay estimates:*

$$\varphi(x) \ll_k \prod_v \min(x_v^k, 1), \text{ for } x = (x_v) \in F_\infty = \prod_{v|\infty} F_v \tag{5.2}$$

this holding for all k*.*

Proof. This is well-known over $\mathbb{Q}$ – see for example the explicit functions in [15]; the result follows over F by taking φ to be a product of the $\mathbb{Q}$-test functions. ☐

As far as I know, one *cannot* achieve this positivity for the larger spectral sum used in Chapter 4 – the restriction to a particular weight (weight 0 in our case) is essential.

In our normalization, the average size of the coefficients $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ is approximately $(h_F D_F \mathrm{Norm}(\mathfrak{q}))^{-1/2}$. Let $\lambda_{\mathbf{f}}(\mathfrak{m})$ denote the $\mathfrak{m}$th Hecke eigenvalue of $\mathbf{f}$, for $\mathfrak{m}$ an integral ideal of F. As in Chapter 4, the coefficients $\lambda_{\mathbf{f}}$ are related to $a_{\mathbf{f}}$ via $a_{\mathbf{f}}(\mathfrak{a}, \alpha) = C_{\mathbf{f}} \lambda_{\mathbf{f}}(\mathfrak{a}\alpha\mathfrak{d})$, where $|C_{\mathbf{f}}|^2$ is essentially the reciprocal of a special value of an L-function.

Lemma 20. *If* $\mathbf{f}$ *satisfies the Ramanujan-Petersson conjecture – in particular, if* $\mathbf{f}$ *is associated to a Galois representation – then the constant* $C_{\mathbf{f}} \gg_{F,\epsilon} \mathrm{Norm}(\mathfrak{q})^{-1/2-\epsilon}$*.*

Proof. This is proved as in [5]. The idea is as follows: $|C_{\mathbf{f}}|^2$ is the reciprocal of a special value of an L-function at 1. This L-function can be estimated with a very short partial sum; the assumption on Ramanujan-Petersson gives a bound on the coefficients that occur, whence an upper bound for the L-function and a lower bound on $C_{\mathbf{f}}$. Indeed, one can establish the required bound on the L-function without using the fact that $\mathbf{f}$ is of Galois type, by using an argument of Iwaniec; the bound is then very slightly weaker (only by $\mathrm{Norm}(\mathfrak{q})^{\epsilon}$). □

The L-series of $\mathbf{f}$ is given in terms of the $\lambda_{\mathbf{f}}$ via $L(s, \mathbf{f}) = \sum_{\mathfrak{J}} \frac{\lambda_{\mathbf{f}}(\mathfrak{J})}{\mathrm{Norm}(\mathfrak{J})^s}$, when the sum is extended over integral ideals $\mathfrak{J}$ of F.

5.3.1 Existence of an amplifier

As in Duke [5] or Wong [18], one can construct an "amplifier" that picks out forms of Galois type. (Recall Remark 15.)

For example, suppose $\mathbf{f}$ is a form with central character χ that is associated to an icosahedral Galois representation. (The composition of the Galois representation with determinant is then the character of $\mathrm{Gal}(\overline{F}/F)$ associated, by class field theory, to χ.)

Fix an integer N, which will be optimized at the end of the argument. Following Duke and Wong, we define a sequence $a_{\mathfrak{J}}^{\mathrm{Ico}}$ (an "amplifier" for icosahedral forms), indexed by integral ideals $\mathfrak{J}$ of F and so that, if $\mathfrak{p}$ denotes a prime ideal of F,

$$a_{\mathfrak{J}}^{\mathrm{Ico}} = \begin{cases} \chi(\mathfrak{p})^6, & \text{for } \mathfrak{J} = \mathfrak{p}^{12}, \mathrm{Norm}(\mathfrak{p}) \leq N^{1/12},\ (\mathfrak{p}, \mathfrak{q}) = 1 \\ -\chi(\mathfrak{p})^4, & \text{for } \mathfrak{J} = \mathfrak{p}^{8}, \mathrm{Norm}(\mathfrak{p}) \leq N^{1/12},\ (\mathfrak{p}, \mathfrak{q}) = 1 \\ -\chi(\mathfrak{p}), & \text{for } \mathfrak{J} = \mathfrak{p}^{2}, \mathrm{Norm}(\mathfrak{p}) \leq N^{1/12},\ (\mathfrak{p}, \mathfrak{q}) = 1 \\ 0 & \text{else.} \end{cases}$$

The point of this is that, for any prime ideal $\mathfrak{p}$ prime to $\mathfrak{q}$, we have the equality

$$-\chi(\mathfrak{p})\lambda_{\mathbf{f}}(\mathfrak{p}^2) - \chi(\mathfrak{p})^4\lambda_{\mathbf{f}}(\mathfrak{p}^8) + \chi(\mathfrak{p})^6\lambda_{\mathbf{f}}(\mathfrak{p}^{12}) = 1$$

This is a consequence of $\mathbf{f}$ being associated to an icosahedral Galois representation. (See [18]).

For convenience, we set T to be the number of prime ideals of norm less than $N^{1/12}$; thus $T \gg_\epsilon N^{1/12-\epsilon}$.

In particular, the sum

$$\sum_{\mathfrak{I}} a_{\mathfrak{I}}^{\mathrm{Ico}} \lambda_{\mathbf{f}}(\mathfrak{I}) \gg \sum_{\substack{\mathrm{Norm}(\mathfrak{p})<N^{1/12} \\ \mathfrak{p}\,\mathrm{prime}}} 1 = T \tag{5.3}$$

Proof. (of Theorem) We form the spectral sum, with $h_{\mathbf{f}}$ and φ chosen, and fixed as $\mathfrak{q}$ varies, according to Lemma 19,

$$S = \sum_{\mathbf{f}\in\mathcal{B}} h_{\mathbf{f}}(\varphi)|C_{\mathbf{f}}|^2 |\sum_{\mathfrak{I}} a_{\mathfrak{I}}^{\mathrm{Ico}} \lambda_{\mathbf{f}}(\mathfrak{I})|^2 + \text{Continuous Spectrum Contribution}$$

where the continuous spectrum contribution is defined analogously, and we suppress it; the only needed fact is that it is positive. See also [15] for a precise expression over $\mathbb{Q}$.

By Lemma 20 and Equation 5.3, we see that:

$$S \gg N_\chi(\mathfrak{q}) T^2 \mathrm{Norm}(\mathfrak{q})^{-1-\epsilon} \tag{5.4}$$

Now, we split up the sum S into h_F sub-sums depending on the ideal class of $\mathfrak{I}$. Recall A is a set of representatives for the ideal classes of F. Each sub-sum involves about $h_F^{-1}T$ ideals. (We are not aiming for uniformity in F, so factors of h_F^{-1} will eventually be neglected.) As in Chapter 4, it becomes expedient to introduce a "twist' by the different in considering the ideal class representatives; we do this in Equation 5.5 below.

$$S = \sum_{\mathbf{f}} h_{\mathbf{f}}(\varphi) | \sum_{\mathfrak{a} \in A} \sum_{\mathfrak{J} \sim \mathfrak{a}\mathfrak{d}} a^{\mathrm{Ico}}_{\mathfrak{J}} C_{\mathbf{f}} \lambda_{\mathbf{f}}(\mathfrak{J})|^2 + \text{Continuous Spectrum Contribution} \tag{5.5}$$

We expand the inner sum and use the Kuznetsov formula Proposition 16, together with the comments preceding Lemma 20.

$$S = \sum_{\mathfrak{a}} \sum_{\substack{r \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^* \\ s \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^*}} a^{\mathrm{Ico}}_{(r)\mathfrak{a}\mathfrak{d}} \overline{a^{\mathrm{Ico}}_{(s)\mathfrak{a}\mathfrak{d}}} \left(h_0 D_F^{1/2} \delta'_{r,s} + \sum_{\epsilon \in \mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2} \sum_{c \in \mathfrak{a}^{-1}\mathfrak{q}} \frac{KS_\chi(r,s;c)}{\mathrm{Norm}(c\mathfrak{a})} \varphi(\frac{rs\epsilon}{c^2}) \right) \tag{5.6}$$

Here $\mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^*$ means elements of $\mathfrak{a}^{-1}\mathfrak{d}^{-1}$ modulo multiplication by units, and $(r), (s)$ denote the ideals generated by r and s.

Each internal sum in Equation 5.6 is over about $h_F^{-1}T$ values of r and s. That is, given $\mathfrak{a}$ there are about $h_F^{-1}T$ classes $[r] \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^*$ for which $a^{\mathrm{Ico}}_{(r)\mathfrak{a}\mathfrak{d}} \neq 0$. Since we are not looking for uniformity in F, we will simply use the bound that there are $O(T)$ values of r, s.

As far as the coefficients $a^{\mathrm{Ico}}_{\mathfrak{J}}$ go, we apply only the trivial estimate $|a^{\mathrm{Ico}}_{\mathfrak{J}}| \ll 1$. For fixed $\mathfrak{a}$, in view of the rapid decay of φ near 0, the c sum extends over all c with $\mathrm{Norm}(c)^2 \ll \mathrm{Norm}(rs)^{1+\epsilon}$. Since the coefficients $a^{\mathrm{Ico}}_{(r)\mathfrak{a}\mathfrak{d}}$ are nonzero only for $\mathrm{Norm}((r)\mathfrak{a}\mathfrak{d}) \ll N$, we see that $\mathrm{Norm}(r) \ll N\mathrm{Norm}(\mathfrak{a})^{-1}D_F^{-1}$, and so the sum includes only c with:

$$\mathrm{Norm}(c\mathfrak{a}) \ll (ND_F^{-1})^{1+\epsilon} \ll_F N^{1+\epsilon} \tag{5.7}$$

Put $\mathfrak{c} = (c)\mathfrak{a}$ – it is an ideal contained in $\mathfrak{q}$. We shall use the "Weil estimate": if $c \in \mathfrak{a}^{-1}$, $r, s \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}$ and $\mathfrak{c} = (c)\mathfrak{a}$, then

$$KS_\chi(r,s;c) \ll_F \mathrm{Norm}((r\mathfrak{a}\mathfrak{d}, s\mathfrak{a}\mathfrak{d}, \mathfrak{c}))^{1/2} \mathrm{Norm}(\mathfrak{c})^{1/2+\epsilon}$$

Here $(\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ means the greatest common factor of the ideals $\mathfrak{B}_i$. The derivation of this is similar to that over $\mathbb{Q}$: appeal to an algebro-geometric estimate at primes; one can use a direct estimate ("stationary phase"!) at prime powers, and use multiplicativity to piece them together. Without the character χ, this bound is contained in [1]. The $\mathrm{Norm}((r\mathfrak{a}\mathfrak{d}, s\mathfrak{a}\mathfrak{d}, \mathfrak{c}))^{1/2}$ factor is entirely harmless, since "very few" r, s have a common factor with $\mathfrak{c}$. (To formally justify this, split the sum in Equation 5.6 up into classes according to the value of $\mathrm{Norm}((r\mathfrak{a}\mathfrak{d}, s\mathfrak{a}\mathfrak{d}, \mathfrak{c}))^{1/2}$.)

We wish to sum over the ideals $\mathfrak{c}$ rather than the elements c. In order to do this, we need to bound the sum over all c corresponding to a given $\mathfrak{c}$, that is, fixing c_0 we wish to sum over all all $c = c_0\epsilon'$ with $\epsilon' \in \mathfrak{o}_F^*$. To do this, we utilize Lemma 15 of Chapter 4 and we use the rapid decay (expressed in Lemma 19) of φ. We obtain, for any $k > 0$ and any $\delta > 0$,

$$|\sum_{c=c_0\epsilon'} \varphi(rsc^{-2})| \ll_{k,\delta} \min(\mathrm{Norm}(rs/c^2)^{\delta}, \mathrm{Norm}(rs/c^2)^{k-\delta})$$

Considerations such as those leading to Equation 5.7 show that this amounts to say that the φ-sum associated to each $\mathfrak{c}$ contributes, in total, at most N^{ϵ} and the sum of Equation 5.6 is essentially restricted to $\mathrm{Norm}(\mathfrak{c}) \ll N^{1+\epsilon}$.

Finally, in Equation 5.6, the sum over $\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2$ is over a finite set and only contributes an additional constant.

Combining all the previously-mentioned estimates, we obtain for all $\epsilon > 0$:

$$S \ll_{F,\epsilon} N^{\epsilon} \left(T + T^2 \sum_{\mathfrak{q}|\mathfrak{c}, \mathrm{Norm}(\mathfrak{c}) \ll N^{1+\epsilon}} \mathrm{Norm}(\mathfrak{c})^{-1/2+\epsilon} \right)$$

and therefore

$$S \ll_{F,\epsilon} \left(T + T^2 \mathrm{Norm}(\mathfrak{q})^{\epsilon} \frac{N^{1/2}}{\mathrm{Norm}(\mathfrak{q})} \right) N^{\epsilon}$$

Combining this estimate with Equation 5.4, we find:

$$N_\chi(\mathfrak{q}) \ll_{F,\epsilon} \left(\text{Norm}(\mathfrak{q})^{1+\epsilon} T^{-1} + \text{Norm}(\mathfrak{q})^{\epsilon} N^{1/2}\right) N^{\epsilon}$$

The optimal value for N is $N = \text{Norm}(\mathfrak{q})^{12/7}$, which gives $T = \text{Norm}(\mathfrak{q})^{1/7-\epsilon}$. This verifies the bound of Theorem 6 for $\mathbf{f}$ that are associated to icosahedral Galois representations. Similarly, one obtains bounds for dihedral, octahedral and tetrahedral which are strictly better (because the corresponding sequences $a^{\text{Dihedral}}, a^{\text{Oct}}, a^{\text{Tetra}}$ are less sparse; this is closely related to the fact that one does not need to go to the *twelfth* symmetric power to find a pole.)

Therefore, Theorem 6 holds for all Galois-type forms together. □

Chapter 6

Appendix

6.1 Classification of Dihedral Forms

We discuss here what is expected, from general theory, about the classification of "dihedral" forms over a number field. Let F be a number field and L_F the conjectural "Langlands group" whose complex representations parameterize automorphic representations. We are only interested in heuristics here, and so the fact that this conjectural is not problematic.

Let π be an automorphic, cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$; suppose it is parameterized by a map $\rho_\pi : L_F \to \mathrm{GL}_2(\mathbb{C})$.

Suppose the symmetric-square L-function of π has a pole. Then, one expects the representation ρ_π parameterizing π to factor through the orthogonal group $O(2, \mathbb{C})$; in particular, by composition with determinant, we find a map $L_F \to \{\pm 1\}$. This map should factor through the Galois group $\mathrm{Gal}(\overline{F}/F)$; in particular, it determines a quadratic extension K, and a map $L_K \to \mathrm{SO}(2, \mathbb{C}) = \mathbb{G}_m(\mathbb{C})$.

One then expects that there is a quadratic extension K of F, and a $GL(1)$ form ω (i.e. a Grossencharacter) of K with the property that $\omega^\sigma = \omega^{-1}$, where σ is the Galois automorphism of K over F, so that the L-function of π and $L(s, \omega, K)$ agree. The central character of π is the Grossencharacter associated naturally with K. Denote by $\pi(\omega)$ the π that one expects to match such an ω in this sense.

One can ask, conversely, given K and ω, when the representation $\pi(\omega)$ is orthogonal. Indeed, all the parameterizing maps in this case factor through the Weil groups: ω determines a character $W_K \to \mathbb{C}^*$, and $\pi(\omega)$ is parameterized by the map $\mathrm{Ind}_{W_K}^{W_F}\omega$. This representation is automatically self-dual, and is orthogonal irreducible if and only if its composition with determinant is nontrivial. To analyze this one can appeal to the following:

Lemma 21. *Let H be a finite index subgroup of G, and let ω be a character of H. Let* Ver *denote the transfer $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$; let* sgn *be the sign character of G acting by permutation on H-cosets. Then:*

$$\det \circ \mathrm{Ind}_H^G \omega = \mathrm{sgn} \cdot (\omega \circ \mathrm{Ver})$$

Since the transfer map $W_F^{\mathrm{ab}} \to W_K^{\mathrm{ab}}$ corresponds under the reciprocity isomorphism to the inclusion $\mathbb{A}_F^\times/F^\times \to \mathbb{A}_K^\times/K^\times$, we conclude that $\pi(\omega)$ is orthogonal precisely when ω is trivial on $\mathbb{A}_F$.

We conclude with the following: one expects orthogonal cuspidal GL_2 forms over F to correspond to pairs (K, ω), where K is a quadratic extension and ω a Grössencharacter of K such that $\omega^\sigma = \omega^{-1}$, such that $\omega^2 \neq 1$. (The last condition is to ensure irreducibility; from an ω whose square is 1 we obtain an Eisenstein series.)

6.2 Concrete Interpretation over $\mathbb{Q}$

Now specialize to $F = \mathbb{Q}$; the results here go through unchanged for a totally real number field. The aim of this section is to give a fairly concrete interpretation of Section 6.1.

As before, let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic extension, σ the nontrivial automorphism of K, and $K_\infty = K \otimes \mathbb{R}$. D is chosen to be a fundamental discriminant, so $|D|$ is the discriminant of K over $\mathbb{Q}$.

Characters of prescribed conductor

We start by explicitly describing the characters ω of $\mathbb{A}_K^\times/\mathbb{A}_\mathbb{Q}^\times K^\times$ such that $\pi(\omega)$ has prescribed conductor.

Note that triviality of ω on $\mathbb{A}_\mathbb{Q}^\times$ implies $\omega^\sigma = \omega^{-1}$. The most convenient phrasing will be in terms of orders of K. We state most of the relevant facts without proof.

Definition 11. *For each finite place v of K, let $\mathfrak{o}_v$ be the maximal compact subring of K_v and $\mathfrak{p}_v$ the maximal ideal. The local conductor of ω at v is defined as $\mathfrak{f}_v = \mathfrak{p}_v^m$, where m is the least non-negative integer such that ω is trivial on $1 + \mathfrak{p}_v^m$. The global conductor of ω is defined as $\mathfrak{f}_\omega = \prod_{v \text{ finite}} \mathfrak{f}_v$.*

The conductor of the GL(2) automorphic form $\pi(\omega)$ is then given by

$$N = |D|\text{Norm}(\mathfrak{f}_\omega)$$

since $|D|$ is the discriminant of K. To distinguish the two, we shall refer to $\mathfrak{f}_\omega$ as the K-conductor of ω.

Lemma 22. *Suppose ω is a character of $\mathbb{A}_K^\times/K^\times$ that is trivial on $\mathbb{A}_\mathbb{Q}^\times$. Then the K-conductor $\mathfrak{f}_\omega = (f)$ for some $f \in \mathbb{Z}$, i.e. it is the extension of an ideal of $\mathbb{Q}$, and the corresponding $\pi(\omega)$ has conductor $|D|f^2$.*

Lemma 23. *Let $f \in \mathbb{Z}$. For v a finite place of K, above the prime p of $\mathbb{Q}$, let $U_v(f)$ be the open subgroup of $K_v^\times$ given by $(1 + f\mathfrak{o}_v)\mathbb{Z}_p^\times$.*

Let $U(f) = \prod_v U_v(f) = \prod_v (1 + f\mathfrak{o}_v) \cdot \prod_p \mathbb{Z}_p^\times$. A character ω, trivial on $\mathbb{A}_Q$, has K-conductor dividing (f) if and only if it is trivial on $U(f)$.

Characters for which $\pi(\omega)$ has conductor dividing $N = |D|f^2$, and equivalently ω has K-conductor dividing (f), are thus just the characters of

$$\tilde{C}(f) = \mathbb{A}_K^\times/(\mathbb{A}_Q^\times K^\times U(f)) = \mathbb{A}_K^\times/\mathbb{R}^\times K^\times U(f)$$

We are interested, then, in understanding $\tilde{C}(f)$. (We include the tilde to remind the reader that $\tilde{C}(f)$ is much larger than the class group, since it includes an archimedean torus.)

Let Λ be the group of units of $\mathfrak{o}_K$, the ring of integers in K. Let $\Lambda(f)$ be the group of units of K that lie in $U(f)$ at finite places. Then $\tilde{C}(f)$ fits into an exact sequence:

$$K_\infty^\times/\mathbb{R}^\times\Lambda(f) \rightarrowtail \tilde{C}(f) \twoheadrightarrow \mathbb{A}_K^\times/(K_\infty^\times K^\times U(f)) \tag{6.1}$$

The final group $C_{D,f} = \mathbb{A}_K^\times/(K_\infty^\times K^\times U(f))$ fits into an exact sequence:

$$\Lambda/\Lambda(f) \to U(1)/U(f) \to C_{D,f} \to \mathbb{A}_K^\times/(K_\infty^\times U(1))$$

Note that the final group is just the class group, with cardinality the class number h_D. The cardinality of $C_{D,f}$ therefore equals:

$$h_D f \prod_{p|f}(1-\chi(p)/p)/[\Lambda : \Lambda(f)]$$

where the product is over primes p dividing f. We shall denote this number by $h_{D,f}$. It is, in fact, the class number of the quadratic order with discriminant Df^2.

On the other hand, we can easily describe the initial group of Equation 6.1: $K_\infty^\times/\mathbb{R}^\times\Lambda(f)$. If K is a *real* field, $K_\infty = \mathbb{R}\oplus\mathbb{R}$ and this quotient is isomorphic to $\mathbb{R}^\times/\Lambda(f)$, where $\Lambda(f)$ is embedded in $\mathbb{R}^\times$ via either of the embeddings of K into $\mathbb{R}$. If $\Lambda(f)$ contains an element of norm -1, this group is a connected torus (indeed an S^1); otherwise, it is disconnected. If K is a complex field, i.e. $K_\infty = \mathbb{C}$, then

$K_\infty^\times/\mathbb{R}^\times\Lambda(f)$ is always a connected torus.

Orders

Let $\mathfrak{o}_{D,f}$ be the (unique) order of discriminant Df^2; the groups mentioned above will be easily interpretable in terms of $\mathfrak{o}_{D,f}$. By definition, $\mathfrak{o}_{D,f} = \mathbb{Z} + f\mathfrak{o}_D$, a subring of $\mathbb{Q}(\sqrt{D})$. $U_v(f)$ is the unit group in the closure $(\mathfrak{o}_{D,f})_v$ of $\mathfrak{o}_{D,f}$ in K_v, and as before $U(f) = \prod_v U_v(f)$. $\Lambda(f)$ is the group of units of $\mathfrak{o}_{D,f}$. The group $C_{D,f}$ is the class group of $\mathfrak{o}_{D,f}$, and $h_{D,f}$ its class number.

6.2.1 Fourier Coefficients

Let $\lambda_m(\omega)$ be the m-th coefficient of the L-series of $\pi(\omega)$. Suppose m is coprime to Df^2. Let (m) be the ideal generated by m. Then

$$\lambda_m(\omega) = \sum_{N(\mathfrak{I})=(m)} \omega(\mathfrak{I}) \tag{6.2}$$

where $\mathfrak{I}$ ranges over ideals and we regard ω as a character of ideals in the natural way.

Now let us sum $\lambda_m(\omega)$ over all ω of K-conductor dividing (f) with a prescribed infinity component ω_∞, i.e. restriction to $K_\infty^\times = (K \otimes \mathbb{R})^\times$. As we have seen, the set of such is a principal homogeneous space for the group of characters of $C_{D,f} = \mathbb{A}_K^\times/K_\infty^\times K^\times U(f)$. When we sum $\lambda_m(\omega)$ over such ω, the sum over ideals $\mathfrak{I}$ in Equation 6.2 is cut down to a sum over *principal* ideals $\mathfrak{I}$ that are generated by an element of $\mathfrak{o}_{D,f}$. Let $X_{\pm m}$ be a set of representatives for elements $x \in \mathfrak{o}_{D,f}$ such that $\mathrm{Norm}(x) = \pm m$, modulo units of $\mathfrak{o}_{D,f}$. ω_∞ is a character of K_∞ trivial on $\Lambda(f)$ and so, for $x \in X_m$, to evaluate $\omega_\infty(x)$ makes sense.

We obtain:

$$\sum_{\omega,\omega_\infty \text{ fixed}} \lambda_m(\omega) = h_{D,f} \sum_{x\in X_{\pm m}} \omega_\infty(x) \tag{6.3}$$

Now, if K is a real field and $\Lambda(f)$ contains no element of norm -1, then characters ω_∞ come in pairs: ω_∞ and ω'_∞ are paired when their restriction to the connected component of $K_\infty^\times$ are identical. Denote by $\omega_{\infty,+}$ the restriction of ω_∞ to the connected component of $K_\infty^\times$.

For $m < 0$, in this setting, define $\lambda_m(\omega) = \pm\lambda_{|m|}(\omega)$ according to whether the Maass form $\pi(\omega)$ is even (positive sign) or odd (negative sign). (That may be computed from ω_∞ as follows: $\mathbb{R}^\times$ sits inside $K_\infty^\times$ as elements of norm one; $\pi(\omega)$ is even or odd depending on whether ω_∞ restricted to this $\mathbb{R}^\times$ factors through absolute value or not.)

One then deduces, where X_m is a set of representatives for $x \in \mathfrak{o}_{D,f}$ of norm m, modulo units in $\mathfrak{o}_{D,f}$ of norm 1:

$$\sum_{\omega,\omega_{\infty,+} \text{fixed}} \lambda_m(\omega) = 2h_{D,f} \sum_{x\in X_m} \omega_{\infty,+}(x) \tag{6.4}$$

6.2.2 Summary of Results

If ω, a Grössencharacter of K, has K-conductor (f), the representation $\pi(\omega)$ has conductor $|D|f^2$. As ω ranges over all Grössencharacters of K, trivial on $\mathbb{A}_Q$, of conductor dividing (f), $\pi(\omega)$ ranges over all representations with central character χ_D and with conductor dividing $|D|f^2$, whose symmetric square has a pole. It is exactly these forms which will be found on $\Gamma_0(|D|f^2)$. Finally, ω_∞ will determine the real type of $\pi(\omega)$ as follows: if K is real and ω_∞ is the character of $\mathbb{R}^\times \times \mathbb{R}^\times$ taking $(x, x^{-1}) \mapsto \text{sgn}(x)^{\epsilon_\omega}|x|^{it_\omega}$, then $\pi(\omega)$ corresponds to a Maass form of eigenvalue $1/4+t_\omega^2$ and parity ϵ_ω; if K is complex and ω_∞ is the character $z \mapsto z^{k_\omega-1}\bar{z}^{-k_\omega+1}$ of $\mathbb{C}^\times$, then $\pi(\omega)$ is holomorphic of weight k_ω.

Real Quadratic Case: First suppose K is real quadratic. Let ϵ_0 be a fundamental unit for $\mathfrak{o}_{D,f}$: a generator for $\Lambda(f)$. There are two cases:

1. $\mathrm{Norm}(\epsilon_0) = -1$. In this case, the t_ω are the integral multiples of $i\pi/2\log(\epsilon_0)$, and so one expects to find Maass forms for $(\Gamma_0(Df^2), \chi_D)$ of eigenvalue $1/4 - r^2$, with $r = \frac{i\pi n}{2\log(\epsilon_0)}$ and $n \in \mathbb{Z}$. If n is even the Maass forms are *even* and if n is odd the Maass forms are odd. Each n occurs with multiplicity $h_{D,f}$. The sum of Fourier coefficients is given by Equation 6.3.

2. $\mathrm{Norm}(\epsilon) = 1$. In this case, the t_χ are the integral multiples of $i\pi/\log(\epsilon_0)$, and so one expects to find Maass forms for $(\Gamma_0(Df^2), \chi_D)$ of eigenvalue $1/4 - r^2$, with $r = \frac{i\pi n}{\log(\epsilon_0)}$ and $n \in \mathbb{Z}$. Each n occurs with multiplicity $2h_{D,f}$, and half the forms are odd, half even. The sum of Fourier coefficients is given by Equation 6.4.

Holomorphic Case: Now suppose that ω is a character of an imaginary quadratic field. Let $|\Lambda(f)| = w_f$; it is the number of roots of unity in the order $\mathfrak{o}_{D,f}$. The possible infinity types of $\pi(\omega)$ are just those discrete series of weight k, with k congruent to 1 mod w_f (in particular, always odd). The multiplicity of each k_ω is again $h_{D,f}$. In this case, m is always positive and the sum of Fourier coefficients is given by Equation 6.3.

6.3 Bessel Transforms and Inversion Questions

6.3.1 Fourier transforms and Bessel Transforms

Let $\varphi(x)$ be a function, which has reasonable decay near 0 and infinity. Here we prove some identities relating φ and its Bessel transform.

Define

$$h^+(t) = \int_0^\infty B_{2it}(x)\varphi(x)x^{-1}dx$$

where $B_\nu(x) = (2\sin(\pi\nu/2))^{-1}(J_{-\nu}(x) - J_\nu(x))$.

From Gradshteyn-Ryzhik, [8], we get:

$$\int_0^\infty B_{2it}(x)\cos(\beta x) = \begin{cases} \frac{1}{\sqrt{\beta^2-1}}\cos(2t\cosh^{-1}(\beta)), & |\beta| \geq 1 \\ 0, & |\beta| \leq 1 \end{cases}$$

Now, let φ' be the *even extension* of φ, so $\varphi'(x) = \varphi(|x|)$, and denote by B'_ν the even extension of B_ν. Then,

$$h^+(t) = \frac{1}{2}\int_{-\infty}^{\infty} \varphi'(x) B'_{2it}(x) x^{-1} dx$$

Let Δ be the Fourier transform of $\varphi(x)$; then the Fourier transform of φ' at β is $\Delta(-\beta) + \Delta(\beta)$. The Fourier transform of the even extension B'_ν at β is

$$\frac{2}{\sqrt{\beta^2 - 1}}\cos(2t\cosh^{-1}(\beta))$$

Therefore $h^+(t)$ equals

$$h^+(t) = \frac{1}{\pi}\int_{\beta=1}^{\infty}(\Delta(-\beta) + \Delta(\beta))\frac{1}{\sqrt{\beta^2 - 1}}\cos(2t\cosh^{-1}(\beta))d\beta =$$

$$= \frac{1}{\pi}\int_0^\infty (\Delta(-\cosh(\theta)) + \Delta(\cosh(\theta)))\cos(2t\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^\infty (\Delta(-\cosh(\theta/2)) + \Delta(\cosh(\theta/2)))\cos(t\theta) \tag{6.5}$$

From this, one sees that the Fourier transform of the even function $2h^+$ evaluated at θ equals $\Delta(-\cosh(\theta/2)) + \Delta(\cosh(\theta/2))$.

Also,

$$\int_0^\infty K_\nu(x)\cos(rx)dx = \frac{\pi}{2\sqrt{1+x^2}}\cosh(\nu\sinh^{-1}(r))\sec(\nu\pi/2)$$

so, in particular, from the definition

$$h^{-}(t) = \frac{4}{\pi}\cosh(\pi t)\int_0^\infty K_{2it}(x)\varphi(x)x^{-1}dx$$

we obtain

$$h^{-}(t) = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{1+\beta^2}}(\Delta(\beta)+\Delta(-\beta))\cos(2t\sinh^{-1}(\beta))d\beta$$

which gives, just as before:

$$h^{-}(t) = \frac{1}{2\pi}\int_0^\infty (\Delta(\sinh(\theta/2)) + \Delta(-\sinh(\theta)/2)\cos(t\theta)d\theta$$

and so:

$$\widehat{h^{-}}(\lambda) = \widehat{h^{-}}(-\lambda) = \frac{1}{2}(\Delta(\sinh(\lambda/2)) + \Delta(-\sinh(\lambda/2)))$$

6.3.2 Inversion problem

The purpose of this section is to verify that one obtains a "large enough" collection of spectral functions h from geometric functions φ with compact support in $(0,\infty)$. We will sketch this in the hardest case (the J-transform); in each case, the key point is that, as in the Paley-Wiener theorem, allowing spectral transforms to have holomorphic extensions forces the geometric function to have good decay. The central result is Theorem 7 below. It also suffices for the implicit application in Chapter 4 by taking a function of the type constructed below at every infinite place. We also will use implicitly various results on Bessel functions, for which we refer to [8].

Note that one must be careful, in these arguments, to avoid complications that arise from exceptional eigenvalues. We do not discuss this issue explicitly, and check our bounds only at those eigenvalues which are non-exceptional, but the arguments given are valid even in the presence of exceptional eigenvalues, as the reader can check.

We note that the formulas for the Sears-Titchmarsh inversion in the appendix of Iwaniec appear to be incorrect, although this does not affect the truth of the Kuznetsov formula presented. Kuznetsov's original paper, [11], provides a treatment and derivation of the formula.

The transform of interest is (note that the $h_{\varphi,k}$ are only of interest for k an *odd* integer):

$$\varphi \to \left(h^+_\varphi(t) = \int B_{2it}(x)\varphi(x)x^{-1}dx, h_{\varphi,k} = \int J_k(x)\varphi(x)x^{-1} \right) \tag{6.6}$$

Here $B_\nu(x) = (2\sin(\pi\nu/2))^{-1}(J_{-\nu}(x) - J_\nu(x))$. In particular, for $\nu = k$, an odd integer, $B_k(x) = (-1)^{(k-1)/2}J_{-k}(x)$. Therefore, for φ of compact support, $(h^+_\varphi, h_{\varphi,k})$ are, up to sign, the values of a holomorphic function $h(z)$ along the imaginary axis and at odd integers.

This is inverted via the Sears-Titchmarsh inversion formula:

$$(h^+(t), h_k) \to \varphi(x) = 4\int_0^\infty B_{2it}(x)h^+(t)t\tanh(\pi t)dt + \sum_{k\equiv 1(2)} 2kh_kJ_k(x) \tag{6.7}$$

(The factor of 4 is missing from Iwaniec.) These transformations can be regarded as an isometry between appropriate Hilbert spaces, but we will not have need of this.

Theorem 7. *Let t_j be a discrete subset of $\mathbb{R}$ with $\#\{j : t_j \leq T\} \ll T^r$ for some r. Let, for each j, there be given a function $c_X(t_j)$ depending on X, so that $c_X(t_j) \ll t_j^{r'}$ for some r' – the implicit constant independent of X; similarly, for each k odd, let there be given a function $c_X(k)$ depending on X so that $c_X(k) \ll k^{r'}$.*

Suppose that one has an equality

$$\lim_{X\to\infty} \left(\sum_j c_X(t_j)h^+_\varphi(t_j) + \sum_{k\, odd} c_X(k)h_{\varphi,k} \right) = 0 \tag{6.8}$$

for all $(h^+_\varphi, h_{\varphi,k})$ *that correspond, under the Bessel transform above, to* φ *of compact support on* $(0,\infty)$*. Then* $\lim_{X\to\infty} c_X(t_j)$ *exists for each* t_j *and equals* 0*, and similarly* $\lim_{X\to\infty} c_X(k)$ *exists for all* k *and equals* 0*. In particular the equality above holds for all functions* h *for which both sides converge.*

The Theorem follows easily from the following "approximation" Proposition:

Proposition 17. *Given* $j_0 \in \mathbb{N}$, $\epsilon > 0$ *and an integer* $N > 0$*, there is* φ *of compact support so that* $h^+_\varphi(t_{j_0}) = 1$*, and for all* $j' \neq j_0$ $h^+_\varphi(t_{j'}) \ll \epsilon(1+|t_{j'}|)^{-N}$*, and for all* k *odd,* $h_{\varphi,k} \ll \epsilon k^{-N}$*.*

Given k_0 *odd,* $\epsilon > 0$ *and an integer* $N > 0$*, there is* φ *of compact support so that* $h_{\varphi,k_0} = 1$ *,* $h_{\varphi,k} \ll \epsilon k^{-N}$ *for* k *odd and* $k \neq k_0$*, and* $h^+_\varphi(t) \ll (1+|t|)^{-N}$ *for all* $t \in \mathbb{R}$*.*

Proof of Theorem 7, given Proposition: We can deduce from the first part of the Proposition the existence of the limits $\lim_{X\to\infty} c_X(t_j)$: they all equal 0.

Now, substituting this result, and using the function φ from the *second* part of the proposition we also deduce the existence of the limits $\lim_{X\to\infty} c_X(k)$: they also equal zero.

(Note the importance of the $c_X(t) \ll t^{r'}$ bound in both these arguments.) □

Proof. (Of Proposition 17) We will often regard h^+ as the values on the *imaginary axis* of a (potentially) holomorphic function h on the complex plane; thus $h^+(t) = h(it)$.

The idea is this: holomorphic extension of h corresponds to rapid decay, near 0, of φ. The properties near ∞ of φ are governed by certain asymptotics of Bessel functions. It is not difficult to arrange, using these facts, a function h with the correct general properties so that the associated φ decays *rapidly* at both 0 and ∞. We then truncate φ to obtain a function of compact support; fortunately this does not modify h too much.

We give the outline of the proof so it is not too convoluted: Lemma 24 and Lemma 25 work out how to compute asymptotic behaviour of φ near 0 and ∞, respectively, given h. Once this is done, Lemma 26 shows that one can obtain the first part of Proposition 17 with a function φ which has at least "very good" decay near 0 and ∞, and Lemma 27 does the same for the second part of Proposition 17. Finally, Lemma 28 and Lemma 29 show that these φ can be *truncated* to have compact support without affecting the functions h too much, thus establishing Proposition 17 as stated.

Throughout we will be dealing with functions so that both $h^+(t)$ and h_k decay very rapidly (faster than any polynomial). Therefore, the convergence of series and the interchange of various sums and series is easy to justify, and we do not explicitly comment on this.

First, we address the issue of holomorphic extension of h and the decay near 0 of φ. We say a holomorphic function f has *rapid decay along vertical lines* if, for fixed σ, $f(\sigma + it)$ decays faster than any polynomial in t, for t real. We say it has *super-exponential* decay if, along the line $s = \sigma + it$, $f(s)$ decays faster than any exponential function in t. For example, $f(s) = e^{s^2}$ has super-exponential decay.

Lemma 24. *Let A be a positive integer. Suppose there exists a function $h(z)$, holomorphic for $|\Re(z)| \leq A + \epsilon$ for some $\epsilon > 0$, and rapidly decaying along vertical lines, so that $h^+(t) = h(it)$ and $h_k = (-1)^{(k-1)/2}h(k)$. Let $\varphi = F_h$ be the function associated to (h^+, h_k) by Sears-Titchmarsh inversion. Then one has an estimate for φ and its derivatives near 0, given by $\varphi^{(l)}(x) \ll x^{A-l}$.*

Proof. Write, noting that $\sin(\pi i t) = i\sinh(\pi t)$,

$$\begin{aligned}\varphi(x) = 2i\int_{t\geq 0} \frac{1}{\cosh(\pi t)} h^+(t) t J_{2it}(x) dt \qquad & -2i\int_{t\geq 0} \frac{1}{\cosh(\pi t)} h^+(t) t J_{-2it}(x) dt \\ & + \sum_{k\equiv 1(2)} 2k h_k J_k(x) \end{aligned} \tag{6.9}$$

$$= 2i\int_{t} \frac{1}{\cosh(\pi t)} h^+(t) t J_{2it}(x) dt + \sum_{k} \cdots$$

One now shifts lines of integration from $\Im(t) = 0$ to $\Im(t) = A$. The residues coming from the integral cancel the contribution of the "discrete sum" $\sum_k 2kh_kJ_k(x)$, and one need only appeal to the power series expansion of the Bessel function to deduce the claimed estimate.

For the derivatives, use the differentiation formulae for Bessel functions. □

If h is a holomorphic function as in the previous Lemma, we will always denote by F_h the function attached to $(h^+(t) = h(it), h_k = (-1)^{(k-1)/2}h(k))$ by Sears-Titchmarsh inversion.

The key to good decay of φ near zero, then, is holomorphic extension. We now must address the question of obtaining good decay for φ near ∞.

Lemma 25. *Suppose $h^+(t)$ decays super-exponentially, that is to say, $h^+(t) \ll_A e^{-A|t|}$ for all A, and similarly for h_k. Then the function φ associated to (h^+, h_k) by Sears-Titchmarsh inversion has the asymptotic expansion:*

$$\varphi(x) = \cos(x - \pi/4)\sum_{n\geq 1} A_n x^{-n/2} + \sin(x - \pi/4)\sum_{n\geq 1} B_n x^{-n/2}$$

The error at truncating at n is of the order of the next term. Here A_n is given by a linear form in (h^+, h_k):

$$A_n(h) = 4\int_0^\infty a_n(2it)h^+(t)t\tanh(\pi t)dt + \sum_j 2ja_n(j)h_j$$

where $a_n(z)$ are certain polynomials in z, and $B_n(h)$ is defined similarly, with corresponding polynomials $b_n(z)$. Each derivative $\varphi^{(l)}(x)$ also has an asymptotic expansion, equal to that obtained by term by term differentiation.

Proof. Indeed, one has an asymptotic for the Bessel functions.

$$B_\nu(x) = \cos(x - \pi/4)\sum_{k=1}^n a_k(\nu)x^{-k/2} + \sin(x - \pi/4)\sum_{k=1}^n b_k(\nu)x^{-k/2} + R$$

where the remainder is $R = O(e^{|3\pi\nu/2|}x^{-(n+1)/2})$ and $a_k(\nu), b_k(\nu)$ is a polynomial in ν. The implicit constant is absolute. (Almost certainly one can improve on this remainder: this is merely what I obtain from analyzing the argument in Whittaker's treatise on Bessel functions, and it suffices for the argument here.)

It follows that $\varphi(x) = 4\int_t B_{2it}(x)h^+(t)t\tanh(\pi t)dt + \sum_k h_k J_k(x)$ also satisfies an asymptotic (and this is easy to justify from the explicit estimate of the remainder term noted above):

$$\varphi(x) = \cos(x - \pi/4)\sum_{k=1}^{n} A_k x^{-k/2} + \sin(x - \pi/4)\sum_{k=1}^{n} B_k x^{-k/2} + O(x^{-(n+1)/2})$$

Here

$$A_k(h) = 4\int_0^\infty a_k(2it)h^+(t)t\tanh(\pi t)dt + \sum_j 2ja_k(j)h_j$$

and $B_k(h)$ is defined similarly. Similar asymptotics hold for the derivatives of φ. (The asymptotics for the derivatives of Bessel functions are easily deduced from the ones above, since one can express the derivative of a Bessel function in terms of other Bessel functions. For example, $J'_\nu(z) = \frac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z))$. □

Lemma 26. *Fix an integer M and $j_0 \in \mathbb{N}, \epsilon, N$. There exists a holomorphic function h, of super-exponential decay along any vertical line, so that $h(k) = 0$ for k integral, and so $h^+(t) = h(it)$ satisfies the conditions of the first part of Proposition 11 (i.e. $h(it_{j_0}) = 1$, and for $j' \neq j_0$ $h'(it_{j'}) \ll \epsilon(1 + |t_{j'}|)^{-N}$), so that additionally the asymptotic constants $A_n(h)$ and $B_n(h)$ from the Lemma 25 vanish for $n < 2M$. In particular, $F_h(x) = O(x^{-M})$.*

Proof. We need to check that there is a function h with the desired properties so that $A_k(h) = B_k(h) = 0$ for $k \leq 2M$. That this should be possible is (loosely speaking) clear, since one has a finite number of compatible linear constraints on an infinite-dimensional space; however, one must be careful in doing this without jeopardizing the necessary growth properties. The idea is to pick some basic localized test functions and then take an appropriate linear combination to satisfy the constraints. We will only sketch the argument.

Fix a real number T and $\delta > 0$. Define $h_{T,\delta}(z) = \frac{2T}{\delta\sqrt{\pi}} e^{-((z^2+T^2)/\delta)^2} \frac{T\sin(\pi z)}{(z\sin(\pi T))}$ – the construction has to be slightly modified for $T = 0$, which we leave to the reader. Then $h_{T,\delta}$ is holomorphic and, as $\delta \to 0$, localized around $t = \pm iT$ – or, to be precise, h restricted to $\mathbb{R} \cup i\mathbb{R}$ is localized there.

It also has zeros for t integral. It has super-exponential decay along vertical lines, and the property that $\int_{t\in\mathbb{R}} h_{T,\delta}(it) \to 1$ as $\delta \to 0$ (the constants in $h_{T,\delta}$ were chosen so that this is true.)

Fix a finite set of real numbers $\{T_j\}$, for $1 \leq j \leq Q$. Eventually we shall fix matters so precisely one of the T_j is a t_j. Now write, for some constants c_j that are to be specified, a new function h_δ as a linear combination of our basic test functions:

$$h_\delta(z) = \sum_{j=1}^{Q} c_j h_{T_j,\delta}(z)$$

for certain T_i. The idea is that by letting $\delta \to 0$ and taking the correct linear combination one can fulfill the linear constraints and the growth conditions simultaneously. h_δ still has zeros at all integral arguments and holomorphic extension to the entire complex plane. Let F_δ be the function defined by Sears-Titchmarsh inversion starting from h_δ (i.e. starting from $h_\delta^+(t) = h_\delta(it), h_k = 0$).

Thus, function F_δ has, on account of the holomorphicity of h_δ, arbitrarily rapid polynomial decay near 0. Its decay near ∞ is governed by the coefficients:

$$A_k(h_\delta) = 4\sum_{j=1}^{Q} c_j \int_{-\infty}^{\infty} h_{T_j,\delta}(it) a_k(2it) t \tanh(\pi t) dt$$

$$B_k(h_\delta) = 4\sum_{j=1}^{Q} c_j \int_{-\infty}^{\infty} h_{T_j,\delta}(it) b_k(2it) t \tanh(\pi t) det$$

Now, as $\delta \to 0$, the localization of h implies that the former integral approaches essentially

$$4\sum_{j=1}^{Q} c_j a_k(2iT_j) T_j \tanh(\pi T_j) + O(\delta)$$

Here, the implicit constant in the $O(\delta)$ is dependent on the choice of T_j; this is harmless, as we will fix the T_j and let $\delta \to 0$. There is a similar statement for the B_k.

Suppose that (for reasons that will be made clear) we wish to prescribe the asymptotic behaviour of the function F_δ. In other words, given sequences $(u_k)_{k=1}^{k=2M}$ and $(v_k)_1^{2M}$, we wish to choose the constants c_j and the "localizing points" T_j so that $A_k(h_\delta) = v_k, B_k(h_\delta) = u_k$ for $1 \leq k \leq 2M$. We focus on how to fix the $A_k(h_\delta)$ correctly; the treatment of B_k is identical. We will assume that $Q = 2M$, so that there exist $2M$ of the T_js.

Let $\mathbf{v}$ be the vector with components $(v_k)_{k=1}^{2M}$. We then wish to solve the system of equations, with $\mathbf{c} = (c_j)_{j=1}^{2M}$, and M_δ a matrix with (j,k)th entry $4a_k(2iT_j)T_j \tanh(\pi T_j) + O(\delta)$,

$$\mathbf{c} \cdot M_\delta = \mathbf{v}$$

Setting $\delta = 0$ for a moment, we see that the determinant of the matrix M_0 is an analytic function of the (T_j); it does not vanish for (T_j) in an open set. Fixing such a (T_j) so that no T_j coincides with a t_j, we see that one has solutions to the above equation, as $\delta \to 0$, with the size of the c_j remaining bounded (using Kramer's rule) by a linear function of the v_k. Replicating this argument (with $Q = 4M$) shows that one may specify, in a similar fashion, the B_k.

We can therefore find a holomorphic function h_δ, so that F_{h_δ} has prescribed asymptotic behaviour, with h_δ arbitrarily small at the t_j, and so $h_\delta(k) = 0$ for $k \in \mathbb{Z}$.

Now, we choose a second function $h(z)$ that is localized at $z = t_j$ and then construct a function as above to "cancel out" its asymptotic behaviour.

So finally, let $h_\delta^{(2)} = h_{t_{j_0},\delta}$, where t_{j_0} is as prescribed in the Proposition 17.

Choose T_j as above and put $h'_\delta = h_\delta^{(2)} + h_\delta$, where h_δ is chosen so the asymptotic behaviour of F_δ matches the asymptotic behaviour of $-F_{h_\delta^2}$.

The function h'_δ finally constructed will have the property that $F_{h'_\delta}(x) \ll x^N$ for all N and $F_{h'_\delta}(x) \ll x^{-k}$ for $k \leq M$ as desired.

Therefore, choosing δ sufficiently small, we can choose $h' = h'_\delta$ so that $F_{h'}$ has the necessary decay conditions and (by construction) $h'(it_{j_0}) = 1$ while $h'(it_j) \ll \epsilon(1+|t_j|)^{-N}$, for any N and any $j \neq j_0$, and $h'(k) = 0$ for k odd (By varying δ one obtains arbitrarily small ϵ.) □

We have the variant of this Lemma, suitable for the second part of the Proposition.

Lemma 27. *Fix an integer M and an integer k_0, and ϵ, N.*

There exists a holomorphic function h, of super-exponential decay along any vertical line, so that $h(\pm k_0) = 1$, $h(k) = 0$ for $k \neq \pm k_0$ and k integral, and satisfying the conditions of the second part of Proposition 11 (i.e. $h(it) \ll (1+|t|)^{-N}$) for $t \in \mathbb{R}$), so that the asymptotic constants A_n and B_n from the Lemma 25 vanish for $n < 2M$. In particular, $F_h(x) = O(x^{-M})$.

Proof. The same as that of the previous Proposition, except at the final stage one replaces the function $h_\delta^{(2)}$ with a function that vanishes at all integers except $\pm k_0$ and is localized around k_0; and then, as before, adds on another function to cancel its asymptotic behaviour. □

The conclusion so far is that one can find φ such that $(h_\varphi^+, h_{\varphi,k})$ satisfies the conclusion of Proposition 17, *except* our φ, rather than being of compact support, has good decay near 0 (in fact $\varphi(x) \ll x^N$ for all N) and ∞ (in fact $\varphi(x) \ll x^{-k}$ for $k \leq M$). We now truncate this φ to produce a function of compact support on $(0, \infty)$.

Thus, let g be a smooth positive C^∞ function on $\mathbb{R}$ of compact support, such that $g(x) = 1$ for $-1 \leq x \leq 1$ and $g(x) = 0$ for $x \geq 2$. If h is a function as in Lemma 26 or Lemma 27, put $\varphi_r(x) = F_h(x)g(\log(x)r)$; the idea is that, as $r \to 0$, $\varphi_r \to \varphi$ pointwise, and it is a smoothly truncated version of φ with compact support in $(0, \infty)$. In order to compare the transform of φ_r with φ one notes that $\varphi_r - \varphi$ is the sum of smooth functions $\varphi_{1,r}$ and $\varphi_{2,r}$, where:

$\varphi_{1,r}$ supported in $x \leq 1$ with growth ϵx^N, for arbitrary N and $\epsilon \to 0$ as $r \to 0$; $\varphi_{2,r}$ is supported in $x \geq 1$ and has growth ϵx^{-M+1}, where again $\epsilon \to 0$ as $r \to 0$. (Note, one can introduce the ϵ by sacrificing a power in the exponent.) Finally, the derivatives $\varphi'_{1,r}$ and $\varphi'_{2,r}$ satisfy the same estimates.

Now apply the following two estimates, Lemma 28 and Lemma 29. These show that truncating φ in this fashion does not vary h "much." In combination with the previous Lemmas, they complete the proof of Proposition 17.

Lemma 28. *Suppose φ is a smooth function supported in $x \leq 1$, such that φ and its derivatives $\varphi^{(l)}(x)$, for $l \leq N$, satisfy $\varphi^{(l)}(x) \ll_m \epsilon x^m$ for all $m > 0$. Then $h^+_\varphi(t) \ll \epsilon(1 + |t|)^{-N}$, and a similar estimate holds for $h_{\varphi,j}$ with j odd.*

Proof. One has the power series representation $J_\nu(z) = z^\nu \sum_k \frac{(-1)^k}{k!\Gamma(k+1+\nu)}(z/2)^{2k}$. Let $M_\varphi(s) = \int_{x>0} \varphi(x)x^{s-1}dx$ be the Mellin transform of φ. The conditions on φ imply that $M_\varphi(it) \ll \epsilon(1 + |t|)^{-l}$ for $l \leq N$.

We may write:

$$h^+_\varphi(t) = \frac{1}{2\sin(\pi it)\Gamma(1+2it)}\left((M_\varphi(2it) - \frac{M_\varphi(2it+2)}{1(1+2it)} + \frac{M_\varphi(2it+4)}{2!(1+2it)(2+2it)} + \dots\right)$$

From this expression one obtains the required estimate. The expression for $h_{\varphi,k}$ follows similarly, replacing $J_k(z)$ by the first few terms in its power series. □

Lemma 29. *Suppose φ is supported in $x \geq 1$ so that, for $0 \leq l \leq k$, $\varphi^{(l)}(x) \ll \epsilon x^{-M}$ for some fixed M. Then $h_{\varphi,j} \ll \epsilon j^{-M}$ and $h_{\varphi}^{+}(t) \ll \epsilon(1+|t|)^{-k+2}$.*

Proof. (Sketch.) First we estimate for k odd

$$h_{\varphi,k} = \int_0^\infty \varphi(x) J_k(x) x^{-1} dx$$

From the power series representation $J_l(z) = (\frac{z}{2})^l \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+l)!}(\frac{z}{2})^{2k}$ we see that, all implied constants being absolute,

$$|J_l(z)| \ll \frac{1}{l!}(\frac{z}{2})^l \sum_{k=0}^\infty \frac{1}{k!}(\frac{z^2}{4l})^k \ll \frac{1}{l!}(\frac{z}{2})^l e^{z^2/4l}$$

In particular,

$$J_l(\alpha l) \ll \frac{1}{\sqrt{l}}(\alpha e^{1+\alpha^2/4})^l$$

Using this, we obtain that $h_{\varphi,k} \ll \epsilon k^{-M}$ (one can essentially work out the integral that computes h_k by integrating from $x = O(k)$ to ∞.)

We now wish to estimate $h_\varphi^+(t)$. For this we use the Mellin-Barnes integral, for $0 < \sigma < 1$,

$$J_{2it}(x) = \frac{1}{4\pi i} \int_{\Re(s)=\sigma} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(it + s/2)}{\Gamma(1 + it - s/2)} ds$$

In particular, we see that

$$h_\varphi^+(t) = \frac{1}{4\pi i \sin(i\pi t/2)} \int_{\Re(s)=\sigma} 2^s M_\varphi(-s) \frac{\Gamma(it + s/2)}{\Gamma(1 + it - s/2)} ds + \ldots$$

where we have only written out one of the two terms, and $M_\varphi(s)$ is the Mellin transform of φ; it is analytic in a left half-plane, $\Re(s) < M$. Integration by parts shows $M_\varphi(\sigma + ir) \ll \epsilon(1+|r|)^{-k}$.

One obtains the estimate $h_\varphi^+(t) \ll \epsilon(1+|t|)^{-k+2}$. Indeed, take $\sigma = 1/2$, and write

$s = 1/2 + ir$, and use Stirling's formula to estimate $\frac{1}{\sin(i\pi t/2)} \frac{\Gamma(i(r+t)+1/2)}{\Gamma(1/2+i(t-r))}$ – it never gets larger than $O(|t|)$ as r varies, and this only happens for $|r| \gg |t|$. Quantifying this gives the required estimate. □

6.4 Trace formulas

The purpose of this section is to indicate why a trace formula restricted over dihedral forms – such as that derived in Chapter 2 – actually suffices to classify and construct such forms. This is relatively standard, and we include a brief discussion, only for completeness.

For simplicity we have always dealt with coefficients prime to the conductor; these, in any case, suffice to determine the form, by strong multiplicity one. It is also possible, with more effort, to carry through our technique for coefficients not prime to the conductor.

Let N be an integer and χ a Dirichlet character to the modulus N. Suppose S is a set of distinct Hecke eigenforms of conductor N and character χ, and denote as usual by $\lambda_n(f)$ the nth Hecke eigenvalue.

Proposition 18. *Suppose that one is also given a set S' and, for each $\alpha \in S'$, a sequence b_m^α which satisfies the Hecke relations relative to χ, so that for each $\alpha \in S$,*

$$b_m^\alpha b_n^\alpha = \sum_{d|(m,n)} b_{mn/d^2}^\alpha \chi(d)$$

Suppose one is also given complex constants C_f for each $f \in S$ and C_α for each $\alpha \in S'$, and that the sequences b_m^α for $\alpha \in S$ are distinct (i.e. given $\alpha, \alpha' \in S'$, there is m such that $b_m^\alpha \neq b_m^{\alpha'}$). Suppose also that:

$$\sum_{f \in S} C_f \lambda_n(f) = \sum_{\alpha \in S'} C_\alpha b_n^\alpha$$

Then there is a bijection $F : S' \to S$ so that $C_{F(\alpha)} = C_\alpha$, and $\lambda_n(F(\alpha)) = b_n^\alpha$ for n prime to N.

Proof. Indeed, let $\mathbb{T}$ be the abstract Hecke algebra generated by all Hecke operators T_p, for p prime to N. (That is, the algebra of operators T_n, for $(n, N) = 1$, subject only to the usual multiplication relations.) Associated to each $f \in S$ is a natural one-dimensional $\mathbb{T}$-module V_f, on which T_n acts by $\lambda_n(f)$. Similarly, to each b_n^α one can associate a one dimensional $\mathbb{T}$-module, V_α, so that T_n acts by b_n^α. The consequence of our conditions is that the virtual $\mathbb{T}$-modules $\oplus_{f \in S} C_f V_f$ and $\oplus_{\alpha \in S'} C_\alpha V_\alpha$ have the same characters; therefore, being semisimple finite-dimensional modules, they are in fact isomorphic as virtual modules. This provides the result. □

6.5 L-functions

6.5.1 Contour Shifting

The aim of this section is to describe the procedure for estimating partial sums of coefficients of L-series, via contour shifting. This procedure is used throughout the main text.

Suppose one has Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $H(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, satisfying bounds $a_n \ll n^A, b_n \ll n^A$; the series are then absolutely convergent in $\Re(s) > 1 + A$.

Let $g(x)$ be a C^∞ function of compact support on $(0, \infty)$, with Mellin transform $G(s) = \int g(x) x^{s-1} dx$. Then $g(x/X)$ has Mellin transform $X^s G(s)$. Now $G(\sigma + it) \ll_N (1 + |t|)^{-N}$, and we may write

$$\sum_n a_n g(n/X) = \int_{\Re(s) \gg 1+A} G(s) L(s) X^s ds$$

In particular, if $L(s)$ has analytic continuation up to $\Re(s) = A'$ in such a way that one has a bound $L(\sigma+it) \ll A(\sigma)(1+|t|)^{B(\sigma)}$ for $\Re(s)=\sigma$ when $\sigma > A'$, then one may shift the contour to the left up to A'. The justification for shifting contours and the absolute convergence of the integrals follows easily from the rapid decay of $G(s)$. In any case, considering the integral along a contour $\Re(s) = A' + \epsilon$ gives the bound:

$$|\sum_{n=1}^{\infty} a_n g(n/X)| \ll_\epsilon X^{A'+\epsilon}$$

For example, if one takes

$$a_n = \begin{cases} 1, & n \,\text{squarefree} \\ 0, & \text{else} \end{cases}$$

then $L(s) = \zeta(s)/\zeta(2s)$, which has meromorphic continuation to $\mathbb{C}$ and is analytic for $\Re(s) > 1/2$ with the exception of a pole at $s = 1$. One obtains $\sum_n a_n g(n/X) = \frac{6}{\pi^2} G(1)X + O_\epsilon(X^{1/2+\epsilon})$, where the leading term comes from the residue at $s = 1$.

6.5.2 Uniformity

Now we discuss how the estimates $L(\sigma+it) \ll A(\sigma)(1+|t|)^{B(\sigma)}$, which are crucial to the method described above, are to be obtained for classes of automorphic L-functions. In particular, we desire a mild uniformity in the parameters of the L-functions. Let $(a_n), (b_n)$, L and H be as in the previous section.

We shall assume that, for some integer N_F, and some complex numbers $\nu_i \in \mathbb{C}$, if one defines

$$\Lambda_F(s) = N_F^{s/2} \prod_i \Gamma(\frac{s+\nu_i}{2}) L(s)$$

and

$$\Lambda_H(s) = N_F^{s/2} \prod_i \Gamma(\frac{s+\overline{\nu_i}}{2}) H(s)$$

then both of these functions continue to analytic functions except for possibly poles at $s = 0, 1$, and also $\Lambda_F(s) = \alpha\Lambda_H(1-s)$, where $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. We will assume that the ν_i have absolutely bounded real part, that is, there is a fixed constant C so that $|\Re(\nu_i)| \leq C$ always.

From the functional equation,

$$L(s) = \alpha N_F^{1/2-s} \prod_i \frac{\Gamma(\frac{1-s+\overline{\nu_i}}{2})}{\Gamma(\frac{s+\nu_i}{2})} H(1-s)$$

We will derive certain estimates for $L(s)$ which will make clear the dependence on ν_i. Since the real parts of the ν_i are confined to a compact set, we are only really varying their imaginary parts. Write $\Im$ for imaginary part. Now one obtains easily, for $\Re(\sigma) \geq 1/2$.

$$\frac{\Gamma(\frac{(1-\sigma)+\overline{(\nu_i+it)}}{2})}{\Gamma(\frac{\sigma+(\nu_i+it)}{2})} \ll \prod_i (1 + |\Im(\nu_i) + t|)^{1/2-\sigma}$$

In particular, for $\sigma \ll -A$, one obtains, using the absolute boundedness of $H(s)$ for $\Re(s) > A$, that:

$$L(\sigma + it) \ll N_F^{1/2-\sigma} \prod_i (1 + |\Im(\nu_i) + t|)^{1/2-\sigma}$$

Let $C_F = N_F \prod_i (1 + |\Im(\nu_i)|)$, the "analytic conductor." It is now a consequence of the Phragmen-Lindelof "convexity" principle that such a bound holds for any σ:

$$L(\sigma + it) \ll C_F^{A(\sigma)} (1 + |t|)^{B(\sigma)}$$

where the $A(\sigma)$ and $B(\sigma)$ are piecewise linear in σ, and *absolute*.

The following special case is important for the text; it follows from the results of this Subsection and the previous one. Let f be a Maass cusp form of eigenvalue $1/4 + \nu^2$ normalized to have L^2 norm 1; then

$$\sum_n g(n/X) a_f(n^2) \ll (1 + |\nu|)^A X$$

for some absolute constant A. This follows by applying the previous discussion to the Dirichlet series $\sum_n a_f(n^2)n^{-s}$, which is the quotient of the (finite part) of the symmetric square L-function by $\zeta(2s)$. Actually, one obtains a still stronger result, with a (possible) main term coming from the residue and a remainder term, but we do not need that.

Notation Index

Since there are a very large number of symbols used in this thesis, we have tried to here to give the section or subsection where many of them are first introduced. Although it is not exhaustive, it will hopefully be of use in navigating this document. There are various notations that are used across different Chapters; by and large, they represent analogous concepts – for example, Δ in Chapters 2 and 4. If a symbol is used in multiple places in a given chapter, those usages are, if not identical, at least compatible (e.g., one is a generalization of the other), unless explicitly otherwise indicated below. The symbol listing for Chapter 5 is sparse, as that Chapter refers explicitly to Chapter 4 for notation. We have omitted the Appendix.

1.2 $\pi, m(\pi, \rho), L(s, \pi, \rho), \lambda(n, \pi, \rho)$.

1.3.1 $a_n(f), \lambda_n(f), h(t_f)$.

1.4 $\ll_\epsilon, x^\epsilon$.

2.2 $\chi, N, S_\chi(m, n, c), g, \sum_{n\sim X}$.

2.2.1 $a_n(f)$.

2.2.2 $\eta_\mathfrak{c}, h^\pm, h_f(\varphi)$.

2.2.3 $\hat{h}^\pm, \Delta$.

2.3 $D, \chi_D, N, f, c(N), m$.

2.3.1 $k, L, \nu, \mathcal{A}(\nu; c, m), \delta(x)$.

2.3.2 $\mathfrak{o}_{D,f}, \mathfrak{o}_{D,f}^{(1)}, \mathfrak{S}', \delta, \epsilon_0, X_m, \tilde{X}_m, w_f$.

2.3.3 $Z(s), Z_p(s), g(p), \beta, \epsilon_p, \gamma_p$.

2.4 $L_{standard}, S_{standard}$.

2.5 $K, K_\infty, \omega, \mathbb{A}_K, C_{D,f}, h_{D,f}, U(f), U_v(f), f_\omega, \lambda_m(f_\omega), t_\omega, \epsilon_\omega, k_\omega$.

2.5.2 Cusp, Eis.

2.5.3 Class, $C_g, \mathcal{B}(g)$.

2.5.4 $\mathfrak{c}_0, \mathfrak{c}_\infty, L_{cst}, \Lambda(\chi, s), L(\chi, s), \eta', H, \hat{H}$.

2.6 b_m^α.

3.1 $L(s, f \times \sigma)$.

3.2 $b_n, S(m; X), C, c(f), \overline{f}, F, \beta$, Diag.

3.2.1 $\hat{g}_q(y), L(d, q; s), \Lambda(d, q; s), R(\rho), \rho_i, E$.

3.2.2 $h(x), f_c(y), \phi(c)$.

3.2.3 $\mathcal{P}(L(d, c, s)), r, \rho, R(\rho), G_c(s)$.

4.1 $h_F, D_F, R_F, A, C_F, \mathfrak{o}_F, d, F_\infty, F_{\infty,+}, v, \mathbb{R}^{[F:\mathbb{Q}]}$, $\mathbb{A}_F$, $\mathbb{A}_{F,f}$, $\mathfrak{o}_{F,v}$, $\mathfrak{o}_v$, $\mathfrak{o}_F^*$, $\mathfrak{o}_F^{*,+}$, F_v, $\widehat{\mathfrak{o}_{F,f}}$, $\widehat{\mathfrak{o}_{F,f}}^\times$, $U_{F,v}$,U_v,$\mathfrak{o}_{F,v}^\times$, $\pi_\mathfrak{a}$, Norm, Z, N, ψ.

4.2 $\Gamma_0(\mathfrak{I}; \mathfrak{a}), \Gamma_0(\mathfrak{I}), K_0(\mathfrak{I}), K_{0,v}(\mathfrak{I}), \mathfrak{d}^{-1}, \mathfrak{d}$.

4.2.1 $\pi, \chi, \chi_\infty, \chi_f, \underline{\det}, \mathcal{X}_\mathfrak{a}, \mathrm{Fun}_\chi, \delta_\mathfrak{a}, L^2_\chi(\mathfrak{I}), L^2_\chi, V_{\chi'}$.

4.2.2 $\sigma_K, \mathrm{SO}_2(F_\infty), \mathbf{f}, \delta(\alpha, \beta), W_{\phi_0}, W_{\phi_0,\infty}, W_\infty, a_\mathbf{f}^{un}, a_f(\mathfrak{a}, \alpha), \lambda_\mathbf{f}(\mathfrak{q}), C_\mathbf{f}, \epsilon$.

4.3.1 $K_0, \chi_{K_0}, \widehat{C_F}, \widehat{C_F(2)}, S, K'$.

4.3.2 $KS_\chi, h_{\mathbf{f}}, J_{\pi_\infty}, \pi(\mathbf{t}, \mathbf{sgn}), h(\mathbf{t}), \hat{h}_\varphi, \Delta$.

4.3.3 $\mathfrak{I}, \mathfrak{D}_{K/F}, \mathfrak{f}, \mathfrak{o}_{K,\mathfrak{I}}, \mathfrak{q}, \alpha, \mathfrak{m}, g, \mu, \Sigma, \Sigma_{\mathfrak{a}}, \Sigma_{\mathfrak{a},\mu}$.

4.3.4 $\mathfrak{c}, k, \sim, \Psi_{c,\epsilon}, \mathcal{A}_{\mathfrak{a}}(\nu; \mathfrak{c}, \mu), || \cdot ||, \delta_{\mathfrak{a}}, \Delta, \mathfrak{S}, \mathfrak{S}', \sigma_K, \mathbf{log}, \mathfrak{o}'_{K,\mathfrak{I}}, X_{K,\mathfrak{a},(\mu)}, X_{K,\mathfrak{a},\mathfrak{m}}$.

4.4.1 $\phi, I(\nu'; \mathfrak{c}', \mu')$.

4.4.2 $\nu', \mu', \nu_0, \mu_0, Z_{v,r}, Z_v(\omega_v, s), Z(\omega, s), C$.

4.4.3 $\mathfrak{f}_\chi, \mathfrak{f}, f_v, f_{\chi,v}, \chi_{\nu'^2-4\mu'}, Q_v, M_s, M_{\geq s}, g(\omega_v, \psi_v), \epsilon(\omega_v, \psi_v)$.

4.4.4 $\beta, \beta_1, \beta_2, K_Q, \pi_K$.

4.5 ω (this usage is essentially different to the usage in (4.4.2)),$\pi(\omega), \mathfrak{I}, U_{\mathfrak{I},v}, U_{\mathfrak{I}}$, U, Λ_U, Λ'_U, $C_{K,\mathfrak{I}}$, $C_{K,\mathfrak{I};F}$, Δ, Δ', Pl_∞, $\Lambda_{U,+}$, $\Lambda'_{U,+}$.

4.5.1 $\mathbf{log}, \omega_\infty, \mathfrak{t}_\omega, R_{K,\mathfrak{I}}, R_F, X_{\mathfrak{a},\mathfrak{I}}$.

4.6 $G, N, B, \overline{N}, A, Z, n(x), a(y_1, y_2), w, \mu_I, \mu_B, L^2_\chi(\Gamma_0(\mathfrak{I}; \mathfrak{a})\backslash \mathrm{GL}_2(F_\infty), \psi_1, \psi_2$, $f_{\eta,\nu}$, $P_{\eta,\nu}$, Z_Γ, α, $F(g)$, Γ_P.

4.6.1 $(P_{\eta,\nu})_\psi, \Omega, S(\omega), KS_\chi(\omega), \hat{\eta}_{\psi_2^{-1}}, \varphi, KS_\chi(\psi_1, \psi_2, c, \epsilon)$.

4.6.2 $\mathrm{Proj}_\pi, \mathcal{W}_{\pi,\psi}, W_{\phi,\psi}, \langle , \rangle_W, c(\pi, \psi_1), F_\pi(f_{\eta,\nu}), J_{\pi,\psi}, \beta, c(\pi : \psi_1 \to \psi_2)$.

4.6.3 a_f^{un}, a_f^{nm}.

5.2 $A_p, \lambda_\pi(n), E(s)$.

5.3 $\rho_\pi, \mathfrak{q}, N_\chi(\mathfrak{q}), \mathcal{B}, \lambda_{\mathfrak{f}}(\mathfrak{p})$

5.3.1 $\mathfrak{I}$ (denotes a generic ideal, as distinct from usage in Chapter 4),$a_{\mathfrak{I}}^{\mathrm{Ico}}, T, N$.

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