ON THE DIMENSION OF THE SPACE OF CUSP FORMS ASSOCIATED TO 2-DIMENSIONAL COMPLEX GALOIS REPRESENTATIONS

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1. Introduction

The aim of this paper is to use the "amplification technique" to obtain estimates on the dimension of spaces of automorphic forms associated to Galois representations; these bounds improve nontrivially on the work of Duke ([D]).

A cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ is *associated* to a 2-dimensional Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ if, for each place v, the local representation π_v is matched with the induced map $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \to \mathrm{GL}_2(\mathbb{C})$ under the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_v)$. In particular, if one fixes a central character, there are only two possibilities for π_∞.

We call a form that is associated to a Galois representation "of Galois type." More precisely, for $G \in$ Types $=$ {Dihedral, Tetrahedral, Octahedral, Icosahedral}, we call a form of "type G" if it associated to a Galois representation whose image in $\mathrm{PGL}_2(\mathbb{C})$ is dihedral, tetrahedral (A_4), octahedral (S_4), or icosahedral (A_5).

Define a real-valued function e on Types via $e(\text{Dihedral}) = 1/2$, $e(\text{Tetrahedral}) = 2/3$, $e(\text{Octahedral}) = 4/5$, and $e(\text{Icosahedral}) = 6/7$. With this definition:

Theorem 1. *Fix a central character χ. The number of GL_2-automorphic forms π with type G, central character χ and conductor q is $\ll_\epsilon q^{e(G)+\epsilon}$.*

From this bound, one can deduce improved bounds on the sizes of various arithmetic objects. In particular, from Deligne-Serre's result ([DS]) one obtains improved bounds on the dimension of weight 1 forms for $\Gamma_0(N)$; or, from the modularity of solvable representation (Langlands-Tunnell) one can obtain bounds on the number of Galois representations with specified conductor, and therefore on the number of field extensions with Galois group A_4 or S_4. We will not touch on these issues and refer to Wong's paper, [W].

Duke's method uses two conflicting properties of the Hecke eigenvalues of Galois-type forms. On the one hand because these forms are associated to complex 2-dimensional representations the Hecke-eigenvalues are "rigid" and can take only very special values; this part of the argument was improved by Wong who studied carefully the possible lifting to $GL_2(\mathbb{C})$ of the groups A_4, S_4, A_5. On the other hand, because of their modular nature the Hecke eigenvalues for different f's are nearly orthogonal. This later property is expressed in Duke's paper by a large sieve inequality over the set of forms of Galois type; this estimate is obtained by duality (Prop. 1 of [D]); the combination of these two properties yield, for instance, to the exponent $e'(\text{Icosahedral}) = 11/12 < 1$.

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In this paper, we consider the set of forms of Galois type as a subset of an orthogonal basis of Maass forms for the *full* spectrum of Laplacian acting on $\Gamma_0(q)\backslash\mathbb{H}$. The restored spectral completeness enable us to use the Petersson-Kuznetsov formula – which does not exist for the subspace generated by forms of Galois type – and this, by Weil's bound on Kloosterman sums, yields a sharper large sieve inequality. Put in this framework, our bound can be suggestively interpreted as an application of the amplification method, with the main difference that a whole set is amplified rather than a single modular form. It is also another example of the advantage of considering modular forms inside broader families than the most obvious one.

Some remarks on generalisations of this result:

The argument we are about to give applies *over any number field*, with some minor modifications. The form of the result over a number field F is as follows: Fix an integral ideal $\mathfrak{q}$ and Grössencharacter χ of F. Then the number of forms π of type G, with central character χ, and F-conductor $\mathfrak{q}$ is $\ll_{F,\epsilon} \mathrm{Norm}(\mathfrak{q})^{e(G)+\epsilon}$. Just as over $\mathbb{Q}$, this result has arithmetic implications: for example, one can obtain, over a totally real field, estimates for the dimension of the space of Hilbert modular forms of weight 1.

The amplification procedure can be made even more precise in the case of dihedral forms: one can essentially recover their classification, due to Langlands-Labesse ([LL]), by using an amplifier of arbitrarily long length. Details may be found in the PhD thesis of the second author: [V].

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2. Set-up and Discussion

For $k \in \{0,1\}$ fixed, $q \geqslant 1$ and χ a Dirichlet character of modulus q such that $\chi(-1) = (-1)^k$, we denote by $S_k(q,\chi)$, the vector space of cusp forms $f(z)$ of weight k, level q and nebentypus χ (relatively to the action of any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ given by $f_{|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) := \left(\frac{cz+d}{|cz+d|}\right)^{-k} f\left(\frac{az+b}{cz+d}\right)$). It is acted on by the Laplacian of weight k

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right) - iky\frac{\partial}{\partial x},$$

and is equipped with the Petersson inner product $(f,f) = \int_{\Gamma_0(q)\backslash\mathbb{H}} |f(z)|^2 \frac{dxdy}{y^2}$. We denote $S^{\mathrm{Artin}}_{1/4,k}(q,\chi)$ the finite subset of $S_k(q,\chi)$ consisting of *primitive Hecke eigenforms* that are associated to some Galois representation ρ_f (in particular, they have Δ_k-eigenvalue $1/4$). For functional purposes, the definition of "associated" can be restricted to primes not dividing q: $\mathrm{Trace}(\rho_f(\mathrm{Frob}_p)) = \lambda_f(p)$, and $\det(\rho_f(\mathrm{Frob}_p)) = \chi(p)$.

This set is a disjoint union:

$$S^{\mathrm{Artin}}_{1/4,k}(q,\chi) = \bigcup_{G\in\mathrm{Types}} S^{G}_{1/4,k}(q,\chi)$$

The sets S^{Dihedral} are well understood, and parameterized by class group characters of quadratic extensions of $\mathbb{Q}$. In particular, one has a uniform upper bound $|S^{\mathrm{Dihedral}}_{1/4,k}(q,\chi)| \ll q^{1/2+\epsilon}$. On the other hand, the "exotic" sets for $G \neq$ Dihedral are much more mysterious. Duke was the first to give a non-trivial upperbound for the cardinality of the exotic sets when q is large [D]: $S^{\mathrm{Exotic}}_{1/4,k}(q,\chi) \ll_\varepsilon q^{1-1/12+\varepsilon}$ for all $\varepsilon > 0$, when χ a real character. Later, Duke's results were generalized (to any χ) and improved (in the tetrahedral and octahedral case) by Wong [W] by more elaborate group theoretical arguments.

In this context, Theorem 1 is the assertion that $S^G_{1/4,k}(q,\chi) \ll_\epsilon q^{e(G)+\epsilon}$, the implied constant depending only on ϵ.

3. Petersson-Kuznetzov formula

A Maass cusp form $f(z) \in S_k(q,\chi)$ has a Fourier expansion:

$$f(z) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \rho_f(n) W_{\frac{n}{|n|}\frac{k}{2},it}(4\pi|n|y)e(nx) \tag{3.1}$$

where $W_{\alpha,\beta}(y)$ is the Whittaker function, and $\lambda = (1/2+it)(1/2-it)$ is the eigenvalue of Δ_k, $t \in \mathbf{R} \cup [-i/4, i/4]$. When f is a primitive Hecke eigenform, we have the equality

$$\sqrt{n}\rho_f(n) = \lambda_f(n) \tag{3.2}$$

where $\lambda_f(n)$ is the n-th Hecke eigenvalue.

Take $B(q,\chi) = \{u_j\}_{j\geqslant 1}$, any orthonormal basis for $S_k(q,\chi)$ compatible with the action of Δ_k, so that u_j has eigenvalue $\lambda_j = 1/4 + t_j^2$ and Fourier coefficients $\rho_j(n)$. We shall use the following version of the Kuznetsov formula, which may be found in [DFI] Proposition 5.2 and section 14. For any real number r, and any integer j we set

$$h(t,r) = \frac{4\pi^3}{|\Gamma(1-\frac{j}{2}-ir)|^2}.\frac{1}{\mathrm{ch}\pi(r-t)\mathrm{ch}\pi(r+t)},$$

and the Kloosterman integral (for $x > 0$)

$$I(x,r) = -2x\int_{-i}^{i}(-i\zeta)^{j-1}K_{2ir}(\zeta x)d\zeta.$$

Note that for t real or purely imaginary, we have $h(t,r) > 0$, and, by Equation (12.7) of [DFI]:

$$I(x,r) \ll \min(x(1+\log|x|), x^{-1/2}(|r|+1)^{1/2}) \tag{3.3}$$

The Kuznetsov formula then states:

Proposition 3.1. *For any positive integers m,n we have*

$$\sum_{j\geqslant 1} h(t_j,r)\sqrt{mn}\,\overline{\rho}_j(m)\rho_j(n) + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbf{R}} h(t,r)\sqrt{mn}\overline{\rho}_{\mathfrak{a}}(m,t)\rho_{\mathfrak{a}}(n,t)dt$$

$$= \delta_{m,n} + \sum_{c\equiv 0(q)} \frac{S_\chi(m,n;c)}{c} I(\frac{4\pi\sqrt{mn}}{c}, r)$$

where $\rho_{\mathfrak{a}}(m,t)$ denotes the Fourier coefficient of the Eisenstein series associated to the cusp $\mathfrak{a}$, $S_\chi(m,n;c)$ is the Kloosterman sum

$$S_\chi(m,n;c) = \sum_{x(c),(x,c)=1} \overline{\chi}(x)e(\frac{m\overline{x}+nx}{c}),$$

For Kloosterman sums we recall the standard bound ($d(c)$ denotes the divisor function)

$$|S_\chi(m,n;c)| \leqslant d(c)(m,n,c)^{1/2}c^{1/2}. \tag{3.4}$$

We can now embed the set $\{\frac{f}{(f,f)^{1/2}}\}_{f\in S^{\mathrm{Artin}}_{1/4,k}(q,\chi)}$ into such an orthonormal basis $\{u_j\}$. Using positivity of $h(t,r)$, (3.2), and the bound (3.3) we deduce the following: for any $N\geqslant 1$ and any sequence of complex numbers $(a_n)_{\substack{n\leqslant N\\(n,q)=1}}$ we may bound:

$$\sum_{f\in S^G_{1/4}(q,\chi)}\frac{h(1/4,r)}{(f,f)}|\sum_{\substack{n\leqslant N\\(n,q)=1}}a_n\lambda_f(n)|^2 \tag{3.5}$$

by the complete spectral sum:

$$\sum_{j\geqslant 1}h(t_j,r)|\sum_{\substack{n\leqslant N\\(n,q)=1}}a_n\sqrt{n}\rho_j(n)|^2+\quad\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{\mathbf{R}}h(t,r)|\sum_{\substack{n\leqslant N\\(n,q)=1}}a_n\sqrt{n}\rho_{\mathfrak{a}}(n,t)|^2dt$$

$$\ll_\varepsilon \sum_{\substack{n\leqslant N\\(n,q)=1}}|a_n|^2+(qN)^\varepsilon\frac{N^{1/2}}{q}\Big(\sum_{\substack{n\leqslant N\\(n,q)=1}}|a_n|\Big)^2 \tag{3.6}$$

4. The Amplification

Fix $G\in$ Types. From group theoretic results (see [W] Lemma 3 and the proof of Theorem 10) one can build an appropriate amplifier for the each family $S^G_k(q,\chi)$: ie. one can choose a sequence $(a^G_n)_{\substack{n\leqslant N\\(n,q)=1}}$ such that, for some exponent $\alpha(G)>0$, we have

$$\sum_{n\leqslant N}|a^G_n|+\sum_{n\leqslant N}|a^G_n|^2\ll N^{\alpha(G)}$$

and such that for any $f\in S^G_k(q,\chi)$ we have

$$|\sum_{n\leqslant N}a^G_n\lambda_f(n)|\gg N^{\alpha(G)}/\log N.$$

For instance, in the icosahedral case, the sequence is constructed from the relation

$$\overline{\chi}^6(p)\lambda_f(p^{12})-\overline{\chi}^4(p)\lambda_f(p^8)-\overline{\chi}(p)\lambda_f(p^2)=1$$

valid for any $f\in S^{\mathrm{Ico}}_k(q,\chi)$ and any $p\not| q$. The a^{Ico}_n defined by

$$a^{\mathrm{Ico}}_n=\begin{cases}\overline{\chi}^6(p),\ \text{for } n=p^{12}\leqslant N,\ (p,q)=1\\ -\overline{\chi}^4(p),\ \text{for } n=p^8\leqslant N^{8/12},\ (p,q)=1\\ -\overline{\chi}(p),\ \text{for } n=p^2\leqslant N^{2/12},\ (p,q)=1\\ 0 \text{ else.}\end{cases}$$

is an amplifier with $\alpha(\mathrm{Ico})=1/12$, and similarly one can find in Wong's paper the construction of the amplifiers in the tetrahedral and octahedral case: in particular, we have $\alpha(\mathrm{Octahedral})=1/8$, $\alpha(\mathrm{Tetrahedral})=1/4$. From the upper bound $(f,f)\leqslant q\log^3 q$ for

$f \in S_k^G(q,\chi)$ (see [D]: note that the forms considered satisfy the Ramanujan-Petersson conjecture) and the fact that Equation 3.5 is bounded by 3.6,

$$|S_k^G(q,\chi)| \ll_\varepsilon (qN)^\varepsilon (qN^{-\alpha(G)} + N^{1/2}) \ll_\varepsilon q^{\varepsilon+1-\frac{2\alpha(G)}{1+2\alpha(G)}}$$

for any $\varepsilon > 0$ by choosing $N = q^{\frac{2}{1+2\alpha(G)}}$. Noting that our definition of $e(G)$ was so that $e(G) = (1+2\alpha(G))^{-1}$, this completes the proof of Theorem 1. ■

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