HOMOLOGICAL STABILITY FOR HURWITZ SPACES AND THE COHEN-LENSTRA CONJECTURE OVER FUNCTION FIELDS, II.

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Abstract. We prove a version of the Cohen–Lenstra conjecture over function fields (completing the results of our prior paper).

This is deduced from two more general theorems, one topological, one arithmetic: We compute the direct limit of homology, over puncture-stabilization, of spaces of maps from a punctured manifold to a fixed target; and we compute the Galois action on the set of stable components of Hurwitz schemes.

Contents

1. Introduction

1.1. **Hurwitz spaces and the Cohen-Lenstra heuristics.** Fix a finite group G and a conjugacy-invariant subset $c \subset G$.

Let $\mathrm{CH}^c_{G,n}$ be the space whose points are given by pairs (E, f), where $E \subset \mathbf{C}$ is a subset of size n, and f is a conjugacy class of surjective morphisms $\pi_1(\mathbf{P}^1_{\mathbf{C}} - E, \infty) \twoheadrightarrow G$, carrying a loop around every point of D into c. Thus $\mathrm{CH}^c_{G,n}$ is "the space of connected G-covers of the Riemann sphere branched at n points of the complex plane, and with monodromy around each branch point inside c."

Clebsch and Hurwitz proved the connectivity of $\mathrm{CH}^c_{G,n}$ for large n, in the case where G is a symmetric group and c the conjugacy class of transpositions. In other words, any two "simply branched" coverings of the plane with the same numerical data can be deformed into each other. The space $\mathrm{CH}^c_{G,n}$ has the natural structure of a complex manifold; this manifold, in turn, can be thought of as the complex points of an algebraic variety over $\mathbf{Q}$, which is defined as a moduli space for branched G-covers in a suitable algebraic sense. This description induces an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the set of connected components of $\mathrm{CH}^c_{G,n}$, and also an action of this Galois group on $H^*(\mathrm{CH}^c_{G,n}, M)$ whenever M is a finite abelian group.

Let us now describe three sample results of this paper. We note that in the body of the paper we work with a slightly larger space, $\mathrm{CHur}^c_{G,n}$, which allows ramification at ∞ and also considers the morphism f not merely up to conjugacy; $\mathrm{CH}^c_{G,n}$ is then a quotient of a certain union of components of $\mathrm{CHur}^c_{G,n}$ (see §1.3 for further discussion). One easily deduces the results below from the results we prove about $\mathrm{CHur}^c_{G,n}$.

Let G be a group of order congruent to 2 mod 4 and let c be the unique conjugacy class of involutions.

> (a) There exists $\alpha > 0$ so that, for all n, each component of $\mathrm{CH}^c_{G,n}$ has vanishing Betti numbers in dimensions between 2 and αn.

and, in the same setting,

> (b) There is a bijection, for n odd and sufficiently large,
>
> $$\pi_0(\mathrm{CH}^c_{G,n}) \longrightarrow H_2(A, \mathbf{Z})_{G/A},$$
>
> equivariant for the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, where A is the unique normal subgroup of G of index 2, the notation $H_2(A, \mathbf{Z})_{G/A}$ means the coinvariants for G/A acting on $H_2(A, \mathbf{Z})$, and where the Galois action on $H_2(A, \mathbf{Z})$ is through the cyclotomic character.[1]

In fact, the theorems proved in this paper are true for quite general groups G – indeed, for our topological results, G need not even be finite. The restriction to the specific class of groups with order congruent to 2 mod 4 is actually only needed for (a), and even there it is imposed only because of the limitations of the previous paper [EVW09]. However, as explained in that paper, results of this kind can be used to prove a version of the Cohen-Lenstra conjecture over rational function fields. In particular we will deduce

> (c) If A is an abelian group of odd order, and q is sufficiently large (depending on A), then the A-part of the Cohen-Lenstra heuristics hold for imaginary quadratic extensions of $\mathbf{F}_q(T)$. (Theorem 12.1.1)

We also obtain results of a similar nature even when A is *nonabelian* of odd order, counting now the average number of unramified A-extensions of imaginary quadratic fields, although we do not discuss these in detail in this paper. (See also §6.1 for a sample application to Cohen-Lenstra-Martinet heuristics for cubic extensions.)

[1] By this we mean the following: Take $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and $\alpha \in H_2(A, \mathbf{Z})$. Choose a primitive root of unity $\zeta \in \overline{\mathbf{Q}}$ of the same order as α. Then $g(\zeta) = \zeta^s$ for some integer s. We define also $g(\alpha) = \alpha^s$.

In any case, statement (c) is a considerable refinement of the main theorem of [EVW09]. For example, there is Q with the following property:

> If $q \geq Q$, the average number of 15-torsion elements in the Jacobian of a random hyperelliptic curve over $\mathbf{F}_q$ is four.

Our prior results showed only that this average *exists* for sufficiently large q, and approaches 4 as $q \to \infty$. In the remainder of the discussion we discuss some of the ingredients in the proof of the above results. See also §1.7 for an overview of the structure of the paper.

1.2. **A classifying space for branched covers.** Let us recall that, for a finite group G, the classifying space BG classifies "G-covers", that is to say: for any finite CW complex X, the set of pointed homotopy classes of maps from X to BG are naturally in bijection with isomorphism classes of G-torsors (i.e. principal G-bundles) over X with a trivialization of the fibre over the basepoint of X.

A key point for us is the existence of a variant of the space BG that classifies *branched* G-covers.

To motivate this space, consider a Riemann surface Σ and a G-cover of Σ branched at points $\{p_1, \ldots, p_n\}$; by this, we mean simply a G-torsor $f : Z \to \Sigma - \{p_1, \ldots, p_n\}$. Such a G-cover defines (at least the homotopy class of) a mapping $\varphi : \Sigma - \{p_1, \ldots, p_n\} \to BG$; under this map, a small circle c_i around each p_i is sent to a loop in BG.

We would like to extend φ to a mapping from Σ to a "compactification" of BG. This can be achieved by adjoining discs to BG in such a way to make each $\varphi(c_i)$ contractible.

More precisely, we make the following definition: For $c \subset G$ any conjugacy-invariant subset, define $A(G, c)$ by adjoining to BG the space $L^c BG \times D^2$. Here $L^c BG \subset \mathrm{Map}(S^1, BG)$ are maps in the free homotopy class of c; we glue these disks to BG along their boundary via the tautological map $L^c BG \times S^1 \to BG$.

Although we will not use this fact, the space $A(G, c)$ can be understood as a classifying space for branched covers of manifolds with trivialized normal bundle to the branch locus. For related results see Brand's paper [Bra80]. For now we explain how $A(G, c)$ controls the topology of Hurwitz spaces.

1.3. **Stable homology of Hurwitz spaces and result (a) of §1.1.** Let us elaborate on (a) from §1.1.

First, as previously mentioned, in the body of this paper we work not with $\mathrm{CH}^c_{G,n}$, as defined in the introduction, but a larger space $\mathrm{CHur}^c_{G,n}$. We will define here only the *topological* version of this larger space; this space is homotopy-equivalent to the complex points of a certain Hurwitz scheme, but we leave the definition of that scheme to §8.

Let D^2 be the unit disc in the plane, and fix a point $*$ on the boundary of D^2. We define $\mathrm{CHur}^c_{G,n}$ to be the space parameterizing pairs (E, f) where $E \subset D^2$ is a subset of size n, and f is a surjective morphism $\pi_1(D^2 - E, *) \twoheadrightarrow G$, carrying a loop around every point of E into c.

In order to formulate our results, we make use of maps

$$V_n : \mathrm{CHur}^c_{G,n} \to \mathrm{CHur}^c_{G,n+D} \tag{1.3.1}$$

obtained by taking "edge-sum" with a certain D-branched cover of a disc; for details see §5.5. For simplicity in this discussion, we will restrict to the setting where c is a single conjugacy class, although we treat the general case in the text.

By means of the ideas sketched above, one constructs a natural map from $\mathrm{CHur}^c_{G,n}$ to the function space $\mathrm{Map}^\partial_n(D^2, A(G,c))$ consisting of those degree n continuous maps, carrying the boundary of D^2 into the subspace $BG \subseteq A(G,c)$. (Here, the notion of "degree n" is defined by composing with the map $A(G,c) \to S^2$ which collapses BG to a point.) Moreover, it is possible to define maps $V_n : \mathrm{Map}_n \to \mathrm{Map}_{n+D}$ which are compatible with those of (1.3.1) and, furthermore, are homotopy equivalences. With these notations we prove:

1.3.1. **Theorem.** *The map*

$$\mathrm{CHur}^c_{G,n} \to \mathrm{Map}^\partial_n(D^2, A(G,c))$$

induces an isomorphism on the direct limit (along the V_n) in homology, $\varinjlim H_i$, for any i.

In our current setting, where c is a single conjugacy class, the space $A(G,c)$ has the rational homotopy type of a 2-sphere, and the limiting rational homology on the right hand side of Theorem 1.3.1 is easily computed: it is zero in degrees greater than 1.

An important caveat is that Theorem 1.3.1 gives us very little information about the topology of $\mathrm{CHur}^c_{G,n}$ for any particular n. It does allow us to compute, for any i, the limit

$$\varinjlim_n H_i(\mathrm{CHur}^c_{G,n})$$

However, this information places at best a lower bound on the size of $H_i(\mathrm{CHur}^c_{G,n})$, which may, a priori, contain any number of classes that vanish when n is increased. In order to get results about the homology of individual moduli spaces, we need a *stabilization theorem* guaranteeing that $H_i(\mathrm{CHur}^c_{G,n})$ is constant when n is sufficiently large relative to i.

1.3.2. **Corollary.** *Let $i \geq 0$. Suppose the maps $V_n : \mathrm{CHur}^c_{G,n} \to \mathrm{CHur}^c_{G,n+D}$ induce isomorphisms in H_i for all sufficiently large n. Then*

$$H_i(\mathrm{CHur}^c_{G,n}) = H_i(\mathrm{Map}^\partial_n(D^2, A(G,c)))$$

for all sufficiently large n.

Stabilization theorems of the desired kind were proved in [EVW09] for groups of order congruent to to 2 mod 4, as described in the previous section. Thus we obtain §1.1 part (a).

1.4. **Configuration–mapping spaces.** Theorem 1.3.1 is actually a special case of a more general result concerning "configuration-mapping spaces." For an n-manifold M with boundary and a topological space X, we will construct a space $\mathrm{CMap}^*_k(M;X)$ consisting of pairs $(\underline{z}, f)$, where $\underline{z}$ is an unordered configuration of k points in the interior of M, and f is a continuous, based function $f : M \setminus \underline{z} \to X$.

The Hurwitz space $\mathrm{CHur}^c_{G,n}$ is our principal example of such a space. It is homotopy equivalent to the subspace of the configuration mapping space $\mathrm{CMap}^*_n(D^2, BG)$ where we insist that the induced map on π_1 be surjective, and that loops around points in the configuration map into the free homotopy type of $c \subseteq G = \pi_1(BG)$.

Concretely: given a branched G-cover $Y \to D^2$, the branch locus defines a configuration $\underline{z} \in D^2$; and the cover defines a map $f : D^2 \setminus \underline{z} \to BG$ recording its isomorphism type.

Our main technical result, Theorem 2.10.1, identifies the limiting homology of $\mathrm{CMap}_k^*(M; X)$; Theorem 1.3.1 is an immediate corollary.

Our proof of Theorem 2.10.1 follows the model set up in [May72, McD75, Sal01], with a few important modifications. We denote by $\mathrm{CMap}_k^\partial(M; X)$ the subspace consisting of those $(\underline{z}, f)$, where f sends the boundary ∂M to the basepoint of X. We first show that for the n-disk, the union

$$\mathrm{CMap}^\partial(D^n; X) := \coprod_{k=0}^{\infty} \mathrm{CMap}_k^\partial(D^n; X)$$

forms an algebra over the little n-disks operad $\mathcal{D}_n$. We are thus entitled to form the n-fold iterated bar construction of this algebra $B_n \mathrm{CMap}^\partial(D^n; X)$. A variant on the group completion theorem then gives

$$H_*(\mathrm{CMap}^\partial(D^n; X))[\pi_0^{-1}] = H_*(\Omega^n B_n \mathrm{CMap}^\partial(D^n; X)).$$

Finally it is possible to explicitly identify $B_n \mathrm{CMap}^\partial(D^n; X)$: Using a quasifibration sequence, this may be described in geometric terms as the quotient of $\mathrm{CMap}^\partial(D^n; X)$ by the relation where points are allowed to disappear on the boundary. In the situation of Theorem 1.3.1 this quotient space is closely related to $A(G, c)$.

The corresponding result for $\mathrm{CMap}^*(D^n; X)$ is proven similarly. We prove the theorem for general domain manifolds M using a handle decomposition of M and the result for disks.

1.5. **Stable H_0 and result (b) from §1.1.** Now let us discuss in more detail the second result of §1.1. We begin by discussing the result without the Galois action; we continue to assume, for simplicity, that c consists of a unique conjugacy class.

In the case of H_0 the stabilization property (as enunciated in Corollary 1.3.2) is known for *any* G, c. That Corollary shows, then, that for sufficiently large n the components of $\mathrm{CHur}_{G,n}^c$ are the same as the components of $\mathrm{Map}_n^\partial(D^2, A(G, c))$, i.e., they are parameterized by degree n elements of the relative homotopy group $\pi_2(A(G, c), BG)$, where "degree" is as discussed after Theorem 1.3.1.

But we prove in part (ii) of Theorem 7.5.1 that the degree zero subgroup of $\pi_2(A(G, c), BG)$ is isomorphic to a certain explicitly computable quotient $H_2(G, c; \mathbf{Z})$ of the second homology group $H_2(G, \mathbf{Z})$. In particular, the degree n elements of $\pi_2(A(G, c), BG)$ are a torsor for $H_2(G, c; \mathbf{Z})$, i.e.

The components of $\mathrm{CH}_{G,n}^c$ are a torsor for $H_2(G, c; \mathbf{Z})$,

at least for sufficiently large n.

We have learned in the course of writing this paper that Fried has an (unpublished) proof of this result. The manuscript [Kul11] also contains arguments somewhat related to our proof. A proof of a special case appears in [FV91], following ideas of Conway–Parker.

One easily computes this group $H_2(G, c; \mathbf{Z})$ to be isomorphic to $H_2(A, \mathbf{Z})_{G/A}$ in the context of §1.1. However, the more significant part of (b) of §1.1 is the computation of the Galois action:

As we have discussed the set of components of $\mathrm{CHur}^c_{G,n}$ carries an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, because $\mathrm{CHur}^c_{G,n}$ is naturally identified with the complex points of a $\mathbf{Q}$-variety. Theorem 8.7.3 computes this Galois action. The basic idea of the computation is simple: we just carry over some topological ideas into algebraic geometry. The implementation is somewhat technical, however.

We do not describe the Galois action in the general case here, except to say that it is a little more complicated than statement §1.1 (b), in that it involves a twist by a certain canonical cocycle in $H^1(\hat{\mathbf{Z}}^\times, H_2(G,c))$.

Here is a simple example of this story: Let G be the semidirect product $(\mathbf{Z}/3\mathbf{Z})^2 \rtimes \{\pm 1\}$, where ± 1 acts by negation on $(\mathbf{Z}/3\mathbf{Z})^2$. Let c be the conjugacy class of involutions. In this case, it turns out that $H_2(G,c)$ is isomorphic to $\mathbf{Z}/3\mathbf{Z}$. The Hurwitz scheme $\mathrm{Hur}^c_{G,n}$ parameterizes hyperelliptic curves X of genus $\lfloor (n-1)/2 \rfloor$ together with an injection $\iota : (\mathbf{Z}/3\mathbf{Z})^2 \hookrightarrow H^1(X, \mathbf{Z}/3\mathbf{Z})$. There are exactly three connected components for n large, differentiated by the pullback to $(\mathbf{Z}/3\mathbf{Z})^2$ of the symplectic pairing ω on $H^1(X, \mathbf{Z}/3\mathbf{Z})$. The component corresponding to $\iota^*\omega \equiv 0$ is defined over $\mathbf{Q}$, and the other two components are defined over $\mathbf{Q}(\sqrt{-3})$.

1.6. **Application to the Cohen–Lenstra heuristics; results (c) and (d) of §1.1.** The example mentioned at the end of §1.5 may suggest a relationship with the Cohen–Lenstra heuristics. Indeed, the idea of how to apply the foregoing results to the Cohen–Lenstra heuristics follows a general philosophy, enunciated in [EVW09, §1.7], that problems in analytic number theory, considered over a function field, become related to problems of stable topology. We refer to [EVW09] for more details on the translation. In the particular context of Cohen–Lenstra, we would like to mention also the work of J.-K. Yu [Yu97], who verified the connectivity of certain Hurwitz spaces and used this to draw conclusions towards the "large characteristic limit" of Cohen–Lenstra.

Although our treatment of the deduction of the Cohen-Lenstra heuristics here is brief, since most of the technical details were handled in [EVW09], we have here tried to state our results in as general a context as possible. For example, as mentioned after result (c) in §1.1, we may obtain *nonabelian* versions of the Cohen–Lenstra heuristics. Moreover, our Theorem 11.1.1 provides a heuristic for the asymptotic number of G-extensions for an arbitrary finite group G, which is in conformity with conjectures of Malle and Bhargava but which extends them to substantially more general contexts.[2] Theorem 11.1.1 also implies that such asymptotics would follow from a homological stability theorem in the sense of Corollary 1.3.2.

In fact, for application to Cohen–Lenstra heuristics, the computation of the Galois action on connected components of the relevant Hurwitz spaces (e.g., those discussed in the example at the end of §1.5) can be avoided by appealing, instead, to known results on the monodromy of hyperelliptic curves ([AH10, Yu97]) and the Weil pairing; we sketch this approach in Prop 6.0.3. On the other hand, as we have mentioned, the general understanding of connected components of Hurwitz spaces allow our results to be much more general; we chose to highlight the Cohen–Lenstra heuristics in §1.1, since they are the best-known from this class of problems.

[2]For instance: it makes precise predictions about the effect of roots of unity in the base field. It is well-known that even the correct formulation of such conjectures in presence of roots of unity can be rather subtle.

1.7. **Structure of the paper.** We now briefly outline the contents of the various sections and how they fit together.

Part 1 is purely topological, centered on the computation of the stable topology of configuration–mapping spaces.

- §2 gives definitions and states the main result (Theorem 2.8.1);
- §3 gives the proof of Theorem 2.8.1.
- §4 contains some arguments necessary for deriving limiting statements on homology from the results in stable topology obtained in the previous two sections.

Part 2 deals with Hurwitz spaces and schemes.

- §5 applies the results of Part 1 to Hurwitz spaces. In particular, assuming stability, the rational homology of *each component* of the Hurwitz space is computed (Corollary 5.8.2).
- §6 is motivational: it sketches how the results of §5 can be applied immediately to the Cohen–Lensta heuristics, under certain simplifying assumptions.
- §7 computes the set of components of the Hurwitz space via combinatorial group theory. (This could also be deduced from Part 1, but the approach given here is easier to translate into algebraic geometry).
- §8 is the technical core of this part. It defines the algebro-geometric version of Hurwitz spaces, the Hurwitz schemes, and computes the Galois action on their set of connected components.
- §9 is meant to be a user-friendly guide to the prior sections §7 and S8, in particular aimed at readers who want to use the results of §8.

Part 3 deals specifically with the asymptotics of counting extensions of function fields (and, by analogy, of number fields.)

- §10 sets up some necessary framework relating (G, c)-extensions of $\mathbf{F}_q(t)$ with $\mathbf{F}_q$-rational points on forms of Hurwitz schemes.
- §11 formulates the main theorem about extensions of $\mathbf{F}_q(t)$ with bounded discriminant. We show (Theorem 11.1.1) that, under an assumption of homological stability, the main theorems of the paper allow us to count the number of such extensions under certain conditions on ramification, and that the results support conjectures of Malle and of Bhargava.
- §12 applies Theorem 11.1.1 to the Cohen–Lenstra setting; in this case, homological stability is known by [EVW09], and one obtains the theorems announced in the introduction.

1.8. **Acknowledgements.** Some technical aspects of this paper are based upon McDuff and Salvatore's beautiful papers [McD75, Sal01] which in turn rest upon foundational work of Boardman-Vogt [BV73] and May [May72]. We wish to thank Søren Galatius and Arun Ram for several useful conversations, Greg Brumfiel for introducing us to the work of Brand [Bra80] on concordance of branched covers, and Mike Fried for explaining to us his unpublished recent work on components of Hurwitz spaces. We also note that some of the results of Part 1 were independently found by Johannes Ebert in unpublished work. The first author was partially supported by the National Science Foundation and by a Romnes Faculty Fellowship; the second-named author was partially supported by the National Science Foundation and the David and Lucile Packard Foundation. The third named author was partially supported by the Australian Research Council under the Discovery Project and Future Fellowship schemes.

1.9. **Notation.**

- S_k is the symmetric group on k letters.
- The k^{th} ordered configuration space of a manifold M is denoted $F_k(M)$. The unordered configuration space is $C_k(M) := F_k(M)/S_k$.
- $\dot{M} := M \setminus \partial M$ is the interior of M.
- For a group G, we will write $G^{ab} = G/[G,G]$ for its abelianization.
- For a union of conjugacy classes $c \subseteq G$, write $\langle c \rangle$ for the (normal) subgroup generated by G.
- If X is a topological space and G a topological group acting on X continuously, the Borel construction is $X_{hG} = X \times_G EG$.
- Throughout, the word *space* will be shorthand for "space having the homotopy type of a CW complex."
- The symbol $\simeq$ will denote homotopy equivalence.
- For topological spaces A and B, $\mathrm{Map}(A,B)$ will denote the space of continuous maps $f : A \to B$, equipped with the compact-open topology. If A and B are equipped with basepoints, $\mathrm{Map}_*(A,B)$ will be the subspace of based maps.

Part 1. Topology of configuration-mapping spaces

2. Configuration-mapping spaces

2.1. Construction. Throughout this section, we let M be a smooth, path-connected, oriented n-manifold, possibly with boundary or corners; we write ∂M for the boundary of M. Write $\dot{M} := M \setminus \partial M$ for the interior of M.

The k^{th} *ordered configuration space*, $F_k(\dot{M})$, is the subspace

$$F_k(\dot{M}) := \{(x_1, \ldots, x_k) \mid x_i \neq x_j \text{ if } i \neq j\} \subseteq \dot{M}^{\times k}$$

The space $F_k(\dot{M})$ admits a free action of S_k by permutation of coordinates; the quotient space $C_k(M) := F_k(\dot{M})/S_k$ is the *unordered configuration space.*

Let $S \subseteq M$ be a subspace of M, and let $(X, *)$ be any based path-connected topological space.

2.2. Definition. The *configuration-mapping space* $\mathrm{CMap}_k(M; X)$ is the space of pairs:

$$\mathrm{CMap}_k(M; X) := \{(\underline{x}, f) \mid \underline{x} \in C_k(\dot{M}),\ f : M \setminus \underline{x} \to X\}$$

Further, let $\mathrm{CMap}_k^S(M; X) \subseteq \mathrm{CMap}_k(M; X)$ be the subspace consisting of $(\underline{x}, f)$ with $f|_{S\setminus(S\cap\underline{x})} = *$. Finally, let $\mathrm{CMap}^S(M; X) := \coprod_k \mathrm{CMap}_k^S(M; X)$. We topologize these spaces as described below.

For all $n > 1$, the diffeomorphism group $\mathrm{Diff}^+(M, \partial M)$ acts on the unordered configuration space $C_k(\dot{M})$ transitively: for any pair $\underline{x} := \{x_1, \ldots x_k\}$ and $\underline{y} := \{y_1, \ldots, y_k\}$ of configurations, one can find disjoint open balls U_i containing x_i and y_i. The diffeomorphism group of a ball is doubly transitive, so we may construct a diffeomorphism of M (the identity away from the U_i) which carries $\underline{x}$ to $\underline{y}$.

Pick a configuration $\underline{z} = \{z_1, \ldots, z_k\} \in C_k(\dot{M})$, and write $\mathrm{Diff}^+(M, \underline{z})$ for the subgroup of $\mathrm{Diff}^+(M, \partial M)$ that fixes $\underline{z}$ as a set (i.e., possibly permuting the points making up $\underline{z}$). The transitivity above yields a homeomorphism

$$C_k(\dot{M}) \cong \mathrm{Diff}^+(M, \partial M)/\,\mathrm{Diff}^+(M, \underline{z}). \tag{2.2.1}$$

2.3. Example. Note that if $M = D^2$,

$$\mathrm{Diff}^+(D^2, \underline{z}) \simeq \pi_0(\mathrm{Diff}^+(D^2, \underline{z})) \cong \mathrm{Br}_k = \pi_1(C_k(\dot{D^2})/S_k)$$

is the Artin braid group on k strands. If we insist that that diffeomorphisms fix the configuration pointwise, we instead obtain the pure braid group PBr_k. These facts follow from the long exact sequence of the fibration (2.2.1), and the fact that $\mathrm{Diff}^+(D^2, \partial D^2)$ is contractible.

Note that $\mathrm{Diff}^+(M, \underline{z})$ acts naturally on $\mathrm{Map}(M \setminus \underline{z}, X)$ by precomposition. There is a natural bijection

$$\mathrm{Diff}^+(M, \partial M) \times_{\mathrm{Diff}^+(M,\underline{z})} \mathrm{Map}(M \setminus \underline{z}, X) \to \mathrm{CMap}_k(M; X) \tag{2.3.1}$$

which carries (ϕ, f) to $(\phi(\underline{z}), f \circ \phi^{-1})$. We may therefore topologize $\mathrm{CMap}_k(M; X)$ via the obvious (compact-open) topology on the left side of the bijection. Further, the map $\mathrm{Diff}^+(M, \partial M) \to C_k(\dot{M})$ given by (2.2.1) has local sections. This is obvious in the case of a single point in D^n, where *global* sections exist, since $C_1(D^n)$ is contractible. The general setting follows "operadically:" produce local sections for an arbitrary configuration in M by gluing local sections on disks containing each point to the identity. Therefore $\mathrm{Diff}^+(M, \partial M) \to C_k(\dot{M})$ is a principal $\mathrm{Diff}^+(M, \underline{z})$-bundle, and so we may conclude

2.3.1. **Proposition.** *The map* $(\underline{x}, f) \mapsto \underline{x}$ *is a fibre bundle* $\mathrm{CMap}_k(M; X) \to C_k(\dot{M})$ *with fibre over* $\underline{x}$ *given by* $\mathrm{Map}(M \setminus \underline{x}, X)$.

2.4. **Restriction of monodromy.** We may define restricted configuration-mapping spaces by controlling the behavior of the function near points of the configuration. Since X is path-connected, we know that $\pi_1(X, *)$ acts on $\pi_k(X, *)$ for all $k \geq 1$ by the path lifting property of the fibration sequence

$$\Omega^k X \to \mathrm{Map}(S^k, X) \to X$$

In the case $k = 1$, this action is via conjugation in $\pi_1(X, *)$. The quotient under this action is the set of free homotopy classes of maps from S^k to X, i.e.,

$$\pi_k(X, *)/\pi_1(X, *) = [S^k, X] = \pi_0(\mathrm{Map}(S^k, X))$$

Let c be a subset of $\pi_0(\mathrm{Map}(S^{n-1}, X)) = \pi_{n-1}(X, *)/\pi_1(X, *)$. We will write $\mathrm{Map}^c(S^{n-1}, X)$ for the union of the components of $\mathrm{Map}(S^{n-1}, X)$ indexed by c.

2.5. **Definition.** Define $\mathrm{CMap}_k^c(M, X) \subseteq \mathrm{CMap}_k(M, X)$ as the subspace $\{(\underline{x}, f)\}$ with the restriction of f to the boundary of an small ball containing each x_i (and no other x_j) to be an element of $\mathrm{Map}^c(S^{n-1}, X)$.

In this definition, we use the fact that our choice of orientation on M gives a well-defined homotopy class of equivalences of S^{n-1} with the boundary of small balls around x_i.

2.6. **Multiplicative structure and boundary monodromy.** When we take $M = D^n$, there is a natural multiplicative structure on the subspace $\mathrm{CMap}^\partial(D^n; X) \subseteq \mathrm{CMap}(D^n; X)$ consisting of those maps which send all of ∂D^n to $*$.

2.6.1. **Proposition.** $\mathrm{CMap}^\partial(D^n; X)$ *is an algebra over the n-dimensional little disks operad $\mathcal{D}_n$; consequently,* $\mathrm{CMap}^\partial(D^n; X)$ *is an A_∞ topological monoid whose group completion* $\Omega B\, \mathrm{CMap}^\partial(D^n; X)$ *is an n-fold loop space.*

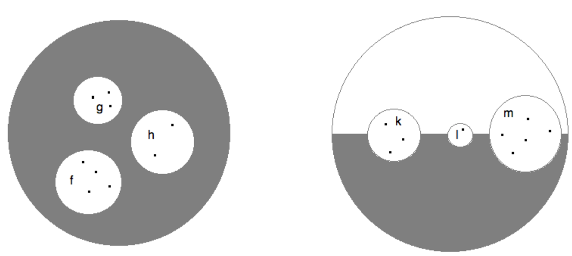


FIGURE 1. On the left, the action of an element of $\mathcal{D}_2(3)$ on a triple of elements of $\mathrm{CMap}^\partial(D^2; X)$, as in Proposition 2.6.1. On the right, the action of an element of $\mathcal{D}_1(3)$ on a triple of elements of $\mathrm{CMap}^H(D^n; X)$, as in Proposition 2.6.2. In both cases, the area in grey is carried to $* \in X$.

This algebra structure lifts the usual $\mathcal{D}_n$-algebra structure on the union $C(D^n)$ of configuration spaces of D^n. Given an element of $\mathcal{D}_n(N)$ – a configuration of non-overlapping discs $D_1, \ldots, D_N$ inside D^n – the corresponding map

$$\mathrm{CMap}^{\partial}(D^n; X)^{\times N} \to \mathrm{CMap}^{\partial}(D^n; X)$$

is defined as in Figure 1. More precisely, given points $(\underline{x}_i, f_i)_{i=1\ldots N}$, we need to construct an $(\underline{x}, f)$ in $\mathrm{CMap}^{\partial}(D^n; X)$. Let $\phi_i : D^n \to D_i$ be the unique homeomorphism which is a composite of a dilation and a translation. Then we can take $\underline{x}$ to be the union of the $\phi_i(\underline{x}_i)$, and f to be the function which is $f \circ \phi_i^{-1}$ on the little disc D_i and $*$ on the complement of the little discs.

Pick a basepoint $* \in D^n$ on the boundary, and write $\mathrm{CMap}^*(D^n; X)$ for the subspace of $\mathrm{CMap}(D^n; X)$ consisting of maps which carry $*$ in D^n to $*$ in X. Then there is a sequence

$$\mathrm{CMap}^{\partial}(D^n; X) \xrightarrow{\subseteq} \mathrm{CMap}^*(D^n; X) \xrightarrow{res} \Omega^{n-1} X \tag{2.6.1}$$

where $res(\underline{z}, f) = f|_{\partial D^n}$; res is a union of fibrations, indexed by the number k of points in a configuration.

2.6.2. **Proposition.** *$\mathrm{CMap}^*(D^n; X)$, $\Omega^{n-1}X$, and $\mathrm{CMap}^{\partial}(D^n; X)$ are homotopy equivalent to $\mathcal{D}_{n-1}$-algebras, and the maps in* (2.6.1) *are maps of $\mathcal{D}_{n-1}$-algebras up to homotopy.*

Proof. First of all, the operad $\mathcal{D}_{n-1}$ embeds in $\mathcal{D}_n$; the configurations of little discs in D^{n-1} are in natural bijection with the configurations of little discs in D^n whose centers lie on the bisecting plane $x_n = 0$. In particular, any $\mathcal{D}_n$-algebra (for instance, $\mathrm{CMap}^{\partial}(D^n; X)$) is a $\mathcal{D}_{n-1}$-algebra by restriction. The loop space $\Omega^{n-1}X$ is also a $\mathcal{D}_{n-1}$-algebra (indeed, these operads were invented to model iterated loop spaces).

As for $\mathrm{CMap}^*(D^n; X)$, it can be identified with the subspace $\mathrm{CMap}^H(D^n; X) \subseteq \mathrm{CMap}(D^n; X)$ which carries the southern hemisphere $H \subseteq \partial D^n$ to $*$. The operad structure is as above: given a configuration of little discs $D_1, \ldots, D_N$ on D^n with centers on $x_n = 0$, and a list of N elements $(\underline{x}_i, f_i)$ of $\mathrm{CMap}^H(D^n; X)$, we can substitute $(\underline{x}_i, f_i)$ into D_i. We now need to extend the resulting map $\amalg D_i \to X$ to all of D^n. We send the points outside the little discs with $x_n \leq 0$ to $*$. If $(x_1, \ldots, x_n) \in D^n$ is a point outside the little discs with $x_n > 0$, we define $f(x_1, \ldots, x_n)$ to be $f(x_1, \ldots, x_n')$, where x_n' is the unique nonnegative value such that $(x_1, \ldots, x_n')$ lies on the boundary of some D_i, if there is such a value, or 0, if there is no such value.

□

We have now shown that $\Omega B\,\mathrm{CMap}^*(D^n; X)$ is an $(n-1)$-fold loop space $\Omega^{n-1}Y$, for some space Y. In the following section we give a model for Y.

A summary of the different configuration mapping spaces:

$\mathrm{CMap}_k(M; X)$	k points $\underline{x} \in M$, and a continuous function $f : M \setminus \underline{x} \to X$
$\mathrm{CMap}_k^c(M; X)$	the restriction of f to a sphere around x_i lies in $c \subseteq [S^{n-1}, X]$
$\mathrm{CMap}_k^{c,S}(M; X)$	furthermore, $f\|_{S \setminus (S \cap \underline{x})}$ is constant at $* \in X$
$\mathrm{CMap}_k^{c,*}(M; X)$	in particular, $f(*) = *$; i.e., f is basepoint-preserving.

2.7. **Stabilization results for configuration-mapping spaces.** As above, take $(X, *)$ to be a based topological space, and $c \subseteq \pi_{n-1}(X)$ a union of orbits under the action of $\pi_1(X)$.

2.8. **Definition.** Let $A_n(X, c)$ be the pushout of the diagram

$$D^n \times \mathrm{Map}^c(S^{n-1}, X) \xleftarrow{i} S^{n-1} \times \mathrm{Map}^c(S^{n-1}, X) \xrightarrow{ev} X$$

Here, $ev(s, f) = f(s)$. There is an inclusion $X \to A_n(X, c)$; define $A'_n(X, c)$ to be the *homotopy fibre* of this map. If $n = 2$ and $X = BG$ is the classifying space of a group G, we will write

$$A(G, c) := A_2(BG, c) \quad \text{and} \quad A'(G, c) := A'_2(BG, c).$$

Lastly, in all of these definitions, the letter c will be dropped when $c = \pi_{n-1}(X)$; e.g., $A_n(X) = A_n(X, \pi_{n-1}(X))$.

The most basic and important configuration-mapping spaces are those for n-dimensional disks D^n. We identify the homotopy type of their group completion:

2.8.1. **Theorem.** *There is a homotopy equivalence*

$$\Omega B\, \mathrm{CMap}^{c,*}(D^n, X) \simeq \mathrm{Map}_*((D^n, S^{n-1}), (A_n(X, c), X))$$

2.8.2. *Remark.* The function space $\mathrm{Map}_*((D^n, S^{n-1}), (A_n(X, c), X))$ may be identified as $\Omega^{n-1} A'_n(X, c)$ by examining the definition of the homotopy fibre.

We prove Theorem 2.8.1 in section 3. The multiplicative structure described in section 2.6 makes $M := \pi_0(\mathrm{CMap}^{c,*}(D^n, X))$ into an associative monoid (commutative when $n > 2$).

2.9. **Definition.** An element $V \in M$ will be called a *central stabilizer* if it lies in the center $Z(M)$, and if, for any $m \in M$, there exist $m', m'' \in M$ and $k, l \geq 0$ such that

$$mm' = V^k \quad \text{and} \quad m''m = V^l$$

A variation on the group completion theorem then gives the following:

2.9.1. **Corollary.** *If* $\mathrm{CMap}^{c,*}(D^n, X)$ *admits a central stabilizer* V*, then there is an isomorphism*

$$\left(\bigoplus H_*(\mathrm{CMap}_k^{c,*}(D^n, X))\right)[V^{-1}] \to H_*(\mathrm{Map}_*((D^n, S^{n-1}), (A_n(X, c), X))).$$

When the monoid M is isomorphic to $\mathbb{N}$, it is easy to conclude from this result that there is an isomorphism

$$\varinjlim_V H_*(\mathrm{CMap}_k^{c,*}(D^n, X)) \cong H_*(\mathrm{Map}_*^0((D^n, S^{n-1}), (A_n(X, c), X)))$$

where the target space consists of maps $(D^n, S^{n-1}) \to (A_n(X, c), X)$ which, when composed with the natural projection from the latter to $(S^n, *)$, are of degree 0. For general M, an analogous result holds, but it will not be necessary for our purposes to have a precise statement.

This result extends to configuration-mapping spaces for general manifolds with boundary. To state the result properly consider the following construction. Let M be a manifold with boundary, and choose a basepoint $* \in \partial M$. Write $D(M)$ for the unit disk bundle of the tangent bundle of M. Define a fibre bundle $E_n(X, c)$ over M, gotten by fibrewise replacement of D^n in $D(M)$ by $A_n(X, c)$. More carefully, equip M with a metric, and let $F(M)$ is the unit frame bundle of M. The group $SO(n)$ acts on $A_n(X,c)$: trivially on X, and on $D^n \times \mathrm{Map}^c(S^{n-1}, X)$ by $\gamma \cdot (x, f) = (\gamma \cdot x, f \circ \gamma^{-1})$.

2.10. **Definition.** Let $E_n(X,c) := F(M) \times_{SO(n)} A_n(X,c)$ and write $\Gamma_k(M; E_n(X,c))$ for the space of continuous sections of $E_n(X,c)$ of degree k which carry points in $m \in \partial M$ to the copy of X in the fibre over m and $*$ to the basepoint of $A_n(X,c)$.

2.10.1. **Theorem.** *If* $\mathrm{CMap}^{c,*}(D^n, X)$ *admits a central stabilizer* V*, then there is an isomorphism*

$$\left(\bigoplus H_*(\mathrm{CMap}_k^{c,*}(M,X))\right)[V^{-1}] \cong H_*(\Gamma(M; E_n(X,c)))$$

Note that if M is parallelizable, then $E_n(X,c) = M \times A_n(X,c)$, and so

$$\Gamma(M; E_n(X,c)) = \mathrm{Map}_*((M, \partial), (A_n(X,c), X)).$$

3. Proof of Theorem 2.8.1

Our method of proof is adapted from [May72, McD75, Sal01]. That is, we first define relative configuration-mapping spaces and prove that they satisfy a quasifibration property. Using this, we construct quasifibrations which model the path-loop fibration. More precisely, we will construct a sequence of spaces $B_1, B_2, \ldots, B_n$, together with "scanning" morphisms $s_k : B_{k-1} \to \Omega B_k$ for $k = 1, 2, \ldots, n$. We will show that s_k is a homotopy equivalence for $1 < k < n$ (Lemma 3.5.1), that B_1 is homotopic to $B(\mathrm{CMap}^S(I^n, X))$ (Lemma 3.3.1) and that B_n is homotopic to $A_n(X)$ (Lemma 3.5.2); together, these yield Theorem 2.8.1.

3.1. **Relative configuration-mapping spaces.** Let $N \subseteq M$ be a closed submanifold of the same dimension as M, possibly with boundary or corners (so that the complement $M \setminus N$ is an open manifold).

3.2. **Definition.** Write $\mathrm{CMap}^S((M,N); X)$ for the quotient of $\mathrm{CMap}^S(M; X) = \coprod_k \mathrm{CMap}^S_k(M; X)$ by the equivalence relation $(\underline{x}, f) \sim (\underline{x}', f')$ whenever

$$\underline{x} \cap (\overline{M \setminus N}) = \underline{x}' \cap (\overline{M \setminus N}) \quad \text{and} \quad f|_{\overline{M\setminus N}} = f'|_{\overline{M\setminus N}}$$

We think of $\mathrm{CMap}^S((M,N); X)$ as the space where configurations disappear when they enter the interior of N, and where functions are defined only up to modification on the interior of N.

Let $n > 1$ be an integer. For the proofs of the main results, we will use the n-cube I^n in place of the disk D^n, in order to make the induction easier to describe. Further, our model for $\mathrm{CMap}^*(I^n, X)$ will be $\mathrm{CMap}^P(I^n, X)$, where $P = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$. In other words, P is the closure of the complement in ∂I^n of the "top" face of ∂I^n; the contractibility of P gives a homotopy equivalence

$$\mathrm{CMap}^P(I^n, X) \simeq \mathrm{CMap}^*(I^n, X)$$

We will use relative configuration-mapping spaces to produce iterated bar constructions for $\mathrm{CMap}^\partial(I^n, X)$ and $\mathrm{CMap}^P(I^n, X)$. We begin by specifying

$$I := [0,1], \quad J := [-1,2], \quad \text{and} \quad DJ := \overline{J \setminus I} = [-1,0] \sqcup [1,2]$$

Here J is a fattened form of I, and DJ is a fattened form of ∂I within J. We may extend this notion to an analogue of ∂I^k by defining $DJ^k = \overline{(J^k \setminus I^k)}$.

For each $t \in I$, let $\tau_k(t) : J^{k-1} \times I^{n-k+1} \to J^k \times I^{n-k}$ be the map

$$\tau_k(t)(s_1, \dots, s_n) = (s_1, \dots, s_{k-1}, s_k + 2t - 1, s_{k+1}, \dots, s_n)$$

which slides the $J^{k-1} \times I^{n-k+1}$ across $J^k \times I^{n-k}$ in the k^{th} coordinate.

Further, for each $\underline{x} \in J^{k-1} \times I^{n-k+1}$, we define a family of functions

$$T_k(t) : \mathrm{Map}(J^{k-1} \times I^{n-k+1} \setminus \underline{x}, X) \to \mathrm{Map}(J^k \times I^{n-k} \setminus \underline{x}, X)$$

by shifting functions in the analogous way:

$$T_k(t)(f)(s_1, \dots, s_n) := \begin{cases} f(s_1, \dots, s_{k-1}, s_k + 1 - 2t, s_{k+1}, \dots, s_n), & 0 \leq s_k + 1 - 2t \leq 1 \\ f(s_1, \dots, s_{k-1}, 0, s_{k+1}, \dots, s_n), & s_k \leq 2t - 1 \\ f(s_1, \dots, s_{k-1}, 1, s_{k+1}, \dots, s_n), & s_k \geq 2t \end{cases}$$

3.3. **Definition.** Let k be an integer in $\{1, \dots, n\}$. We now define the k-fold bar construction that will provide our models for loop spaces of $A(X)$. There are two cases, which we treat simultaneously; we may let $S = \partial I^n$ and $S' = \partial(J^k \times I^{n-k})$, or $S = P$ and (with the extra hypothesis $k < n$)

$$S' = \partial(J^k \times I^{n-k-1}) \times I \cup J^k \times I^{n-k-1} \times \{0\}$$

We now define

$$B_k(\mathrm{CMap}^S(I^n, X)) := \mathrm{CMap}^{S'}((J^k \times I^{n-k}, DJ^k \times I^{n-k}); X).$$

Furthermore, define the *scanning maps* $s_k : B_{k-1}(\mathrm{CMap}^S(I^n, X)) \to \Omega B_k(\mathrm{CMap}^S(I^n, X))$ by

$$s_k(\underline{x}, f)(t) := (\tau_k(t)(\underline{x}), T_k(t)(f))$$

Note that when $t = 0$ or $t = 1$, the configuration $\tau_k(t)(\underline{x})$ has all x_k coordinates in $J \setminus I$, and the function $T_k(t)(f)$ sends $I^k \times J^{n-k}$ to $*$; this is just to say that in $\mathrm{CMap}^{S'}((J^k \times I^{n-k}, DJ^k \times I^{n-k}); X)$ we have $s_k(\underline{x}, f)(0) = s_k(\underline{x}, f)(1) = (\emptyset, *)$.

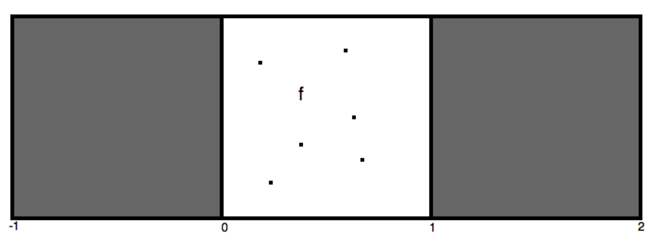


FIGURE 2. A picture of an element of $B_1(\mathrm{CMap}(I^2, X))$; configurations in the grey area are dropped, and function data ignored. The scanning map s_1 "slides" a configuration-mapping on I^2 from left to right horizontally across the rectangle $J \times I$.

If $S = P$ and $k = n$, we can define the scanning map s_n in an analogous way, its image is not a loop in $B_n(\mathrm{CMap}^P(I^n, X))$, but rather a path ending at the basepoint.

The terminology "k-fold bar construction" is justified below by Lemma 3.5.1, but for now, we content ourselves with:

3.3.1. **Lemma.** *For $S = P$ or ∂I^n, there is a homotopy equivalence*

$$B_1(\mathrm{CMap}^S(I^n, X)) \simeq B(\mathrm{CMap}^S(I^n, X))$$

Proof. There is a strictly associative monoid M which is weakly multiplicatively homotopy equivalent to $\mathrm{CMap}^S(I^n, X)$. M is defined as the set of pairs

$$\begin{aligned} M &:= \{(r, (\underline{z}, f)) \mid \underline{z} \subseteq [0, r] \times I^{n-1},\ f|_{(r,\infty)\times I^{n-1}} = *\} \\ &\subseteq [0, \infty) \times \mathrm{CMap}^S([0, \infty) \times I^{n-1}, X). \end{aligned}$$

We note that M fibres over $[0, \infty)$ with fibre over r identifiable as $\mathrm{CMap}^S([0, r] \times I^{n-1}, X)$ The multiplicative structure on M is given by

$$(r, (\underline{z}, f)) \cdot (s, (\underline{x}, g)) = (r + s, (\underline{z} \cup T_r(\underline{x}), f \cup T_r(g))).$$

where T_r is the operation that translates configurations and functions to the right by r (or $T_1(r)$ in the notation used above).

There is an interpretation of the simplicial classifying space BM: it consists of ordered configurations $0 \le t_1 \le t_2 \le \cdots \le t_k \le 1$ of points in $[0, 1]$ labelled by elements of M. Points are allowed to collide, and when they do, their labels multiply. Furthermore, points and their labels are dropped if they are equal to 0 or 1. More carefully,

$$BM := \coprod_k \Delta^k \times M^{\times k} / \sim$$

where the equivalence relation $\sim$ is generated by

(1) $(0, t_2, \ldots, t_k; m_1, \ldots, m_k) \sim (t_2, \ldots, t_k; m_2, \ldots, m_k)$
(2) $(t_1, \ldots, t_{k-1}, 1; m_1, \ldots, m_k) \sim (t_1, \ldots, t_{k-1}; m_1, \ldots, m_{k-1})$
(3) If $t_i = t_{i+1}$, then

$$(t_1, \ldots, t_k; m_1, \ldots, m_k) \sim (t_1, \ldots, t_i, \widehat{t_{i+1}}, \ldots, t_{k-1}; m_1, \ldots, (m_i \cdot m_{i+1}), \ldots, m_k)$$

Note that BM is homotopy equivalent to a similarly defined space

$$B'(M) := \coprod_k \mathcal{C}_1(k) \times M^{\times k} / \sim$$

Here $\mathcal{C}_1(k)$ is the space of configurations of k little intervals $(c_1, \ldots, c_k)$ in $\mathbf{R}$ with disjoint interiors, with c_i to the left of c_{i+1} for all i. The equivalence relation $\sim$ is generated by

(1) If the right boundary of c_1 is less than or equal to 0, then
$$(c_1, \ldots, c_k, m_1, \ldots, m_k) \sim (c_2, \ldots, c_k, m_2, \ldots, m_k)$$
(2) If the left boundary of c_k is greater than or equal to 1, then
$$(c_1, \ldots, c_k, m_1, \ldots, m_k) \sim (c_1, \ldots, c_{k-1}, m_1, \ldots, m_{k-1})$$
(3) If the right boundary of c_i equals the left boundary of c_{i+1}, then
$$(c_1, \ldots, c_k, m_1, \ldots, m_k) \sim (c_1, \ldots, c_i \cup c_{i+1}, \ldots, c_k, m_1, \ldots, (m_i \cdot m_{i+1}), \ldots, m_{k-1}),$$
where $c_i \cup c_{i+1}$ is the little interval whose image is the union of c_i and c_{i+1}.

An equivalence $\phi : B'(M) \to B_1(\mathrm{CMap}^S(I^n, X))$ is given by rescaling elements of M and placing them inside the boxes defined by the configuration of little intervals. Explicitly, if $c_i : I \to \mathbf{R}$ is a little interval in $\mathbf{R}$ and $(r_i, \underline{z}_i, f_i) \in M$, we make the following definitions:

- $d_i = c_i \times \mathrm{id}_{I^{n-1}} : I^{\times n} \to \mathbf{R} \times I^{n-1}$.

- $\underline{x}_i$ is the configuration in I given by $d_i(S_{r_i}(\underline{z}_i))$, where $S_r(t_1,\dots,t_n) = (t_1/r, t_2, \dots, t_n)$.
- g_i is the function on $\mathrm{im}\,(d_i)$ given by $f_i \circ S_{r_i}^{-1} \circ d_i^{-1}$.

These extend in a natural way to the case $r_i = 0$. Note that the functions g_i are constant at $*$ on $\mathrm{im}\,(d_i(\{0,1\} \times I^{n-1}))$; thus they extend (by $*$) to a continuous function g on I^{n-1}. Define

$$\phi(c_1,\dots,c_k,(r_1,\underline{z}_1,f_1),\dots,(r_k,\underline{z}_k,f_k)) = (\cup_i \underline{x}_i, g).$$

This has image in $\mathrm{CMap}^S((\mathbf{R}\times I^{n-1}, [\mathbf{R}\setminus(0,1)]\times I^{n-1}), X)$, which is homeomorphic to $B_1(\mathrm{CMap}^S(I^n, X))$. The definition of ϕ preserves the equivalence relations defining $B'(M)$, and so gives a well-defined map. A homotopy inverse carries $(\underline{y}, h)$ to $(\mathrm{id}, (1, \underline{y}, h))$, where $\mathrm{id} \in \mathcal{C}_1(1)$ is the identity little interval $\mathrm{id} : I \to I$. $\square$

3.4. **The quasifibrations.**

3.4.1. **Lemma.** *Let M be a connected, compact n-manifold, $S \subseteq M$ a subspace, $M' \subseteq M$ a compact n-submanifold, and N a closed n-submanifold such that $(M', N\cap M')$ is connected. Then there is a quasifibration sequence*

$$\begin{array}{ccc} \mathrm{CMap}^{(S\cap M')\cup\partial M'}((M', N\cap M'); X) & \longrightarrow & \mathrm{CMap}^S((M,N);X) \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathrm{CMap}^S((M, M'\cup N); X) \end{array}$$

Here the quasifibration π simply quotients by the defining equivalence relation of $\mathrm{CMap}^S((M, M'\cup N); X)$, which is stronger than that of $\mathrm{CMap}^S((M,N);X)$.

Proof. For brevity, we write Z for the closure $Z := \overline{M \setminus (M' \cup N)}$. Consider the fibre

$$\pi^{-1}(\underline{x}, f) = \{(\underline{y}, g)\} \subseteq \mathrm{CMap}^S((M,N);X),$$

where we pick a representative $\underline{x} \subseteq Z$, and $f : M \setminus \underline{x} \to X$. This consists of configurations $\underline{y} \subseteq \dot{M}'$ (modulo those lying in the interior of $N\cap M'$), and functions $g : M' \setminus \underline{y} \to X$ which agree with f on $\partial M'$ (modulo those that differ only in $N \cap M'$). This is precisely the subspace

$$\pi^{-1}(\underline{x}, f) = \{(\underline{y}, g) \mid g|_{\partial M'} = f|_{\partial M'}\} \subseteq \mathrm{CMap}^{(S\cap M')}((M', N\cap M'); X).$$

Note that this does not depend upon $\underline{x}$. Further, the fibre over $(\underline{x}, c_*)$, where c_* is the constant map at $* \in X$, is $\mathrm{CMap}^{(S\cap M')\cup\partial M'}((M', N\cap M'); X)$.

As is standard in proofs like these, we use the Dold-Thom criterion for quasifibrations. The space $\mathrm{CMap}^S((M, M'\cup N); X)$ has a filtration by the image

$$C_k := \mathrm{im}\,(\mathrm{CMap}^S_k(M;X)) \subseteq \mathrm{CMap}^S((M, M'\cup N); X)$$

of $\mathrm{CMap}^S_k(M;X)$ in $\mathrm{CMap}^S((M, M'\cup N); X)$. To use the Dold-Thom criterion, we must show:

1. The map $\pi : \pi^{-1}(C_k \setminus C_{k-1}) \to C_k \setminus C_{k-1}$ is a fibration, with fibre $\mathrm{CMap}^{(S\cap M')\cup\partial M'}((M', N\cap M'); X)$.
2. There is an open subset $U_k \subseteq C_k$, containing C_{k-1} and homotopies

$$h_t : C_k \to C_k \quad \text{and} \quad H_t : \pi^{-1}(C_k) \to \pi^{-1}(C_k)$$

such that

(a) $h_0 = \mathrm{id}$, $h_t(C_{k-1}) \subseteq C_{k-1}$, and $h_1(U_k) \subseteq C_{k-1}$;
(b) $H_0 = \mathrm{id}$, $\pi \circ H_t = h_t \circ \pi$;
(c) $H_1 : \pi^{-1}(\underline{x}, f) \to \pi^{-1}(h_1(\underline{x}, f))$ is a homotopy equivalence for each $(\underline{x}, f) \in U_k$.

There is a map

$$C_k \setminus C_{k-1} \to \mathrm{CMap}_k^{S\cap Z}(Z; X)$$

which sends $(\underline{x}, f)$ to $(\underline{x} \cap \dot{Z}, f|_Z)$. This is a homeomorphism onto its image, which consists of those $(\underline{z}, g)$, where g admits an extension to M, constant on S.

Define restriction maps r_M and $r_{M'}$

$$\mathrm{CMap}^S((M, M' \cup N); X) \xrightarrow{r_M} \mathrm{Map}(\partial M', X) \xleftarrow{r_{M'}} \mathrm{CMap}^{(S\cap M')}((M', N \cap M'); X)$$

both given by the equation $r(\underline{x}, f) = f|_{\partial M'}$. Then we may define a homeomorphism

$$h : (C_k \setminus C_{k-1}) \times_{\mathrm{Map}(\partial M', X)} \mathrm{CMap}^{(S\cap M')}((M', N \cap M'); X) \to \pi^{-1}(C_k \setminus C_{k-1})$$

(where the fibre product is over r_M and $r_{M'}$), by the formula

$$h((\underline{x}, f), (\underline{y}, g)) = (\underline{x} \cup \underline{y}, f \cup_{\partial M'} g)$$

Now, r_M, restricted to $C_k \setminus C_{k-1}$, is a fibration (with fibre over c_* given by $\mathrm{CMap}_k^{(S\cup \partial M')\cap Z}(Z; X)$. Therefore the projection from the domain of h to $C_k \setminus C_{k-1}$ is a fibration. Since h is a homeomorphism, this proves item 1.

To construct the homotopies, we choose a collar neighborhood U of M' with the property that there is an isotopy which retracts U into M'. That is, we require a path of homeomorphisms

$$J_t : M \to M$$

such that J_0 is the identity, $J_1(U) \subseteq M'$, $J_t(S) \subseteq S$, and $J_t(N \cap U) \subseteq N$ for all t. Let U_k be defined as

$$U_k = \{(\underline{x}, f) \mid \#(\underline{x} \cap (M \setminus U)) \le k-1\} \subseteq C_k.$$

Since $M' \subseteq U$, this contains $C_{k-1} = \{(\underline{x}, f) \mid \#(\underline{x} \cap (M \setminus M')) \le k-1\}$. Define h_t and H_t by the same formula

$$h_t(\underline{x}, f) = (J_t(\underline{x}), f \circ J_t^{-1})$$

By construction, (a) and (b) of item 2 are immediately satisfied. As in Lemma 3.4 in [McD75] and Lemma 6.1 in [Sal01], to show (c), we identify the fibre over $(\underline{x}, f)$ with $\mathrm{CMap}^{(S\cap M')\cup \partial M'}((M', N \cap M'); X)$. Then H_1 pushes configurations away from $\partial M'$, and glues in the configuration given by intersecting $h_1(\underline{x}, f)$ with U. But since U meets N, there are paths from $h_1(\underline{x}) \cap U$ into N; we may homotope the added points along these paths (and deform f concurrently) where they are dropped. Thus H_1 is a homotopy equivalence.

$\square$

3.5. **The proof of Theorem 2.8.1.**

3.5.1. **Lemma.** *For $S = P$ or ∂I^n, the scanning maps $s_k : B_{k-1}(\mathrm{CMap}^S(I^n, X)) \to \Omega B_k(\mathrm{CMap}^S(I^n, X))$ are weak equivalences for $1 < k < n$ if $S = P$ or $1 < k \le n$ if $S = \partial I^n$.*

Proof. We define $M := J^k \times I^{n-k}$, and two subspaces

$$N := (DJ^{k-1} \times J \times I^{n-k}) \cup (J^{k-1} \times [-1,0] \times I^{n-k}) \quad \text{and} \quad M' := J^{k-1} \times [1,2] \times I^{n-k}$$

Together, $N \cup M' = DJ^k \times I^{n-k}$, so $\mathrm{CMap}^S((M, N \cup M'); X) = B_k(\mathrm{CMap}^S(I^n, X)$. Further, the shift (in the k^{th} coordinate) by 1 is a homeomorphism from $(M', N \cap M')$ to $(J^{k-1} \times I^{n-k+1}, DJ^{k-1} \times I^{n-k+1})$, so

$$\mathrm{CMap}^S((M', N \cap M'); X) \cong B_{k-1}(\mathrm{CMap}^S(I^n, X))$$

There is a commutative diagram

$$\begin{tikzcd}
B_{k-1}(\mathrm{CMap}^S(I^n,X)) \arrow[r] \arrow[d, "s_k"'] & \mathrm{CMap}^S((M,N);X) \arrow[r, "\pi"] \arrow[d, "s"'] & B_k(\mathrm{CMap}^S(I^n,X)) \arrow[d, "="'] \\
\Omega B_k(\mathrm{CMap}^S(I^n,X)) \arrow[r] & PB_k(\mathrm{CMap}^S(I^n,X)) \arrow[r] & B_k(\mathrm{CMap}^S(I^n,X))
\end{tikzcd}$$

Here s is defined by the same formula as s_k. The lower row is the path-loop fibration. The upper row is a quasifibration by Lemma 3.4.1. Both spaces in the middle are contractible. For the path space, this fact is classical. For $\mathrm{CMap}^S((M,N);X)$, we note that there is an isotopy of M into N that "pushes out from M'." Thus $\mathrm{CMap}^S((M,N);X)$ deformation retracts into the subspace consisting of configurations of 0 points in M and a constant function f whose value is determined on M'. So, if $M' \subseteq S$ (which is always the case for $S = \partial I^n$, and for $k < n$ if $S = P$), this subspace is a point. We conclude that s_k is a weak equivalence using the long exact sequence of homotopy groups associated to a (quasi-)fibration and the five lemma. □

We may identify the iterated bar constructions for these configuration-mapping spaces:

3.5.2. **Lemma.** *There are equivalences*

$$A_n(X) \simeq B_n(\mathrm{CMap}^\partial(I^n, X)) \quad \text{and} \quad A'_n(X) \simeq B_{n-1}(\mathrm{CMap}^P(I^n, X))$$

Proof. Let us start with the first equivalence. We note that, since $\partial J^n \subseteq DJ^n$,

$$B_n(\mathrm{CMap}^\partial(I^n, X)) = \mathrm{CMap}^\partial((J^n, DJ^n); X) = \mathrm{CMap}((J^n, DJ^n); X)$$

has no restriction on the values that functions take on the boundary of J^n.

Write z for the center of J^n, and consider a configuration $\underline{x} = (x_1, \dots x_k)$. We may non-uniquely order the configuration so that x_1 is closest to z, and x_2 is next closest. Define $r = r(\underline{x}) = d(z, x_2) > 0$ to be the distance from z to x_2. Radial expansion out from z at a rate of $1/r$ then gives a deformation retraction of $\mathrm{CMap}((J^n, DJ^n); X)$ onto C_1, the subspace consisting of pairs $(\underline{x}, f)$, where $\underline{x} = x_1$ consists of (at most) a single element within I^n. Now, C_1 evidently breaks up into a disjoint union of two spaces: C'_1, where x_1 lies in the interior of I^n, and C''_1, where $x_1 \in DJ^n$. There is a homotopy equivalence

$$C'_1 \to D^n \times \mathrm{Map}(S^{n-1}, X)$$

which sends (x_1, f) to $(\phi(x_1), f')$ where $\phi : I^n \to D^n$ is a homeomorphism, and f' is the restriction of f to a small sphere around x_1. Further, C''_1 is homeomorphic to $\mathrm{Map}(I^n, X)$, which is in turn homotopy equivalent to X. These spaces are glued together by restriction along the boundary of I^n; this is precisely the way that $A_n(X)$ is defined.

For the second equivalence, use the quasifibration sequence of the proof of Lemma 3.5.1:

$$B_{n-1}(\mathrm{CMap}^P(I^n, X)) \longrightarrow \mathrm{CMap}^P((M,N);X) \xrightarrow{\pi} B_n(\mathrm{CMap}^P(I^n, X))$$

Again, we may drop the P in the last term, so $B_n(\mathrm{CMap}^P(I^n, X))$ is equivalent to $B_n(\mathrm{CMap}(I^n, X)) = A_n(X)$. However, $\mathrm{CMap}^P((M,N);X)$ is no longer contractible; rather, the retraction above gives homotopy equivalence to $\mathrm{Map}(I^n, X) \simeq X$. The result follows.

□

We note that the same proofs of Lemmas 3.3.1, 3.5.1 and 3.5.2 hold upon restriction of monodromy to $c \subseteq \pi_{n-1}(X)$. Theorem 2.8.1 then follows by the string of weak equivalences

$$\begin{aligned}
\Omega B\,\mathrm{CMap}^{c,*}(I^n, X) &\simeq \Omega B_1(\mathrm{CMap}^{c,*}(I^n, X)) \\
&\simeq \Omega^2 B_2(\mathrm{CMap}^{c,*}(I^n, X)) \\
&\vdots \\
&\simeq \Omega^{n-1} B_{n-1}(\mathrm{CMap}^{c,*}(I^n, X)) \\
&\simeq \Omega^{n-1} A'_n(X, c)
\end{aligned}$$

afforded by these Lemmas.

3.6. **Other source manifolds.** Let M be a connected, compact manifold with nonempty boundary, and $D^- \subseteq M$ be an embedded cube meeting ∂M. That is, there is an embedding of $[-1,0] \times I^{n-1}$ into M with image D^-, carrying the subset $\{0\} \times I^{n-1}$ into ∂M.

Recall that we write $\Gamma(M; E_n(X,c))$ for the space of sections of the bundle $E_n(X,c)$ over M with fibre $A_n(X,c)$ that carry points $m \in \partial M$ into the copy of X in the fibre over m (and carry the basepoint of M to the basepoint of $A_n(X,c)$ in the fibre over it). We will write $\Gamma((M,D^-); E_n(X,c))$ for the larger space of sections where this restriction only holds for $m \in \partial M \setminus (\partial M \cap D^-)$.

3.6.1. **Theorem.** *There is a weak equivalence*

$$\mathrm{CMap}^{c,*}((M,D^-), X) \to \Gamma((M,D^-); E_n(X,c)).$$

Proof. One may give a formally identical to the proof of Theorem 2.6 in [McD75], which implies this result when X is a point. Alternatively, we may proceed by handle induction. Every n-manifold M with boundary has a handle decomposition using handles of index $k < n$. The proof of Lemma 3.5.1 gives the base of the induction, when $M = D^n$ consists of a single cell; then M deforms into D^-, so both $\mathrm{CMap}^{c,*}((M,D^-), X)$ and $\Gamma((M,D^-); E_n(X,c))$ are contractible.

Now assume that the result is true for a manifold M_1, and that M is gotten from M_1 by attaching a k-handle $D^k \times D^{n-k}$ with $0 < k < n$ along $S^{k-1} \times D^{n-k}$. Write M_2 for a "fattened" version of the handle; that is, we may assume that there is a subspace $M_2 \subseteq M$, homeomorphic to $J^k \times I^{n-k}$ with $M_1 \cap M_2 = DJ^k \times I^{n-k} \cong S^{k-1} \times I^{n-k+1}$. Furthermore, we may arrange this handle so that $D^- \cap M_2 = \varnothing$.

Consider the diagram

$$\begin{tikzcd}
\mathrm{CMap}^{c,*}((M_1, D^-), X) \arrow[r] \arrow[d] & \mathrm{CMap}^{c,*}((M, D^-), X) \arrow[r] \arrow[d] & \mathrm{CMap}^{c,*}((J^k, DJ^k) \times I^{n-k}, X) \arrow[d] \\
\Gamma((M_1, D^-); E_n(X, c)) \arrow[r] & \Gamma((M, D^-); E_n(X, c)) \arrow[r] & \Gamma((J^k, DJ^k) \times I^{n-k}; E_n(X, c)).
\end{tikzcd}$$

The upper row is the quasifibration sequence of Lemma 3.4.1 given by quotient by M_1. Likewise, the lower row is the fibration sequence coming from restriction of functions to M_2. Further, the left and right vertical maps are weak equivalences, by induction (on the left), and by Lemmas 3.5.1 and 3.5.2 (on the right), since

$$\Gamma((J^k, DJ^k) \times I^{n-k}; E_n(X, c)) \simeq \Omega^{n-k} A_n(X, c).$$

The result for M then follows by the five lemma.

□

4. Group completion arguments

4.1. **The proof of Corollary 2.9.1.** We have shown that $\mathrm{CMap}^{c,*}(D^n, X)$ admits the structure of an E_{n-1}-algebra; in particular it is an A_∞ H-space. Let A denote a rectification of $\mathrm{CMap}^{c,*}(D^n, X)$; that is, A is a strictly associative, unital, topological monoid, equipped with an equivalence $A \to \mathrm{CMap}^{c,*}(D^n, X)$ which is multiplicative up to homotopy.

Assume that A (and hence $\mathrm{CMap}^{c,*}(D^n, X)$) admits a central stabilizer V; pick a representative of this element of π_0, which we will also write as V. Define A_∞ as the telescope on multiplication by V:

$$A_\infty := \mathrm{Tel}(A \xrightarrow{V} A \xrightarrow{V} \cdots)$$

Since V is central, A_∞ admits an action of A. Consider the map

$$p : A_\infty \times_A EA \to BA.$$

The domain is contractible, as it may be identified with the telescope

$$A_\infty \times_A EA := \mathrm{Tel}(A \times_A EA \xrightarrow{V} A \times_A EA \xrightarrow{V} \cdots)$$

Thus the homotopy fibre of p is ΩBA.

If $g \in A$, there is an induced map $g_* : H_*(A_\infty) \to H_*(A_\infty)$. By assumption, there are elements g' and g'' in A with

$$g_* g'_* = V_*^k \quad \text{and} \quad g''_* g_* = V_*^l.$$

As powers of V are automorphisms of $H_*(A_\infty)$, we see that A acts on A_∞ through homology isomorphisms. The results of [MS76] implies that p is a homology fibration; thus the comparison map

$$A_\infty \to \Omega BA$$

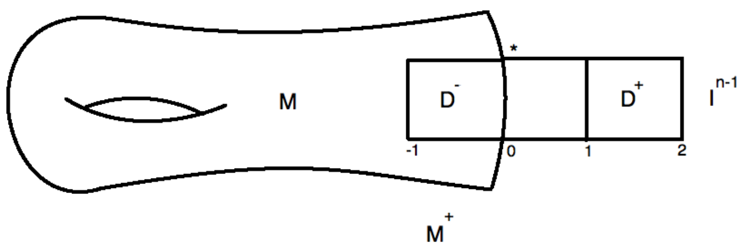
from the actual fibre of p to its homotopy fibre is a homology equivalence. Since $H_*(A_\infty) \cong H_*(A)[V^{-1}]$, we conclude (using Theorem 2.8.1) that there is an isomorphism

$$H_*(\mathrm{CMap}^{c,*}(D^n, X))[V^{-1}] \cong H_*(\mathrm{Map}_*((D^n, S^{n-1}), (A_n(X, c), X)))$$

4.2. **The proof of Theorem 2.10.1.** Let M, D^- be as in section 3.6. Write M^+ for the manifold (with corners)

$$M^+ = M \cup_{D^-} (J \times I^{n-1}),$$

and let $D^+ = [1,2] \times I^{n-1}$.



Note that $\mathrm{CMap}^{c,*}(M,X)$ admits an action of the E_{n-1}-monoid $\mathrm{CMap}^c(I^n,X)$by attaching a cube along ∂M at $*$ and compressing the new boundary into M. Write H for a rectification of $\mathrm{CMap}^{c,*}(M,X)$ to a strict A-module. Then one can form the telescope H_∞ of the action of V on H, and H_∞ retains a A-action.

4.2.1. **Proposition.** *There is a commutative diagram*

$$\begin{tikzcd}
H_\infty \times_A EA \arrow[r, "\simeq"] \arrow[d] & \mathrm{CMap}^{c,*}((M^+,D^+);X) \arrow[r, "\simeq"] \arrow[d] & \Gamma((M^+,D^+);E_n(X,c)) \arrow[d, "res"] \\
BA \arrow[r, "\simeq"] & \mathrm{CMap}^{c,*}((J,DJ)\times I^{n-1};X) \arrow[r, "\simeq"] & \Gamma((J,DJ)\times I^{n-1};E_n(X,c))
\end{tikzcd}$$

in which the rows are homotopy equivalences.

The middle vertical map is the quotient wherein configurations in M are allowed to disappear. The right vertical map is the fibration given by restricting sections to $J \times I^{n-1} \subseteq M^+$. The same arguments as in the previous section give a homology isomorphism of H_∞ with the the fibre of res. But the fibre of res over the constant map at the basepoint of $A_n(X,c)$ is precisely the space of sections over M^+ which are constant on $J \times I^{n-1}$. There is clearly a homotopy equivalence $(M^+, J \times I^{n-1}) \to (M,*)$ of pairs, so we conclude that the map

$$H_\infty \to \Gamma(M; E_n(X,c))$$

is a homology isomorphism; this concludes the proof of Theorem 2.10.1.

Proof. The fact that the horizontal maps of the square on the right are homotopy equivalences are the content of Theorems 3.6.1 and 2.8.1. Its commutativity is obvious. Lemma 3.3.1 gives the lower left equivalence; further, its proof immediately generalizes to give a homotopy equivalence $H \times_A EA \to \mathrm{CMap}^{c,*}((M^+,D^+);X)$.

Moreover, this map commutes with stabilization by V, and so defines the required map from $H_\infty \times_A EA$. But the inclusion $H \times_A EA \to H_\infty \times_A EA$ is a homotopy equivalence, since the map induced on $H \times_A EA$ by V is homotopic to the identity (push V into D^+).

□

Part 2. Stable components of Hurwitz spaces and Hurwitz schemes

5. Hurwitz spaces

Our primary application of the machinery developed so far will be to the stable homology of *Hurwitz spaces*, which parametrize branched covers of a disk with specified Galois group G. (We expect our results carry over, with natural modifications, to analogous questions where the disc is replaced by an arbitrary oriented surface with boundary.)

In this section, we recall some of the results of Part 1 in the specific context of Hurwitz spaces. In particular, we will deduce from those results an identification (Corollary 5.8.2) of the homology of components of the Hurwitz space.

5.1. **Notation.** *Although the definitions of Hurwitz spaces make sense in the case when G is discrete, we assume for the rest of the paper that G is a finite group.* We denote by c a generating subset of G invariant under conjugation, by c/G the set of conjugacy classes making up c, and by $\mathbf{Z}^{c/G}$ the free abelian group on c/G; for $\mathcal{C} \in c/G$ we denote by $e_{\mathcal{C}}$ the corresponding element of $\mathbf{Z}^{c/G}$. For $x \in \mathcal{C}$ we write simply e_x for $e_{\mathcal{C}}$. Thus, with this notation, $e_x = e_y$ if and only x is conjugate to y.

There is a morphism

$$\mathbf{Z}^{c/G} \to G^{\mathrm{ab}} \tag{5.1.1}$$

sending e_x to the image of x in G^{ab}, for any $x \in c$. This homomorphism is *surjective* because c generates G.

We typically denote an element of $\mathbf{Z}^{c/G}$ by $\underline{m}$; then we write

$$|\underline{m}| = \sum_{\mathcal{C} \in c/G} \underline{m}(\mathcal{C}); \ \mathrm{mindeg}(\underline{m}) = \min_{\mathcal{C} \in c/G} \underline{m}(\mathcal{C}).$$

We sometimes say "for sufficiently large $\underline{m}$" as an abbreviation for "there exists N such that, if $\mathrm{mindeg}(\underline{m}) > N$, ..."

We denote by D^2 the unit disc $\{z \in \mathbf{C} : |z| \leq 1\}$, and by $\dot{D}^2$ its interior. We fix a basepoint $*$ on the boundary of D^2.

5.2. **Hurwitz spaces.** Let Conf_n be the configuration space of n un-ordered points in the interior of the disc. In the notation of the first part of this paper, $\mathrm{Conf}_n = C_n(\dot{D}^2)$.

We will consider n-branched G-covers of the disc. By that we mean: a configuration $\{P_1, \ldots, P_n\}$ inside the interior of the disc D^2, together with a homomorphism

$$\rho : \pi_1(D^2 - \{P_1, \ldots, P_n\}, *) \to G, \tag{5.2.1}$$

where $*$ is the fixed point on the boundary of D^2.

Such data arises from a branched G-covering: Suppose given a Riemann surface T with boundary, and a point $\dagger \in T$, together with

$$f : (T, \dagger) \to (D^2, *) \text{ together with } G \hookrightarrow \mathrm{Aut}(f).$$

We also suppose that G acts simply transitively on the fiber of a general point, f is branched at exactly n (unlabelled) points in $\dot{D}^2$ and $f(\dagger) = *$. Then the effect of monodromy on $\dagger$ defines ρ as in (5.2.1). (Note that modifying $\dagger$ by the action of G has the effect of conjugating ρ.)

We denote by $\mathrm{Hur}_{G,n}$ the space of such coverings (considered up to isomorphism). This admits a natural map to the configuration space of n points in $\dot{D}^2$ and we topologize $\mathrm{Hur}_{G,n}$ such that this is a covering map:

$$\pi : \mathrm{Hur}_{G,n} \to \mathrm{Conf}_n$$

Note that $\mathrm{Hur}_{G,n}$ carries a G-action by conjugation. Some authors reserve the term "Hurwitz space" for the *quotient* of $\mathrm{Hur}_{G,n}$ by this G-action.

The image of a counterclockwise traversal of ∂D^2 based at $*$ defines an element of G, which we call the *boundary monodromy.* The boundary monodromy induces a continuous map $\mathrm{Hur}_{G,n} \to G$, thus separating $\mathrm{Hur}_{G,n}$ into components.

5.3. **Variations.**

- We denote by $\mathrm{Hur}^c_{G,n} \subseteq \mathrm{Hur}_{G,n}$ the Hurwitz moduli space parameterizing covers such that the monodromy around every branch point belongs to c.
- For $\underline{m} \in \mathbf{Z}^{c/G}$, we denote by $\mathrm{Hur}^c_{G,\underline{m}} \subset \mathrm{Hur}_{G,|\underline{m}|}$ the subspace parametrizing covers with $m(\mathcal{C})$ branch points of type $\mathcal{C}$, for each conjugacy class $\mathcal{C}$ in c/G. (We sometimes refer to $\underline{m}$ as the "multidiscriminant" of the covering.)
- By prefixing a C to Hur, we denote the subspace that parameterizes only *connected* coverings (i.e., ρ surjective or T connected, with the notation above). Thus, for example, $\mathrm{CHur}^c_{G,n}$ parameterizes connected covers of the disc, with n branch points and all monodromy of type c.
- Let Conf^c_n be the *colored* configuration space, whose points parametrize subsets $S \subset \dot{D}^2$ endowed with a function $S \to c/G$; and finally let $\mathrm{Conf}^c_{\underline{m}}$ be the subset of colored configurations where exactly $m(\mathcal{C})$ points are colored with $\mathcal{C}$ for every $\mathcal{C} \in c/G$ (and all points are colored, so that $\sum_{\mathcal{C}} m(\mathcal{C}) = n$).

These definitions agree with the notation of [EVW09, 2.2]. There are natural covering maps

$$\pi : \mathrm{Hur}_{G,n} \to \mathrm{Conf}_n$$
$$\pi : \mathrm{Hur}^c_{G,n} \to \mathrm{Conf}^c_n$$
$$\pi : \mathrm{Hur}^c_{G,\underline{m}} \to \mathrm{Conf}^c_{\underline{m}}.$$

and similar mappings for the subspaces parameterizing connected covers:

$$\pi : \mathrm{CHur}_{G,n} \to \mathrm{Conf}_n$$
$$\pi : \mathrm{CHur}^c_{G,n} \to \mathrm{Conf}^c_n$$
$$\pi : \mathrm{CHur}^c_{G,\underline{m}} \to \mathrm{Conf}^c_{\underline{m}}.$$

Clearly $\mathrm{Hur}^c_{G,n}$ and $\mathrm{Hur}^c_{G,\underline{m}}$ are unions of connected components of $\mathrm{Hur}_{G,n}$. Similarly, Conf^c_n and $\mathrm{Conf}^c_{\underline{m}}$ are unions of finite degree covers of Conf_n.

5.4. The map $\mathrm{Hur}_{G,n} \to \mathrm{Conf}_n$ is a covering space, and so it is determined by the action of the fundamental group of Conf_n on a fiber.

Let $\underline{z}$ be a configuration of n points in D^2.

The fundamental group of $D^2 \setminus \underline{z}$ is free of rank n; the fiber of $\mathrm{Hur}_{G,n} \to \mathrm{Conf}_n$ above $\underline{z}$ is the set of homomorphisms

$$\pi_1(D^2 \setminus \underline{z}, *) \longrightarrow G.$$

If we choose a system of loops around punctures (disjoint except at their source $*$) this induces a set of generators $\gamma_1, \ldots, \gamma_n$ for $\pi_1(D^2 \setminus \underline{z}, *)$ satisfying $\gamma_1 \ldots \gamma_n = \gamma_\partial$, where γ_∂ is the class of a loop around ∂D^2; in this way, the $\mathrm{Hom}(\pi_1(D^2 \setminus \underline{z}, *), G)$ can be identified with the set $G^{\times n}$ of n-tuples of elements of G. This identification is not canonical, for it depends on the choice of an arc system as mentioned.

The fundamental group $\pi_1(\mathrm{Conf}_n)$ is isomorphic to the Artin braid group Br_n, and the action of $\pi_1(\mathrm{Conf}_n)$ on the fiber can be described quite explicitly by "braiding:" the standard generator σ_i for the braid group (which pulls strand i past strand $i+1$) acts on $G^{\times n}$ through the formula

$$\sigma_i \cdot (g_1, \ldots g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n) \tag{5.4.1}$$

The subspace $\mathrm{CHur}_{G,n} \subset \mathrm{Hur}_{G,n}$ can then corresponds to the subset of $(g_1, \ldots, g_n) \in G^n$ which *generate* G, whereas the subspace $\mathrm{Hur}^c_{G,n} \subset \mathrm{Hur}_{G,n}$ corresponds to those n-tuples where each g_i belongs to c.

In particular, we have an identification

$$\pi_0 \mathrm{Hur}_{G,n} \xrightarrow{\sim} G^n/\mathrm{Br}_n$$

between the component set of $\mathrm{Hur}_{G,n}$, and the orbits of Br_n on n-tuples. Even better: if $E\mathrm{Br}_n$ is a contractible space on which Br_n acts freely, $\mathrm{Hur}_{G,n}$ is homotopy equivalent to $G^n \times E\mathrm{Br}_n/\mathrm{Br}_n$. Similarly,

$$\pi_0 \mathrm{Hur}^c_{G,n} \xrightarrow{\sim} c^n/\mathrm{Br}_n. \tag{5.4.2}$$

5.5. Monoidal structure and central stabilizer. The Hurwitz spaces admit a natural monoidal structure, i.e. there is a multiplication map (well defined up to homotopy):

$$\mathrm{Hur}^c_{G,n} \times \mathrm{Hur}^c_{G,m} \to \mathrm{Hur}^c_{G,n+m}$$

If we identify $\mathrm{Hur}^c_{G,n}$ with the (homotopy-theoretic) quotient of c^n by the braid group Br_n, in the sense discussed after (5.4.1), then this multiplication is induced by the maps $c^n \times c^m \to c^{n+m}$ and $\mathrm{Br}_n \times \mathrm{Br}_m \hookrightarrow \mathrm{Br}_{n+m}$.

In particular, $\mathcal{H} := \coprod_n \pi_0 \mathrm{Hur}^c_{G,n}$ has the structure of a monoid; we can identify it with

$$\mathcal{H} \xrightarrow{\sim} \coprod_n c^n/\mathrm{Br}_n,$$

where the multiplication on the right-hand side is by concatenation.

Moreover, any $U \in \mathcal{H}$ induces a homotopy class of maps: $U : \mathrm{Hur}_{G,n} \to \mathrm{Hur}_{G,n+d}$, where d is such that $U \in \pi_0 \mathrm{Hur}_{G,d}$. In fact, U corresponds to a component of $\mathrm{Hur}_{G,\underline{m}}$ for some $\underline{m} \in \mathbf{Z}^{c/G}$ with $|\underline{m}| = n$; we write $\underline{\deg}(U) = \underline{m}$, so U in fact induces

$$U : \mathrm{Hur}_{G,\underline{k}} \to \mathrm{Hur}_{G,\underline{k}+\underline{\deg} U}. \tag{5.5.1}$$

Again, only the homotopy class of this map is well-defined.

For $g \in c$, let $[g]$ be the associated element of $\mathcal{H}$ (thus living in $\pi_0 \mathrm{Hur}^c_{G,1}$). Set

$$V := \prod_{g \in c} [g]^{\mathrm{ord}(g)} \in \mathcal{H},$$

where $\mathrm{ord}(g)$ denotes the order of $g \in G$. Then V is central (and is in fact independent of the ordering of c chosen), and for any $g \in c$,

$$\text{For any } g \in c \text{ there are } A, A' \in \mathcal{H} \text{ with } [g]A = V = A'[g]. \tag{5.5.2}$$

As we will see, V is indeed a central stabilizer for $\pi_0 \mathrm{Hur}_{G,n}$ in the sense of Definition 2.9.

5.5.1. *Remark.* The monoid $\mathcal{H}$ is equipped with a natural map $\mathcal{H} \to G$, given by multiplication. It is easy to see that the kernel of this map lies in the center of $\mathcal{H}$.

Let $\mathcal{H}[V^{-1}]$ be the monoid obtained by adjoining a formal inverse of V. Then (5.5.2) shows that $\mathcal{H}[V^{-1}]$ is a *group*: it is generated by the images of $[g]$, for $g \in c$, but these are all invertible by (5.5.2).

In this way we obtain a *central extension*

$$\mathcal{H}[V^{-1}] \twoheadrightarrow G$$

equipped with an extra structure, namely, for every $x \in c$ we are given a certain canonical element in its preimage, namely, the image of $[x]$. We will later on show that $\mathcal{H}[V^{-1}] \twoheadrightarrow G$ is universal among central extensions with this extra structure.

More generally, for any $a : c/G \to \mathbb{N}$, we may define

$$V(a) = \prod_{g \in c} g^{a(g)\mathrm{ord}(g)}; \tag{5.5.3}$$

then $V(a)$ has the same properties. The set of all such $V(a)$ form a monoid in the center of $\mathcal{H}$; we denote it by $\mathcal{V}$, and will use it later.

Note that, if M is a module under $\mathcal{V}$ (i.e., a module under the monoid algebra $\mathbf{Z}\mathcal{V}$) then inverting V and inverting $\mathcal{V}$ have the same effect. They can both be described as the operation of tensoring over $\mathbf{Z}\mathcal{V}$ with the group algebra of the group completion of $\mathcal{V}$.

5.6. A review of $A(G,c)$ and $A'(G,c)$. Let us recall the definition of the spaces $A(G,c)$ and $A'(G,c)$ from the introduction in the present context:

There is a natural map from the free loop space $LBG = \mathrm{Map}(S^1, BG)$ to the set of conjugacy classes in G: to each loop we associate its monodromy. This in fact identifies the connected components of LBG with the set of conjugacy classes in G. Let $L^c BG$ denote the union of those components indexed by conjugacy classes in c.

The space $A(G,c)$ is obtained by adjoining to BG a disc with boundary γ for every $\gamma \in L^c BG$; that is to say, the following is a pushout diagram:

$$\begin{tikzcd} S^1 \times L^c BG \arrow[r, "\subseteq"] \arrow[d, "ev"'] & D^2 \times L^c BG \arrow[d] \\ BG \arrow[r] & A(G,c) \end{tikzcd} \tag{5.6.1}$$

Now $A(G,c)$ is equipped with a natural map

$$p : A(G,c) \to \bigvee_{c/G} (S^2), \tag{5.6.2}$$

which collapses BG to a point, and projects each component of $D^2 \times L^c BG$ to $S^2 = D^2/\partial$. Since G is finite, BG and each component of $L^c BG$ have the rational homology of a point. Thus p_* is an isomorphism in rational homology.

Finally, recall that $A'(G,c)$ is, by definition, the homotopy fiber of the map $A(G,c) \to BG$.

5.7. There is a natural map:

$$\mathrm{Hur}^c_{G,n} \longrightarrow \mathrm{Map}\left((D^2, \partial D^2), (A(G,c), BG)\right). \tag{5.7.1}$$

Indeed, any branched G-cover of the disc – branched, say, at $P_1, \dots, P_n \in \dot{D}^2$ – yields a map from $D^2 - \{P_1 \dots, P_n\}$ to BG. Let r be one-half of the minimal distance between any two P_i, or between any P_i and ∂D^2. Then a circular loop of radius $r/2$ around each P_i defines an element of $L^c BG$, which can be canonically filled in $A(G,c)$.

In terms of the homotopy fiber $A'(G,c)$ of $BG \to A(G,c)$, we may rewrite (5.7.1) as a natural map (in the homotopy category)

$$\mathrm{Hur}^c_{G,n} \longrightarrow \Omega A'(G,c). \tag{5.7.2}$$

The map (5.7.1) interacts well with p: we have a natural diagram that commutes up to homotopy:

$$\begin{tikzcd} \mathrm{Hur}^c_{G,n} \arrow[r] \arrow[d] & \mathrm{Conf}^c_n \arrow[d] \\ \mathrm{Map}\left((D^2, \partial D^2), (A(G,c), BG)\right) \arrow[r] & \Omega^2 \bigvee_{c/G} S^2 \end{tikzcd} \tag{5.7.3}$$

where the bottom horizontal arrow is induced by p, and the right vertical arrow is the "approximation map."

5.8. A review of the prior theorems. In the notation of the previous part, there is *by definition* an evident homotopy equivalence:

$$\mathrm{Hur}_{G,n} \xrightarrow{\sim} \mathrm{CMap}^*_n(D^2, BG).$$

(Recall that the latter space is, by definition, a configuration of n points in D^2, together with a map from their complement to BG). This family admits a central stabilizer (in the sense of Definition 2.9, with $X = BG$), by the construction of V in §5.5. Indeed, one only needs to verify that the monoidal structure on the left-hand side defined in §5.5 coincides with the monoidal structure on the right-hand side constructed in Part 1.

Then we have proved that the map (5.7.2) induces, first of all, a homotopy equivalence

$$\Omega B\left(\coprod \mathrm{Hur}^c_{G,n}\right) \xrightarrow{\sim} \Omega A'(G,c), \tag{5.8.1}$$

and (by a variation on the group-completion theorem) we have an isomorphism

$$\left(\bigoplus H_* \mathrm{Hur}^c_{G,n}\right)[V^{-1}] \xrightarrow{\sim} H_* \Omega A'(G,c). \tag{5.8.2}$$

Note the implication for H_0: there is an isomorphism of groups $\mathcal{H}[V^{-1}] \xrightarrow{\sim} \pi_1 A'(G,c)$.

5.8.1. Theorem. *Fix $j \geq 0$. Let Y be an irreducible component of $\mathrm{Hur}^c_{G,\underline{m}}$. For any* sufficiently large *multidegree $\underline{m}$ – that is to say, every coordinate of $\underline{m}$ is sufficiently large – the map from $\mathrm{Hur}^c_{G,\underline{m}}$ to the* colored configuration space *of $\underline{m}$ points in D^2 induces a map on H_j that factors through the image of $H_j(Y, \mathbf{Q})$ in $\bigoplus H_j(\mathrm{Hur}^c_{G,n}, \mathbf{Q})[V^{-1}]$; and the factored map is an isomorphism:*

$$\left(\ \textit{image of } H_j(Y, \mathbf{Q}) \textit{ in } \bigoplus H_j(\mathrm{Hur}^c_{G,n}, \mathbf{Q})[V^{-1}]\ \right) \xrightarrow{\sim} H_j(\mathrm{Conf}^c_{\underline{m}}, \mathbf{Q}) \tag{5.8.3}$$

where Y *is any component of* $\mathrm{Hur}^c_{G,\underline{m}}$.

If we have a homological stability theorem available, the statement simplifies:

5.8.2. Corollary. *Fix* $j \geq 0$. *Suppose that* V *induces an isomorphism*

$$H_j(\mathrm{CHur}^c_{G,\underline{m}}, \mathbf{Q}) \xrightarrow{\sim} H_j(\mathrm{CHur}^c_{\underline{m}+\underline{\deg}(V)}, \mathbf{Q})$$

whenever $\mathrm{mindeg}(\underline{m}) \geq Q$. *Then, for any component* Y *of* $\mathrm{CHur}^c_{G,\underline{m}}$, *the induced map*

$$H_j(Y, \mathbf{Q}) \to H_j(\mathrm{Conf}^c_{\underline{m}}, \mathbf{Q})$$

is an isomorphism whenever $\mathrm{mindeg}(\underline{m}) \geq Q$.

The corollary follows immediately from the theorem – noting that V maps $H_j(\mathrm{Hur}^c_{G,\underline{m}})$ to $H_j(\mathrm{CHur}^c_{\underline{m}+\underline{\deg}(V)})$, the left-hand side of (5.8.3) can be replaced by $\left(\bigoplus H_j(\mathrm{CHur}^c_{G,n}, \mathbf{Q})\right)[V^{-1}]$.

Proof. (of Theorem 5.8.1) Visibly, $A(G,c)$ is connected.

By the theorem of Seifert and van Kampen, the fundamental group of $A(G,c)$ is isomorphic to the quotient of G by the subgroup generated by c. But we are supposing that c generates G, so that $A(G,c)$ is simply connected.

We shall use the following

> (*) If $f : X \to Y$ is a continuous map of connected, simply connected spaces, then it induces isomorphisms on rational homology if and only if it induces isomorphisms on rational homotopy. In that case, f induces a rational homology isomorphism of loop spaces.

The first statement is due to Serre [Ser53] and the second follows from the Eilenberg–Moore spectral sequence $\mathrm{Tor}_{H^*(X,\mathbf{Q})}(\mathbf{Q},\mathbf{Q}) \implies H^*(\Omega X, \mathbf{Q})$ which converges under the connectivity hypothesis.

Then $A(G,c) \to \bigvee_{c/G} S^2$ is a rational homology isomorphism between simply connected spaces, and so by (*) also a rational homotopy equivalence. Consider the induced map

$$\Omega A(G,c) \longrightarrow \Omega(\bigvee_{c/G} S^2).$$

These spaces are connected. Moreover this map induces an isomorphism on rational homotopy $\pi_j \otimes \mathbf{Q}$, for $j \geq 2$. Pass to their universal covers, which we denote by a superscript $\sim$:

$$\tilde{\Omega} A(G,c) \longrightarrow \tilde{\Omega}(\bigvee_{c/G} S^2).$$

This is a map of connected, simply connected spaces, and is still an isomorphism on $\pi_j \otimes \mathbf{Q}$ for $j \geq 2$. What this means is that the induced map between identity components

$$\Omega^2 A(G,c)_0 \longrightarrow \Omega^2(\bigvee_{c/G} S^2)_0$$

induces a rational homology isomorphism (using (*) above). It follows (because both sides have compatible H-group structures) that the induced map from *any* component of $\Omega^2 A(G,c)$, to the image component in $\Omega^2(\bigvee_{c/G} S^2)_0$, is also a rational homology isomorphism.

We deduce from this that the map

$$\operatorname{Map}\big((D^2,\partial D^2),(A(G,c),BG)\big) \longrightarrow \Omega^2 \bigvee_{c/G} S^2 \tag{5.8.4}$$

(here obtained by collapsing both ∂D^2 and BG to a point) is *also* a rational homotopy equivalence between any component of the source and its image component. In fact, the left-hand side is identified with $\Omega A'(G,c)$, and it's enough to check this, again, for the identity component of both sides. But the identity component of $\Omega A'(G,c)$ coincides with that of $\Omega^2 A(G,c)$: the fibration sequence $A'(G,c) \to BG \to A(G,c)$ induces a fibration sequence $\Omega^2 A(G,c) \longrightarrow \Omega A'(G,c) \longrightarrow G$.

Choose now a component Y of $\mathrm{Hur}^c_{G,\underline{m}}$ and put $n = |\underline{m}|$. In the diagram that follows, a subscript $[Y]$ denotes "the component corresponding to Y." For instance, $\left(\mathrm{Conf}^c_{\underline{m}}\right)_{[Y]}$ has the following meaning: the image component of Y in $\mathrm{Conf}^c_{\underline{m}}$.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & (\coprod_n \mathrm{Conf}^c_n)_{[Y]} \arrow[d] \\ \left(\Omega B \coprod_n \mathrm{Hur}^c_{G,n}\right)_{[Y]} \arrow[r] \arrow[d, "\simeq"'] & (\Omega B \coprod \mathrm{Conf}^c_n)_{[Y]} \arrow[d, "\simeq"] \\ \operatorname{Map}\big((D^2,\partial D^2),(A(G,c),BG)\big)_{[Y]} \arrow[r, "\simeq"] & \Omega^2(\bigvee_{c/G} S^2)_{[Y]}. \end{tikzcd} \tag{5.8.5}$$

Here, B refers to the classifying space, for the monoidal structure noted in §5.5.

Now we apply $H_p(-,\mathbf{Q})$ to this diagram. For typographical reasons, we omit the $\mathbf{Q}$ and put the $[Y]$ subscript outside $H_p(\dots)$, although the meaning is in the same as the previous diagram.

$$\begin{tikzcd} H_p(Y) \arrow[r, "f"] \arrow[d, "k"'] & H_p(\coprod_n \mathrm{Conf}^c_n)_{[Y]} \arrow[d, "j"] \\ H_p\left(\Omega B \coprod_n \mathrm{Hur}^c_{G,n}\right)_{[Y]} \arrow[r] \arrow[d, "\cong"'] & H_p\left(\Omega B \coprod \mathrm{Conf}^c_n\right)_{[Y]} \arrow[d, "\cong"] \\ H_p \operatorname{Map}\big((D^2,\partial D^2),(A(G,c),BG)\big)_{[Y]} \arrow[r, "\cong"] & H_p\Omega^2(\bigvee_{c/G} S^2)_{[Y]}. \end{tikzcd} \tag{5.8.6}$$

Note that the bottom lower maps are isomorphisms by virtue of our Theorems from Part 1.

We need only verify that the map j is injective and the map k is surjective., at least when the multidegree $\underline{m}$ of the original component is *large enough.* Once it is so, the map f automatically factors through $\left(H_*\Omega B \coprod \mathrm{Hur}^c_{G,n}\right)_{[Y]}$, and moreover that factored map is an isomorphism.

We prove the injectivity of j using a "colored" version of the argument at the end of p. 103 in [McD75]. For each $g \in c/G$, there is a stabilization map σ^g_n : $\mathrm{Conf}^c_{n-1} \to \mathrm{Conf}^c_n$ which adds a point near the boundary of D^2 and colors it g. The injectivity of j follows if each σ^g_n induces an injection in homology.

Write $SP^n(X)$ for the n^{th} symmetric product of X, and recall the Dold-Thom theorem that $\pi_*(SP^\infty(X)) \cong H_*(X)$. There are maps $P_{k,n} : \mathrm{Conf}_n^c \to SP^{\binom{n}{k}} \mathrm{Conf}_k^c$ which carry a colored configuration to the $\binom{n}{k}$-tuple of colored subsets of size k. Define maps $\tau_{k,n} : H_*(\mathrm{Conf}_n^c) \to H_*(\mathrm{Conf}_k^c)$ in homotopy by the composite

$$SP^\infty(\mathrm{Conf}_n^c) \xrightarrow{SP^\infty(P_{k,n})} SP^\infty(SP^{\binom{n}{k}} \mathrm{Conf}_k^c) \xrightarrow{+} SP^\infty(\mathrm{Conf}_k^c)$$

It is easy to verify that $\tau_{k,n} \circ (\sigma_n^g)_* = \tau_{k,n-1} \bmod \mathrm{im}\,(\sigma_k^g)_*$, so by Lemma 2 of [Dol62], each $(\sigma_n^g)_*$ is a split injection.

We are left to check the surjectivity of k. Consider the map

$$\coprod \mathrm{Hur}_{G,n}^c \to \Omega B \coprod \mathrm{Hur}_{G,n}^c \tag{5.8.7}$$

Let A be the set of connected components of the right-hand side. Being the set of components of a loop space, it has the natural structure of group; moreover, it is equipped with a homomorphism $\psi : A \to \mathbf{Z}^{c/G}$ that sends the image of $\mathrm{Hur}_{G,\underline{m}}^c$ to $\underline{m}$. Note that there also a unique homomorphism $\varphi : \mathcal{V} \to A$ with the property that $V \in \mathcal{V}$ sends component Y_α to component $Y_{\alpha+\varphi(V)}$; we are using the notation introduced around (5.5.3).

For $\alpha \in A$, let X_α (resp. Y_α) be the corresponding connected component of the right-hand side of (5.8.7) (resp. preimage of X_α on the left-hand side of (5.8.7)).

Let R be the rational monoid ring of the monoid $\mathcal{V}$ and S the rational group ring of its group-completion $\mathcal{V}_{\text{gp}}$.

As we have shown, the induced map on H_p gives an isomorphism:

$$\left(H_p \coprod \mathrm{Hur}_{G,n}^c\right)[\mathcal{V}^{-1}] = \bigoplus_\alpha H_p(Y_\alpha) \otimes_R S \xrightarrow{\sim} H_p \Omega B \coprod \mathrm{Hur}_{G,\underline{m}}^c \tag{5.8.8}$$

Note that, in this isomorphism, all objects are $\mathbf{Z}^{c/G}$-graded, i.e. admit a direct sum decomposition $M = \bigoplus_{t \in \mathbf{Z}^{c/G}} M_t$ compatible with their group or ring structures: the rings S, R are graded by means of the map φ, whereas we grade the Hurwitz space by $\underline{m}$ and the homology of $\Omega B \ldots$ by means of its connected components and the map ψ. Also observe that, with this grading, any element of S which is supported on *sufficiently large* elements of $\mathbf{Z}^{c/G}$ in fact belongs to R.

The left hand side of (5.8.8) is *finitely generated* as S-module: There exists collection of $h_\alpha \in H_p(Y_\alpha)$ such that the h_α generate as S-module. The statement reduces, using (5.8.4), to the statement that $H_p(\Omega^2 \bigvee_{c/G} S^2)$ is finitely generated as S-module. But the action of $\mathcal{V}_{\text{gp}}$ on $H_p(\Omega^2 \bigvee_{c/G} S^2)$ factors through $\mathcal{V}_{\text{gp}} \to \mathbf{Z}^{c/G} = \pi_0(\Omega^2 \bigvee_{c/G} S^2)$; since the image of $\mathcal{V}_{\text{gp}}$ has finite index there, our statement reduces to the following:

> For D the double-loop space of a wedge of k distinct 2-spheres, $H_p(D, \mathbf{Q})$ is finitely generated as a $\mathbf{Q}[\pi_0(D)]$-module, where the action is through the group structure on D.

This follows from the Hilton-Milnor theorem [Hil55]; in this instance, it gives a homotopy equivalence between D and the product

$$\prod_{b \in B} \Omega^2 S^{|b|+1}$$

where B is a monomial basis of a free Lie algebra on k generators, and $|b|$ is the length of the monomial. Each of the terms in this product has rational homology in at most two nonzero degrees, and these grow linearly with $|b|$: $|b|-1$ and if $|b|$ is odd, also $2|b|-1$. Further, the number of monomials of a given length is finite, and so the homology of a single component is finitely generated in each degree.

Now take $x \in H_p(X_\beta)$. It is the image, under the morphism of (5.8.8), of some $\tilde{x} = \sum_{\alpha \in E} s_\alpha h_\alpha$ on the left-hand side. Let s'_α be the projection of s_α onto the $\psi(\beta-\alpha) \in \mathbf{Z}^{c/G}$ component of the $\mathbf{Z}^{c/G}$-graded ring S. Then x is in fact the image of

$$\tilde{x}' = \sum_{\alpha \in E} s'_\alpha h_\alpha.$$

It remains to observe that, if $\psi(\beta)$ is sufficiently large, then s'_α automatically belongs to R, and so $\tilde{x}'$ actually belongs to $\bigoplus H_p(Y_{\beta'})$, where the sum is taken over those β' with $\psi(\beta') = \psi(\beta)$. Now replace $\tilde{x}'$ by its projection to $H_p(Y_\beta)$; its image is $x \in H_p(X_\beta)$, just as desired. □

6. A sketch of a simple case of Cohen–Lenstra

We begin by explaining how the results of the previous section can be used to prove results towards the Cohen-Lenstra conjectures over $\mathbf{F}_q(t)$, even without the detailed analysis of connected components that occupies §7 and §8. In particular, we compute the average size of the ℓ-part of the class number of a quadratic extension of a rational function field over a finite field, in the simple case when ℓ does not divide $q-1$. What we show is only a special case of our later Theorem 11.1.1; we are sketching this case separately in order to express the main idea of the proof before taking on the somewhat heavy notational load required to state the theorem precisely in the general case.

Of course, in order to be complete this argument requires that we have in hand the definition of the Hurwitz scheme, which will appear below in §8.6. For now, the reader may take for granted (or recall from [EVW09]) that the number of points over $\mathbf{F}_q$ of the Hurwitz scheme CHur appearing in Proposition 6.0.3 are in bijection with nontrivial ℓ-torsion points on hyperelliptic curves over $\mathbf{F}_q$, and that the complex points of CHur are homotopic to the space CHur appearing in the first part of the paper.

6.0.3. Proposition. *There is a function $Q(\ell) > 0$ with the following property. Let ℓ be an odd prime and $\mathbf{F}_q$ a finite field of odd order $q > Q(\ell)$, such that ℓ does not divide $q(q-1)$.*

Let $C(d)$ be the number of squarefree monic polynomials $f(t)$ in $\mathbf{F}_q[t]$ of degree $2d+1$, and let $N(d)$ be the sum, over all such f, of $h-1$, where h is the number of ℓ-torsion elements in the class group of $\mathbf{F}_q(t, \sqrt{f})$.

Then

$$\lim_{d \to \infty} \frac{N(d)}{C(d)} = 1.$$

We note that this prediction conforms with the Cohen-Lenstra heuristics for quadratic imaginary number fields. We proved in [EVW09, Th 8.7] that $|\frac{N(d)}{C(d)} - 1|$ was bounded above and below by constants as $d \to \infty$; the topological results of the first half of this paper are what allow us to go further and show that this quantity decays to 0 as d grows.

Proof. (Sketch)

The number $N(d)$ can be interpreted as the number of $\mathbf{F}_q$-rational points on the Hurwitz scheme $\mathsf{CHur}^c_{G,2d+1}$, where G is the dihedral group of order 2ℓ, and c is the conjugacy class of involutions. The argument of [EVW09, Proposition 7.5] shows that the étale cohomology of $\mathsf{CHur}^c_{G,2d+1}$ is "identical" in characteristic p and characteristic 0 (see *loc. cit.* for the precise statement.)

The scheme CHur is a scheme over $\mathbf{Z}[\frac{1}{2\ell}]$; we denote by $\mathsf{CHur}_{\mathbf{C}}$ its base-change to $\mathbf{C}$, and so on.

It is known that $(\mathsf{CHur}_{G,2d+1})_{\mathbf{C}}$ is connected, as follows from results of Yu, Achter-Pries, and Hall (see the discussion in the proof of [EVW09, Thm 8.7]). It follows from the [EVW09, Theorem 6.1] that the condition of Corollary 5.8.2 holds. Thus, the projection on H^i_{et} induced by projection of $\left(\mathsf{CHur}^c_{G,2d+1}\right)_{\mathbf{C}}$ to $(\mathsf{Conf}_{2d+1})_{\mathbf{C}}$ is an isomorphism once d exceeds some linear function of i. By the comparison between characteristic p and characteristic 0, the same statement holds for $(\mathsf{CHur}_{G,2d+1})_{\overline{\mathbf{F}_q}}$ and $(\mathsf{Conf}_{2d+1})_{\overline{\mathbf{F}_q}}$.

It is thus natural to expect from the Lefschetz trace formula that the two varieties should asymptotically have the same number of $\mathbf{F}_q$-rational points, as the proposition asserts, and this is indeed the case. To prove this fact requires an a priori bound on the contribution of cohomology outside the stable range; the argument is exactly as in the proof of [EVW09, Thm 8.7]. (It is this a priori bound that necessitates the hypothesis that $q > Q(\ell)$.) In the earlier paper, the contribution to $\frac{N(d)}{C(d)}$ of the H_i for $i > 0$ was merely shown to remain bounded as $d \to \infty$; the results of the first half of this paper demonstrate that the limit is in fact 0. $\square$

The argument of Proposition 6.0.3 shows that, when we are in a situation where we can prove a stability theorem as required in Corollary 5.8.2, the problem of proving counting theorems in the vein of 6.0.3 comes down to an analysis of the $\mathbf{F}_q$-rational connected components of a Hurwitz scheme. In the context of Proposition 6.0.3, the Hurwitz space was connected, so this problem presented no difficulty. In general, though, we need to describe the Galois action on the geometric components of $\mathsf{CHur}_{G,n}$, which is not accessible by purely topological means. The remainder of this section is devoted to producing such a description.

6.1. Cohen-Lenstra for cubic fields. As an example of the generality of the method, we remark that the argument of Proposition 6.0.3 can also be used to show that Cohen-Lenstra heuristics (in this context more properly called Cohen-Lenstra-Martinet heuristics) hold for *cubic* extensions of $\mathbf{F}_q(t)$:

In this case, one takes G to be a semidirect product $V_\ell \rtimes S_3$, where ℓ is a prime greater than 3, and V_ℓ is the 2-dimensional reduced permutation representation of S_3. The group G has a unique conjugacy class of involutions, which we denote c. The corresponding Hurwitz space $\mathsf{CHur}^c_{G,2d+1}$ is connected – this follows for instance from the substantially more general results of Allcock and Hall [AH10, Theorem 3] concerning this space. Since G is in the class of groups to which [EVW09, Proposition 7.5] applies, one once again finds that

$$\frac{|\mathsf{CHur}^c_{G,2d+1}(\mathbf{F}_q)|}{|\mathsf{Conf}_{2d+1}(\mathbf{F}_q)|} \longrightarrow 1$$

as $d \to \infty$, for all q sufficiently large relative to ℓ.

It follows from Proposition 6.0.3 and the usual identification of cubic extensions with 3-torsion in class groups of quadratic extensions that the number of cubic extensions of $\mathbf{F}_q(t)$ with discriminant f also averages to 1, as f ranges over squarefree polynomials of odd degree. Putting these together, one can show that if $L/\mathbf{F}_q(t)$ is a cubic extension, chosen randomly from those with squarefree discriminant including ∞, the number of nontrivial $\mathbf{F}_q$-rational ℓ-torsion points on the Jacobian of the curve $C/\mathbf{F}_q$ with function field L is on average 1.

7. Stable components of Hurwitz spaces

In this section we identify the stable π_0 of Hurwitz spaces by combinatorial group theory. The results of the present section were known to M. Fried (unpublished), and the special case where the union of conjugacy classes c consists of all nonidentity elements was treated in [FV91]. The proof we present of Theorem 7.6.1 is based on ideas of Conway–Parker, as presented in the paper of Fried and Völklein. (The paper [Kul11] also contains arguments in a related vein.)

We note that the methods of the first part of this paper give another proof, which we do not explicate in full.

7.1. Suppose that $f : \tilde{G} \to G$ is a central extension, endowed with a bijective lift of c to $\tilde{G}$: that is to say, a conjugacy-equivariant section $x \mapsto \tilde{x}$ from c to $\tilde{G}$ such that $f(\tilde{x}) = x$ for all $x \in c$.

Then the function

$$(\gamma_1, \ldots, \gamma_n) \in c^{\times n} \mapsto \prod \widetilde{\gamma_i} \in \tilde{G} \tag{7.1.1}$$

is constant on orbits of the braid group, where the action is as in (5.4.1). The value lies in $\ker(f)$ if $\prod \gamma_i = 1$. In this way we obtain an invariant of covers of $\mathbf{P}^1$, first described by M. Fried, which we will describe as a "lifting invariant." For example ([Ser90]) the space of connected degree n covers of $\mathbf{P}^1$ such that every monodromy element is a 3-cycle is usually disconnected, owing to the existence of the double cover of A_n to which the conjugacy class of 3-cycles lifts bijectively.

In order to get the most mileage out of this construction, we apply it to the universal central extension of G to which c lifts bijectively. In this section we shall show that such a universal extension exists (see Theorem 7.5.1 for the precise formulation); this contrasts with the fact that, in general, a group does not a have a universal central extension. The lifting invariant corresponding to such a universal central extension, which we will call the "universal lifting invariant," separates components of the Hurwitz space when the number of branch points is large (see Theorem 7.6.1).

7.2. Schur coverings. Let G be any group and $c \subset G$ a union of conjugacy classes. Take a central extension

$$(*) \quad A \longrightarrow \tilde{G} \longrightarrow G$$

of G by a finite abelian group A. It is classified by an element of $H^2(G, A)$; that group fits into a short exact sequence

$$\mathrm{Ext}^1(H_1(G, \mathbf{Z}), A) \longrightarrow H^2(G, A) \xrightarrow{\pi} \mathrm{Hom}(H_2(G, \mathbf{Z}), A). \tag{7.2.1}$$

A *Schur cover* is, by definition, an extension of G by A so that the image of the extension class under π above is an *isomorphism* $H_2(G, \mathbf{Z}) \to A$. (We will prove below that this is equivalent to the definition of "Schur covering" that is usually given in finite group theory.) If G is perfect, i.e. $G^{\mathrm{ab}} = H_1(G, \mathbf{Z})$ is trivial, then this is a *universal* central extension, that is to say, maps uniquely to any other central extension. In general, a Schur cover is not unique; for example, the dihedral group and quaternion group both admit the structure of Schur covers of $(\mathbf{Z}/2\mathbf{Z})^2$. On the other hand, one has at least:

Lemma. *Suppose G^{ab} is free abelian. Then a Schur cover $\mathcal{K} \to \tilde{G} \to G$ maps (not necessarily uniquely) to any other central extension $B \to G' \to G$.*

Proof. We must show that the pullback of the second sequence to $\tilde{G}$ splits, i.e. there is a section of $G' \times_G \tilde{G} \to \tilde{G}$; equivalently, the class of the extension $G' \to G$ in $H^2(G, B)$ maps to zero in $H^2(\tilde{G}, B)$.

The latter map fits into the inflation-restriction exact sequence:

$$H^1(G,B) \to H^1(\tilde{G},B) \longrightarrow \mathrm{Hom}(\mathcal{K},B)^G \xrightarrow{\iota} H^2(G,B) \longrightarrow H^2(\tilde{G},B) \tag{7.2.2}$$

We claim that the induced morphism $H^2(G,B) \to H^2(\tilde{G},B)$ is zero; in fact, we will show that the map ι is an isomorphism. Note that, because G acts trivially on B and $\mathcal{K}$, every element of $\mathrm{Hom}(\mathcal{K}, B)$ is automatically G-invariant.

Consider the composite

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{K},B) \arrow[r, "\iota"] & H^2(G,B) \arrow[d, "\pi_B"] \\ & \mathrm{Hom}(H_2(G,\mathbf{Z}),B) \end{tikzcd} \tag{7.2.3}$$

where the second morphism π_B is from the universal coefficient sequence (7.2.1), and is an isomorphism because G^{ab} is free. The composite map coincides with the map[3] induced from $H_2(G,\mathbf{Z}) \to \mathcal{K}$. Because $\tilde{G}$ is supposed a Schur cover, $H_2(G,\mathbf{Z}) \to \mathcal{K}$ is an isomorphism. So ι is also an isomorphism. $\square$

Let us note, for later use, that the inflation-restriction sequence also implies that the induced map $H^1(G,B) \to H^1(\tilde{G},B)$ is an isomorphism. Since B was arbitrary, this can be rephrased: the induced map

$$\tilde{G}^{\mathrm{ab}} \to G^{\mathrm{ab}} \tag{7.2.4}$$

is an isomorphism. In the terminology of group theory, $\tilde{G} \to G$ is a stem extension.

We now prove that our notion of "Schur cover" agrees with the usual definition in finite group theory:

Lemma. *Suppose that G is finite and that $\mathcal{K} \to \tilde{G} \to G$ is a stem extension with $|\mathcal{K}|$ maximal amongst such extensions. Then this is a Schur cover (in the sense discussed after* (7.2.1)*).*

Proof. The inflation-restriction sequence (7.2.2) with $B = \mathbf{Q}/\mathbf{Z}$ simplifies to:

$$\mathrm{Hom}(\mathcal{K},\mathbf{Q}/\mathbf{Z}) \overset{j}{\hookrightarrow} H^2(G,\mathbf{Q}/\mathbf{Z}) \overset{\iota}{\to} H^2(\tilde{G},\mathbf{Q}/\mathbf{Z}).$$

This shows that $|\mathcal{K}| \leq |H^2(G,\mathbf{Q}/\mathbf{Z})| = |H_2(G,\mathbf{Z})|$; since we have seen there is a stem extension of this size, we must have $|\mathcal{K}| = |H_2(G,\mathbf{Z})|$ and the map j is an

[3]Starting with $f \in \mathrm{Hom}(\mathcal{K},B)$, one pushes out $\tilde{G} \to G$ via f to obtain a central extension by B, and then takes its image under the universal coefficient sequence. This is (by naturality) the same as the image under f of the class $\mathrm{Hom}(H_2(G,\mathbf{Z}),\mathcal{K})$ associated to $\tilde{G} \to G$.

isomorphism; this map is dual to the natural map $H_2(G, \mathbf{Z}) \to \mathcal{K}$, which must therefore be an isomorphism. $\square$

Fix an extension (*); we now analyze when c lifts bijectively to $\tilde{G}$.

Take two commuting elements $x, y \in G$. Then, if $\hat{x}, \hat{y}$ are arbitrary lifts of x, y to $\tilde{G}$, the commutator $[\hat{x}, \hat{y}]$ lies in A and is independent of the choice of $\hat{x}, \hat{y}$. We denote it by $\langle x, y\rangle_{\tilde{G}}$ for short.

Thus $x, y \mapsto \langle x, y\rangle_{\tilde{G}}$ defines a function from commuting pairs in G to A. Then c lifts bijectively to $\tilde{G}$ *if and only if* $\langle x, y\rangle_{\tilde{G}} = \mathrm{id}_A$ whenever $x \in c$ and y commutes with c. (When we write the symbol $\langle x, y\rangle_{\tilde{G}}$, we always suppose that x, y commute.)

There is a universal way to understand this "commutator" pairing: Take commuting elements $x, y \in G$. Define the "universal commutator"

$$\langle x, y\rangle \in H_2(G, \mathbf{Z}) \tag{7.2.5}$$

as the image of the canonical generator for $H_2(\mathbf{Z}^2, \mathbf{Z})$, under the map ϕ_* induced by $\phi : \mathbf{Z}^2 \to G,\ \phi : (m, n) \mapsto x^m y^n$. With this understanding $\langle x, y\rangle_{\tilde{G}}$ is the image of $\langle x, y\rangle \in H_2(G, \mathbf{Z})$ under the morphism

$$H_2(G, \mathbf{Z}) \longrightarrow A,$$

induced by the extension class of $(*)$.

We conclude that, given a central extension $(*)$ with extension class $\alpha \in H^2(G, A)$, a necessary and sufficient condition for c to lift to $\tilde{G}$ is that the image of class α in $\mathrm{Hom}(H_2(G, \mathbf{Z}), A)$, under the universal coefficients map (7.2.1), vanishes on all commutators $\langle x, y\rangle$ with $x \in c$; equivalently, the morphism

$$H_2(G, \mathbf{Z}) \longrightarrow A$$

associated to (*) via the universal coefficients sequence (7.2.1) factors through the *quotient* of $H_2(G, \mathbf{Z})$ generated by all such commutators. This quotient will play a fundamental role:

7.3. Definition. [Reduced Schur multiplier; reduced Schur cover.] Set $H_2(G, c; \mathbf{Z})$ to be the quotient of $H_2(G, \mathbf{Z})$ by the subgroup $Q_c \subset H_2(G, z)$ generated by all universal commutators $\langle x, y\rangle$ (see (7.2.5)), where x, y commute and $x \in c$.

By a *reduced Schur cover* $\widetilde{G}_c \to G$ we mean: the quotient of a Schur cover $H_2(G, \mathbf{Z}) \to \tilde{G} \to G$ by all commutators $[x, y] \in H_2(G, \mathbf{Z})$ where x, y commute and $x \in c$. The kernel of a reduced Schur cover is isomorphic to $H_2(G, c; \mathbf{Z})$. However (like a Schur cover) *it need not be unique.*

Note that, if $f : \widetilde{G}_c \to G$ is a reduced Schur cover, then the image of $H_2(\widetilde{G}_c, \mathbf{Z})$ inside $H_2(G, \mathbf{Z})$ equals Q_c. Indeed, by the homology version of (7.2.2), this image is the kernel of a connecting map

$$H_2(G, \mathbf{Z}) \to \ker(f)$$

which is none other than the map associated to f by the universal coefficients sequence (7.2.1).

7.4. Marked central extensions. A *c-marked extension* of (G, c) is a triple $(\tilde{G}, f, \tilde{c})$ where $f : \tilde{G} \to G$ is a extension and $\tilde{c} \subset \tilde{G}$ is a conjugacy-invariant set (i.e., a union of conjugacy classes for $\tilde{G}$) such that $f : \tilde{c} \to c$ is a bijection. (We sometimes refer to giving $\tilde{c}$ as giving a *marking* of the central extension $(\tilde{G}, f)$.) We will be only concerned with *c-marked central extension*: that is, such a triple where $\ker(f)$ is central.

We will often say simply "marked central extension" when the class c is understood.

If A is a group, a marked central A-extension of G is a marked central extension $(\tilde{G}, f, \tilde{c})$ together with an isomorphism $\ker f \cong A$.

By a *homomorphism* of marked central extensions $(\tilde{G}, f, \tilde{c}) \to (\tilde{G}', f', \tilde{c}')$ we shall mean a group homomorphism $\phi : \tilde{G} \to \tilde{G}'$ covering the identity map on G, carrying $\tilde{c}$ to $\tilde{c}'$, and such that $f' \circ \phi = \phi \circ f$.

Observe that if ϕ, ϕ' are *two such morphisms*, then in fact

$$\phi \text{ and } \phi' \text{ agree on the subgroup generated by } \tilde{c}. \tag{7.4.1}$$

By *universal marked central extension* we mean $(\tilde{G}, f, \tilde{c})$ with the property that it admits a unique homomorphism to any central marked extension (G', f', c').

Clearly, universal marked central extensions are unique up to unique isomorphism, if they exist.

7.5. Existence of universal marked central extensions. It is not hard to see that such universal central marked extensions *exist*:

Consider the group Γ with one generator for each element of c – call these $[g], g \in c$ – and subject to the relation

$$[x][y][x]^{-1} = [xyx^{-1}]. \tag{7.5.1}$$

The map $[g] \mapsto g$ extends to a homomorphism $\Gamma \to G$ and clearly the $[x], x \in c$ give a lifting of c. Now the kernel of $\Gamma \to G$ is *central* in Γ: for every choice of $g_i \in c$, $y \in c$, and $a_i \in \mathbf{Z}$,

$$[g_1]^{a_1}[g_2]^{a_2} \dots [g_k]^{a_k}[y] = [y^{g_1^{a_1} \dots g_k^{a_k}}][g_1]^{a_1}[g_2]^{a_2} \dots [g_k]^{a_k}$$

In particular, if $g_1^{a_1} \dots g_k^{a_k} = \mathrm{id}_G$, then $[y]$ and $[g_1]^{a_1}[g_2]^{a_2} \dots [g_k]^{a_k}$ commute. The extension $\Gamma \to G$ is universal, because the relation (7.5.1) is satisfied by any lifting of c in any marked central extension.

There is a natural variation of the prior discussion: Recall that we have defined a monoid $\mathcal{H}$ and a central $V \in \mathcal{H}$ in §5.5; the monoid $\mathcal{H}$ is equipped with a monoid morphism $\mathcal{H} \to G$ and an element $[x] \in \mathcal{H}$ lifting any $x \in c \subset G$. Then $\mathcal{H}[V^{-1}]$ *is a group and* $\mathcal{H}[V^{-1}] \to G$ *is a universal marked central extension with respect to the marking* $[x], x \in c$.

Here is the proof. We already saw (Remark 5.5.1) that $\mathcal{H}[V^{-1}]$ is a group. There are unique morphisms

$$\mathcal{H}[V^{-1}] \xrightarrow{f} \Gamma, \quad \Gamma \xrightarrow{g} \mathcal{H}[V^{-1}]$$

commuting with the projections to G and characterized by the fact they preserve the liftings of c to $\mathcal{H}[V^{-1}]$ and Γ: Indeed, one justs checks that the defining relations for Γ (resp. the braid equivalences defining $\mathcal{H}$) are satisfied in $\mathcal{H}[V^{-1}]$) (resp. in Γ). Thus f, g are inverses, for the lifts of c generate both Γ and $\mathcal{H}[V^{-1}]$.

Theorem 7.5.1 gives two more constructions of a universal marked central extension:

7.5.1. Theorem. *Let G be a finite group and c a generating conjugacy-invariant set. Then (G, c) admits a universal marked central extension. Its kernel – to be denoted by $H_2'(G, c; \mathbf{Z})$ – is non-canonically isomorphic to $H_2(G, c; \mathbf{Z}) \times \mathbf{Z}^{c/G}$.*

Indeed, any of the following extensions admits (not uniquely) the structure of universal marked central extension:

(i) *The group with presentation* $\langle [g] : g \in c \mid [x][y][x]^{-1} = [xyx^{-1}] \rangle$*, where the morphism to* G *sends* $[g]$ *to* g*.*

(i)' *The morphism* $\mathcal{H}[V^{-1}] \to G$*, notation of §5.5.*

(ii) *The map* $\pi_1 A'(G,c) \to G$ *induced on fundamental groups by* $A'(G,c) \longrightarrow BG$*;*

(iii) *(if* G *is finite): The map*[4] $\mathbf{Z}^{c/G} \times_{\mathrm{G^{ab}}} \widetilde{G}_c \to G$*, where* $\widetilde{G}_c \to G$ *is a reduced Schur cover and the morphism* $\mathbf{Z}^{c/G} \to G^{\mathrm{ab}}$ *is as in* (5.1.1)*.*

We have already proved (i) and (i)'. (ii) follows from (i)' and (5.8.2), we note it only to make contact between this part and the results of Part 1. The part of this Theorem that is most useful for "practical computation" will be (iii). Indeed, in what follows, we denote by $\hat{G}$ the group defined there:

$$\hat{G} = \mathbf{Z}^{c/G} \times_{G^{\mathrm{ab}}} \tilde{G}_c$$

Proof. Fix a lift $x \mapsto \tilde{x}$ of c to $\tilde{G}_c$, and put

$$\hat{c} = \{(e_x, \tilde{x}) : x \in c\},$$

so that $\hat{c}$ is a conjugacy-invariant subset of $\hat{G}$ lifting c.

7.5.2. Lemma. *The morphism* $\mathbf{Z}^{\hat{c}/\hat{G}} \longrightarrow \hat{G}^{\mathrm{ab}}$ *is an isomorphism.*

Proof. The induced maps $\tilde{c}/\tilde{G} \to c/G$ and $\tilde{G}_c^{\mathrm{ab}} \to G^{\mathrm{ab}}$ are both isomorphisms.

We are reduced to verifying that, for any finite group F and any conjugacy-invariant subset $c \subset F$, the abelianization of

$$\mathbf{Z}^{c/F} \times_{F^{\mathrm{ab}}} F$$

is free on generators (e_x, x), where we choose one representative x from each conjugacy class in c. We leave this to the reader. $\square$

We may now complete the proof of part (iii):

Suppose that $A \to G' \to G$ is a central extension to which c lifts bijectively, and let $\alpha \in H^2(G, A)$ be its extension class.

We claim, first of all, that the pullback of α to $H^2(\hat{G}, A)$ is zero.

The image of α in $\mathrm{Hom}(H_2(G, \mathbf{Z}), A)$ factors through $H_2(G, c; A)$. Consider the commutative diagram

$$\begin{array}{ccc} H^2(G, A) & \longrightarrow & \mathrm{Hom}(H_2(G, \mathbf{Z}), A) \\ \downarrow & & \downarrow \\ H^2(\hat{G}, A) & \xrightarrow{\sim} & \mathrm{Hom}(H_2(\hat{G}, \mathbf{Z}), A) \end{array}$$

The bottom row is an isomorphism by virtue of the Lemma that we have just proven and (7.2.1). Our claim follows from the fact that the composite

$$H_2(\hat{G}, \mathbf{Z}) \to H_2(G, \mathbf{Z}) \to H_2(G, c; \mathbf{Z})$$

[4] If X_1, X_2 are groups equipped with homomorphisms $\phi_i : X_i \to Y$, then the fiber product group $X_1 \times_Y X_2$ is, by definition, the subgroup of $X_1 \times X_2$ consisting of pairs (x_1, x_2) with $\phi_1(x_1) = \phi_2(x_2)$.

is *zero.* In fact, that map $\hat{G} \to G$ factors through $\tilde{G}_c$, and the composite

$$H_2(\tilde{G}_c, \mathbf{Z}) \to H_2(G, \mathbf{Z}) \to H_2(G, c; \mathbf{Z})$$

is zero (see remark after Definition 7.3).

We have shown, therefore, that the pullback of α to $\hat{G}$ is zero; equivalently, there is a morphism of extensions $\phi : \hat{G} \to G'$ from $\hat{G} \to G$ to $G' \to G$. This homomorphism is not unique; however, it is unique up to addition of homomorphisms in $\mathrm{Hom}(\hat{G}, A)$. The Lemma now implies that the latter group is identified (by restriction) with functions $\hat{c}/\hat{G} \to A$; thus there exists a homomorphism which is compatible with the markings, and that this homomorphism is unique. □

7.6. Summary. Let us summarize where we are: To a marked central extension $(f : \tilde{G} \to G, \tilde{c})$ of a pair (G, c) we associate the *lifting invariant*; it is a braid-invariant function on n-tuples from c, taking values in $\tilde{G}$. In other terms (see the discussion of (5.4.2)) it is a function

$$\pi_0 \mathrm{Hur}^c_{G,n} \longrightarrow \tilde{G}$$

On the other hand, there exists a universal c-marked central extension, which we have denoted as $H_2'(G, c; \mathbf{Z}) \to \tilde{G} \to G$ (see Theorem 7.5.1) and we refer to the corresponding lifting invariant as the *universal lifting invariant.*

Recall that the natural marking on the extension $G \times \mathbf{Z}^{c/G} \to G$ (namely, $\tilde{c} = \{(x, e_x) : x \in c\}$) gives rise to a morphism $\tilde{G} \to G \times \mathbf{Z}^{c/G}$, and, by projection, to a homomorphism induces a morphism

$$\tilde{G} \to \mathbf{Z}^{c/G}. \tag{7.6.1}$$

7.6.1. Theorem. *The universal lifting invariant induces, for any sufficiently large multidegree $\underline{m} \in \mathbf{Z}^{c/G}$, a bijection*

$$\pi_0(\mathrm{CHur}^c_{G,\underline{m}}) \xrightarrow{\sim} \textit{preimage of } \underline{m} \textit{ in } \tilde{G} \textit{ under } (7.6.1). \tag{7.6.2}$$

Explicitly, fix a reduced Schur cover $H_2(G, c; \mathbf{Z}) \to \tilde{G}_c \to G$. The lifting invariant associated to this central extension gives a function

$$\pi_0(\mathrm{CHur}^c_{G,\underline{m}}) \xrightarrow{\sim} \tilde{G}_c$$

and in any sufficiently large degree $\underline{m}$ this map is a bijection.

"Sufficiently large" means here, as always, that $\underline{m}$ contains sufficiently many representatives of *each* conjugacy class making up c, i.e. $\mathrm{mindeg}(\underline{m}) \geq A$ for some sufficiently large A.

7.6.2. *Remark.* Theorem 7.6.1 can be seen as a generalization of the result of Fried and Völklein, which is the special case where $c = G - \mathrm{id}$ and $H_2(G, c; \mathbf{Z})$ is trivial.

In fact, Theorem 7.6.1 has been proved in unpublished work by Fried.

Proof. We have already proved that

$$\left(\bigoplus_{\underline{m}} \pi_0 \mathrm{Hur}^c_{G,\underline{m}} \right) [V^{-1}] \twoheadrightarrow G. \tag{7.6.3}$$

has the structure of universal marked central extension. In this identification, the map (7.6.1) sends every element of $\pi_0 \mathrm{Hur}^c_{G,\underline{m}}$ to $\underline{m}$.

Note that the image of $\pi_0\mathrm{Hur}^c_{G,\underline{m}}$ under V lies in $\pi_0\mathrm{CHur}^c_{G,\underline{m}+\underline{\deg}(V)}$, and the bijection (7.6.2) follows from (7.6.3) and:

> *Claim:* For any sufficiently large $\underline{m}$, the map V induces an isomorphism $\pi_0\mathrm{CHur}^c_{G,\underline{m}} \to \pi_0\mathrm{CHur}^c_{G,\underline{m}+\underline{\deg}(V)}$;

Proof of claim – By an elementary argument, there exists A with the following property: For any $y \in c$ and any $\underline{m}$ with $\mathrm{mindeg}(\underline{m}) \geq A$, any $(g_1, \ldots, g_n) \in c^n$ which generates G and has multidiscriminant $\underline{m}$ is equivalent, under Br_n, to $(y, g_2', \ldots, g_n')$ for some $g_i' \in c$. In other words, the map "left multiplication by $[y]$" yields a surjection

$$\pi_0(\mathrm{CHur}^c_{G,\underline{m}}) \to \pi_0(\mathrm{CHur}^c_{G,\underline{m}+e_y})$$

whenever $\mathrm{mindeg}(\underline{m}) \geq A$. This must therefore also be a *bijection* for $\mathrm{mindeg}(\underline{m}) \geq A'$, for suitable A'. That implies the claim.

We have now proven the first assertion of the Theorem. The second assertion of the theorem – "Explicitly..." – follows easily from the explicit construction of a universal marked central extension given in Theorem 7.5.1, part (iii).

□

8. The Galois action on stable components of Hurwitz schemes

So far, we have concentrated on Hurwitz spaces, which parametrize topological branched coverings of a disc. In fact, the Hurwitz space can be interpreted as the space of complex points of a *Hurwitz scheme*, which parametrizes branched coverings of the affine line over a general base scheme. Our goal in the present section is to upgrade Theorem 7.6.1, which describes the connected components of Hurwitz spaces in the "many branchpoint limit," to a description of the connected components of the Hurwitz scheme. In this setting, the connected components carry a Galois action, the explicit description of which is the main goal of this section.

We note that the study of rationality properties of connected components of Hurwitz spaces has been studied previously in the context of the Inverse Galois Problem, with special attention paid to connected components defined over $\mathbf{Q}$; see for instance the work of Fried and Debes-Emsalem on Harbater-Mumford components [DE06].

We now explain in more detail the contents of this section. The reader may wish to begin by glancing ahead at §9 which gives a "user-friendly" summary of the results.

After some notational preliminaries (§8.1) we recall in §8.2 the notion of a *tangential basepoint* for the fundamental group in the sense of Deligne. To see why we need it, recall that the branched covers parametrized by our Hurwitz spaces $\mathrm{CHur}^c_{G,n}$ (see §5.2) are endowed with a marked point in the fiber over a specified point in the boundary of the compact disc. In particular, it is important that we are studying covers of a surface with a boundary component as opposed to a surface with a puncture. Constructing the analogous set-up on the algebraic side requires a bit of work, since the complex points of $\mathbf{A}^1$ are naturally a punctured sphere, not a disc. The tangential basepoint provides the necessary analog.

With the notion of tangential basepoint in hand, we make an algebraic analog to the notion of branched G-cover in §8.3. In §8.3 we also define algebraic analogs of the multidiscriminant and boundary monodromy (see §5 for these notions in the topological context). For example, in topology, we may assign to a branched covering the element $\sum m(\mathcal{C})e_{\mathcal{C}}$ of $\mathbf{Z}^{c/G}$, where $m_{\mathcal{C}}$ is the number of branch points of monodromy type $\mathcal{C}$, and the sum is taken over all conjugacy classes $\mathcal{C}$ in c. The multidiscriminant also makes sense in the algebro-geometric context, but requires some care, thanks to the fact that inertia groups in the arithmetic setting do not come equipped with a canonical generator. This means, in particular, that the behavior of the multidiscriminant under "Galois conjugation" is not transparent.

After some algebraic preliminaries (§8.4) we make in §8.5 an algebraic analog of the assignment

$$\text{branched topological } G\text{-cover} \rightarrow \text{element of the universal marked central extension } \tilde{G}.$$

that was described in Theorem 7.6.1. It remains then, to prove the analog of Theorem 7.6.1, i.e. that this assignment in fact separates components of the Hurwitz scheme. This is done in the final §8.7 by "comparison between characteristic 0 and characteristic p", after recalling the definition of the Hurwitz scheme in §8.6.

8.1. Notation. This section contains a significant amount of notation, for which we apologize to the reader.

8.1.1. *Fields.* Let k be a separably closed field. Let $k_0 \subset k$ be a subfield such that k is the separable closure of k_0, and define Γ_k to be the Galois group of k over k_0. We will almost exclusively have in mind the case where k_0 is a finite field and k its algebraic closure. We denote by p the characteristic of k, if nonzero.

8.1.2. *Schemes.* Unless otherwise specified, all schemes are k-schemes. (There will be several points where we consider schemes over other rings, notably §8.2.1 where we work over a discrete valuation ring and §8.2.3 where we work over k_0. In each case this will be indicated.)

We will work throughout this section with the projective line $\mathbf{P}^1$. Our $\mathbf{P}^1$ will always be endowed with a choice of a closed point which we call ∞; we denote by $\mathbf{A}^1$ the open subscheme of $\mathbf{P}^1$ obtained by deleting ∞.

8.1.3. *Tate twists.* We adopt the notation $\hat{\mathbf{Z}}(1)$ for the group $\varprojlim \mu_n$ and $\hat{\mathbf{Z}}$ for the ring $\varprojlim(\mathbf{Z}/n\mathbf{Z})$, where the limit is taken over all n relatively prime to the characteristic of k. The subset of topological generators in $\hat{\mathbf{Z}}(1)$ will be denoted $\hat{\mathbf{Z}}(1)^\times$. It is a torsor for the units $\hat{\mathbf{Z}}^\times$ of $\hat{\mathbf{Z}}$. It also carries an action of Γ_k, by virtue of its description in terms of roots of unity.

Notation warning: It is more customary to denote the group we call $\hat{\mathbf{Z}}$ by $\hat{\mathbf{Z}}'$, to emphasize the fact that we have removed the pro-p-part of the group. We have deviated from this custom in order to keep superscripts to a minimum.

8.1.4. *The power action.* Let X be a set endowed with an action of $\hat{\mathbf{Z}}^\times$. We denote by

$$X\langle -1\rangle := \mathrm{Mor}_{\hat{\mathbf{Z}}^\times}(\hat{\mathbf{Z}}(1)^\times, X) \tag{8.1.1}$$

the set of functions $\hat{\mathbf{Z}}(1)^\times \to X$ that are equivariant for the $\hat{\mathbf{Z}}^\times$-actions. Of course, if we fix one element $\mu \in \hat{\mathbf{Z}}(1)^\times$, then an element of $X\langle -1\rangle$ is uniquely specified by its value on μ.

The Galois group Γ_k acts on $X\langle -1\rangle$ through its action on $\hat{\mathbf{Z}}(1)^\times$. Explicitly, for $f \in X\langle -1\rangle$, we have

$$f^\sigma(\alpha u) = f(u)$$

where $u \in \hat{\mathbf{Z}}(1)^\times$, and $\sigma \in G_k$ has image $\alpha \in \hat{\mathbf{Z}}^\times$ under the cyclotomic character.

If X is a finite group, $\hat{\mathbf{Z}}^\times$ has a natural action on the set of elements of X, defined as follows. Given $\alpha \in \hat{\mathbf{Z}}^\times$ and $x \in X$, we set $x^\alpha = x^k$, where k is the image of α under $\hat{\mathbf{Z}}^\times \mapsto (\mathbf{Z}/\chi\mathbf{Z})^\times$, where χ is the order of $x \in X$. We refer to this as "the power action" of $\hat{\mathbf{Z}}^\times$ on X; if X is abelian, it is an action by automorphisms of X.

When X is a group, $X\langle -1\rangle$ is identified with $\mathrm{Hom}_{\hat{\mathbf{Z}}}(\hat{\mathbf{Z}}(1), X)$, which is the "usual" definition of $X(-1)$: any equivariant function $\hat{\mathbf{Z}}(1)^\times \to X$ extends uniquely to an element of $\mathrm{Hom}_{\hat{\mathbf{Z}}}(\hat{\mathbf{Z}}(1), X)$. If X is an *abelian group* of order N, we have an identification of Γ_k-modules

$$X\langle -1\rangle = \mathrm{Hom}(\mu_N, X) \quad \text{(group homomorphisms).}$$

where the Galois action on the right-hand side is inherited from that on μ_N.

8.1.5. *Universal marked central extension and multidiscriminant.* Now let G be a finite group whose order is prime to the characteristic of k, together with a conjugacy-invariant subset $c \subset G$ which is *rational*, i.e. $c^\alpha = c$ for $\alpha \in \hat{\mathbf{Z}}^\times$.

In the prior section we have shown (Theorem 7.5.1) the existence of a universal marked central extension $(\tilde{G}, f, \tilde{c})$. Let $H_2'(G, c) = \ker(f)$. The trivial central extension $G \times \mathbf{Z}^{c/G} \to G$ admits a natural marking (obtained by lifting $g \in c$ to (g, e_g), where e_g is the coordinate vector corresponding to g). The universal property of $\tilde{G}$ induces a morphism

$$\tilde{G} \to G \times \mathbf{Z}^{c/G} \tag{8.1.2}$$

which restricts to a homomorphism

$$H_2'(G, c) \to \mathbf{Z}^{c/G}, \tag{8.1.3}$$

which we shall sometimes call "the multidiscriminant morphism."

The group $\hat{\mathbf{Z}}^\times$ acts on $\mathbf{Z}^{c/G}$ by virtue of its action on c/G; explicitly,

$$\alpha \in \hat{\mathbf{Z}}^\times : e_x \to e_{x^\alpha}, \tag{8.1.4}$$

where e_x is the coordinate vector corresponding to $x \in c$. We call this action the *permutation action* of $\hat{\mathbf{Z}}^\times$ on $\mathbf{Z}^{c/G}$. It will be important for us to lift this action to $\tilde{G}$: see §8.1.7.

There is a natural homomorphism

$$|\cdot| : \mathbf{Z}^{c/G} \to \mathbf{Z}$$

which sends each e_x to 1. Note that $|\alpha\mathfrak{m}| = |\mathfrak{m}|$ for any $\alpha \in \hat{\mathbf{Z}}^\times$ and any $\mathfrak{m} \in \mathbf{Z}^{c/G}$.

8.1.6. *The profinite $\hat{\mathbf{Z}}^\times$ action on the profinite completion of $\tilde{G}$.* Recall (discussion after (8.1.4)) that we wish to lift the $\hat{\mathbf{Z}}^\times$-action on $\mathbf{Z}^{c/G}$ to a $\hat{\mathbf{Z}}^\times$-action on $\tilde{G}$.

We now carry out a preliminary construction: we produce an action (the "profinite action") of $\hat{\mathbf{Z}}^\times$ on the profinite completion of $\tilde{G}$. In §8.1.7 we will extract from this the desired $\hat{\mathbf{Z}}^\times$-action on $\tilde{G}$ itself (sometimes called the "discrete action" when we need to clearly distinguish it from the first action).

The group $\hat{\mathbf{Z}}^\times$ acts on the functor[5] that sends a finite central extension (G', f') to its set of markings: given a marking c' we replace it by c'^α. (This is indeed a marking by virtue of the assumption that c is rational, i.e. setwise fixed under the action of $\hat{\mathbf{Z}}^\times$.) Explicitly speaking, if $x \mapsto \tilde{x}$ is the lifting of c associated to c', then the corresponding lifting for c'^α is given by

$$x \mapsto \widetilde{(x^{\alpha^{-1}})}^{\alpha}.$$

Now let $(\tilde{G}, f, \tilde{c})$ be a universal marked central extension. Then the profinite completion of $\tilde{G}$ – call it $\tilde{G}^\wedge$ – comes equipped with a map to G and a lifting of c; by abuse of notation, we continue to denote these as $(\tilde{G}^\wedge, f, \tilde{c})$. It is easy to see that this admits a unique morphism to any *finite* marked central extension of G.

In other terms – considered as a functor from the category of finite central extensions of G to the category of sets – the "set of markings" is represented by $(\tilde{G}^\wedge, f)$. Thus, from the action on markings, we get a group action of $\hat{\mathbf{Z}}^\times$ on $(\tilde{G}^\wedge, f)$ itself.

Using the description of $\tilde{G}$ from part (i) of Theorem 7.5.1, this action is described by the rule[6]

$$\alpha \in \hat{\mathbf{Z}}^\times : [g_1]^{k_1} \dots [g_r]^{k_r} \longrightarrow \left([g_1^{\alpha^{-1}}]^{k_1\alpha}[g_2^{\alpha^{-1}}]^{k_2\alpha} \dots\right) \tag{8.1.5}$$

(The exponents k_i can be taken to be ± 1 if desired; they are there simply so that the definition is not restricted to the semigroup generated by the $[g_i]$.) We refer to this action as the *profinite action* and denote it by

$$\alpha \cdot \aleph \quad (\alpha \in \hat{\mathbf{Z}}^\times, \aleph \in \tilde{G}^\wedge).$$

8.1.7. *The discrete action of $\hat{\mathbf{Z}}^\times$ on $\tilde{G}$ and $H_2'(G, c)$.* We now define the action of $\hat{\mathbf{Z}}^\times$ on $\tilde{G}$ itself.

We define a new action $*$ of $\hat{\mathbf{Z}}^\times$ on $\tilde{G}^\wedge$ via the rule

$$\alpha * X := (\alpha^{-1} \cdot X)^\alpha \quad (X \in \tilde{G}^\wedge)$$

The map from $\tilde{G}$ to its profinite completion $\tilde{G}^\wedge$ is injective, realizing $\tilde{G}$ as a subgroup of $\tilde{G}^\wedge$. We show below that the the discrete action $\hat{\mathbf{Z}}^\times$ in fact preserves this subgroup, so that we can promote $*$ to an action of $\hat{\mathbf{Z}}^\times$ on $\tilde{G}$ itself.

Consider the morphism $\tilde{G} \to G \times \mathbf{Z}^{c/G}$ of (8.1.2), and the induced morphism $\tilde{G}^\wedge \to G \times \hat{\mathbf{Z}}^{c/G}$ of profinite completions. This is equivariant for $\hat{\mathbf{Z}}^\times$, where $\hat{\mathbf{Z}}^\times$ acts on $\tilde{G}$ by means of $*$, it acts on G by the power action, and it acts on $\mathbf{Z}^{c/G}$ by the permutation action (8.1.4). In other terms the following commutes for any $\alpha \in \hat{\mathbf{Z}}^\times$:

[5] In other words, there is an embedding of $\hat{\mathbf{Z}}^\times$ into the group of natural automorphisms of this functor.

[6] To see directly that this defines a self-homomorphism, we need to verify the defining relation $[x][y][x]^{-1} = [xyx^{-1}]$ remains valid under the above rule. In other words, we need to check that

$$[x^{\alpha^{-1}}]^\alpha [y^{\alpha^{-1}}]^\alpha [x^{\alpha^{-1}}]^{-\alpha} = [x][y^{\alpha^{-1}}]^\alpha [x]^{-1}$$

But indeed $[x^{\alpha^{-1}}]^\alpha$ and $[x]$ differ by a *central* element, because they have the same image in G.

$$\begin{tikzcd} \tilde{G}^{\vee} \arrow[r] \arrow[d, "\alpha*"'] & G \times \hat{\mathbf{Z}}^{c/G} \arrow[d, "\alpha"'] \\ \tilde{G}^{\vee} \arrow[r] & G \times \hat{\mathbf{Z}}^{c/G} \end{tikzcd} \tag{8.1.6}$$

On the other hand, $\tilde{G}$ is the preimage, inside $\tilde{G}^{\wedge}$, of $G \times \mathbf{Z}^{c/G}$. Since $\hat{\mathbf{Z}}^{\times}$ (in the right-hand action, i.e., the power action and permutation action) preserves $G \times \mathbf{Z}^{c/G} \subset G \times \hat{\mathbf{Z}}^{c/G}$, it follows that the * action also preserves $\tilde{G}$ inside $\tilde{G}^{\wedge}$.

One can write down the discrete action rather explicitly, in terms of a certain cocycle $\hat{\mathbf{Z}}^{\times} \to H_2(G, c)$; we refer to §9 for this and some examples.

8.2. Tangential basepoints. For reasons explained in the introduction to this section, we will need to adopt the viewpoint of tangential basepoint; the next two subsections are devoted to this task. The material contained therein is in some sense familiar to experts, but lacks a standard reference, so we include it here in the interest of self-containment.

8.2.1. *Setup.* Let A be a ring drawn from one of two classes; either A is a complete dvr with residue field k, maximal ideal $\mathfrak{m}$, and fraction field K; or A is a field k, in which case we understand $\mathfrak{m} = 0$ and $K = k$.

In what follows we regard $\mathbf{P}^1$ as a scheme over $\operatorname{Spec}(A)$. We fix an open subscheme $U \subset \mathbf{A}^1/\operatorname{Spec} A$, whose complement in $\mathbf{P}^1$ is finite and etale over $\operatorname{Spec} A$.

For any point $p \in \mathbf{P}^1(A)$ we denote by $\mathcal{O}_p$ the completed local ring of $\mathbf{P}^1$ at $p(\mathfrak{s})$, where $\mathfrak{s}$ is the special point of $\operatorname{Spec}(A)$. The map $\operatorname{Spec}(A) \to \mathbf{P}^1$ factors through $\operatorname{Spec}(\mathcal{O}_p)$ and we denote by $\mathfrak{m}_p$ the prime ideal of $\mathcal{O}_p$ that is the image of the generic point of $\operatorname{Spec}(A)$. Finally, we denote by K_p the localization of $\mathcal{O}_p$ away from $\mathfrak{m}_p$, that is to say, obtained by inverting a generator for $\mathfrak{m}_p$.

Note that $\mathcal{O}_p$ is isomorphic to $A[[t]]$ in such a way that $\mathfrak{m}_p$ corresponds to the ideal generated by t; and K_p is isomorphic to $A((t))$. *In particular K_p is not a field in general,* although it is a field in the degenerate case where A is a field.

Thus we have a morphism $\operatorname{Spec}(K_p) \longrightarrow U$ which, after a choice of a geometric generic point η of $\operatorname{Spec}(K_p)$, induces a morphism[7]

$$\pi_1^{\mathrm{t}}(\operatorname{Spec}(K_p), \eta) \cong \hat{\mathbf{Z}}(1) \to \pi_1^{\mathrm{t}}(U, \eta)$$

Here the notation π_1^{t} denotes the *tame* fundamental group; more precisely, it refers to the group classifying étale covers of the scheme $\mathbf{P}^1$ over A that are tame along the divisor $\mathbf{P}^1 - U$.

If $y : \operatorname{Spec}(L) \to U$ is another geometric basepoint, the tame fundamental group $\pi_1^{\mathrm{t}}(U, y)$ can be identified with $\pi_1^{\mathrm{t}}(U, \eta)$; this identification is canonical only up to conjugation. So associated to η is a conjugacy class of homomorphisms from $\hat{\mathbf{Z}}(1)$ to $\pi_1^{\mathrm{t}}(U, y)$. The image of this homomorphism depends only on p, and we call it a *tame inertia group at p.* When $p \in U(A)$, or even when the restriction of p to $\mathbf{P}^1(K)$ lies in $U(K)$, the tame inertia group at p is trivial.

[7]Indeed, $\pi_1(\operatorname{Spec}(K_p), \eta)$ is isomorphic to the absolute Galois group of K_p, and the latter is isomorphic to $\hat{\mathbf{Z}}(1)$ via the morphism

$$g \in \mathrm{Gal} \mapsto \frac{g(\alpha)}{\alpha} \in \mu_n,$$

where α is an nth root of a uniformizer.

We will also sometimes use the notation π_1' to denote the maximal prime-to-p quotient of the étale fundamental group, which is a quotient of π_1^{t}; by a slight abuse of notation, the projection of a tame inertia group from the tame fundamental group to its prime-to-p quotient will also be called a tame inertia group.

8.2.2. *The tangential basepoint via Puiseaux series.* This discussion, of course, depends on the choice of a geometric generic point η. We now explain how to remove this ambiguity. Choose an element $\pi \in \mathcal{O}_p$ which generates the ideal $\mathfrak{m}_p$.

Define the ring of Puiseaux series $K_p[\pi^{1/\infty}]$ to be the "union" $\bigcup_{(N,p)=1} K_p[\pi^{1/N}]$; more precisely, as the direct limit of rings $E_N := K_p[t_N]/(t_N^N - \pi)$ under the transition maps $E_N \to E_{Nk}$ induced by $t_N \mapsto t_{Nk}^k$. This ring is equipped with a distinguished Nth root of π for every $(N,p) = 1$, namely t_N itself. Write η_π for the embedding of K_p into $K_p[\pi^{1/\infty}]$.

We have used the notation $K_p[\pi^{1/\infty}]$ to remind the reader that K_p need not be, in general, a field. However, in cases where K_p is a field (i.e., when $A = k$ is a field) we allow ourselves to use the notation $K_p(\pi^{1/\infty})$ for $K_p[\pi^{1/\infty}]$.

If $f : Y \to U$ is a tame cover of degree d, and $Y_p \to \operatorname{Spec}(K_p)$ its restriction, then the pullback $Y_p \times_{\operatorname{Spec} K_p} \operatorname{Spec} K_p[\pi^{1/\infty}]$ *splits into a set of* d *copies of* $\operatorname{Spec} K_p[\pi^{1/\infty}]$. This fact is fundamentally important for us. It follows from Abhyankar's lemma [SGA03, Corollaire 5.3, XIII] applied to $\mathcal{O}_p \cong A[[t]]$.

Denote this set of d copies by $\mathrm{fib}_\pi(f)$; now fib_π is a fiber functor from the category of tame covers to the category of finite sets, and the automorphism group of this functor is a group we call $\pi_1^{\mathrm{t}}(U, \eta_\pi)$. In other words, η_π can be thought of as a basepoint for the tame fundamental group of U; in fact it is a *tangential basepoint* in the sense of Deligne, as we discuss in the next subsection.

We have also a canonical morphism

$$\hat{\mathbf{Z}}(1) \to \pi_1^{\mathrm{t}}(U, \eta_\pi).$$

which arises from the action of $\hat{\mathbf{Z}}(1)$ as ring automorphisms of $K_p[\pi^{1/\infty}]$ over K_p.

The next step is to understand the dependence of this data on the choice of π. We claim there is a canonical isomorphism between $\pi_1(U, \eta_{\pi_1})$ and $\pi_1(U, \eta_{\pi_2})$ whenever $\pi_1 \equiv \pi_2$ modulo $\mathfrak{m}_p^2$. We will demonstrate this by constructing a distinguished isomorphism

$$f : K_p[\pi_1^{1/\infty}] \xrightarrow{\sim} K_p[{\pi_2}^{1/\infty}] \tag{8.2.1}$$

over K_p, which identifies the two fiber functors.

We construct f as follows. We have already remarked that π_1 has a distinguished Nth root t_N in $K_p[\pi_1^{1/\infty}]$. In order to specify the isomorphism f, we need only specify $f(t_N)$ for every N. By definition,

$$K_p({\pi_2}^{1/\infty}) = \lim_N K_p[u_N]/(u_N^N - \pi_2)$$

We define $f(t_N)$ to be the unique Nth root of π_1 in $K_p[{\pi_2}^{1/\infty}]$ that differs from u_N by an element of $\mathfrak{m}_p$-adic valuation strictly greater than $1/N$.

8.2.3. *Galois action on fundamental groups with tangential basepoint.* When the affine curve over k whose fundamental group we are studying arises by base change from a curve over k_0, the entire picture carries an action of Γ_k. To explain how this works, especially with respect to the tangential basepoint, requires setting up some of the algebraic mechanisms of section 8.2.2 over the smaller field k_0.

To this end, let U_0 be an open subscheme of $\mathbf{A}^1/k_0$. (In this section, all schemes are over k_0 unless otherwise indicated.) Let π be a uniformizer in the completed local ring of U_0 at p, and write $K_{0,p}$ for the localization of this completed local ring away from p. In this case $K_{0,p}$ is a field and we allow ourselves to use the notation $K_{0,p}(\pi^{1/\infty})$ instead of $K_{0,p}[\pi^{1/\infty}]$.

The inclusion of $K_{0,p}$ into $K_p(\pi^{1/\infty})$ yields a map

$$\eta_\pi : \operatorname{Spec} K_{0,p}(\pi^{1/\infty}) \to U_0$$

As above, we can define a fiber functor, which we again denote fib_π, on the category of tame covers of U_0 as follows. If $f : Y \to U_0$ is a tame cover, we let Y_p be the restriction of Y to $\operatorname{Spec} K_{0,p}$, and then define $\mathrm{fib}_\pi(f)$ to be the set of connected components of $Y_p \times_{K_{0,p}} K_p(\pi^{1/\infty})$. We denote the automorphism group of this fiber functor by $\pi_1(U_0, \eta_\pi)$.

Now the family of covers $U \times_{k_0} k'$, as k' ranges over separable extensions of k_0, yields a surjective homomorphism from $\pi_1(U_0, \eta_\pi)$ to $\pi_1(\operatorname{Spec} k_0) = \Gamma_k$, whose kernel is the group $\pi_1(U, \eta_\pi)$ defined in the previous section. Moreover, there is a natural map

$$\mathrm{Gal}(K_p(\pi^{1/\infty})/K_{0,p}) \to \pi_1(U_0, \eta_\pi)$$

coming from the action of $\mathrm{Gal}(K_p(\pi^{1/\infty})/K_{0,p})$ on $Y_p \times_{K_{0,p}} K_p(\pi^{1/\infty})$.

On the other hand, we can also define a field $K_{0,p}(\pi^{1/\infty})$ by adjoining formal roots of π to $K_{0,p}$ as above. Then we have an exact sequence of Galois groups

$$1 \to \mathrm{Gal}(K_p(\pi^{1/\infty})/K_p) \to \mathrm{Gal}(K_p(\pi^{1/\infty})/K_{0,p}) \to \mathrm{Gal}(K_p/K_{0,p}) \to 1$$

where the latter surjection has a section given by the identification of $\mathrm{Gal}(K_p/K_{0,p})$ with the Galois group of $K_p(\pi^{1/\infty})$ over $K_{0,p}(\pi^{1/\infty})$. Both of these groups are also naturally identified with Γ_k.

In particular, the tangential basepoint π gives rise to a section $\Gamma_k \to \pi_1(U_0, \eta_\pi)$, just as would an ordinary basepoint defined over k_0. This section endows $\pi_1(U, \eta_\pi)$ with an action (not merely an outer action) of Γ_k, which preserves the tame inertia group at p.

8.2.4. *Relation with Deligne's tangential basepoint.* We have seen above that, by means of a choice of a germ of a function near a puncture of U, we can construct a fundamental group for U which is "based" at that germ and which contains a decomposition group that is canonical (i.e. not only defined up to conjugacy.) We now explain the relation between this fundamental group and the tangentially based fundamental group defined by Deligne.

In this section, we work over the field k, but we note that Deligne asserts in [Del89, §15.19] that a similar definition has the desired properties over a more general base scheme.

Let $y = y_\infty$ be a nonzero tangent vector at ∞. We are going to associate to y a fiber functor

$$\mathrm{fib}_y : \text{ étale coverings of } U \longrightarrow \text{ finite set.}$$

This fiber functor is equipped with a canonical action of $\hat{\mathbf{Z}}$ ("monodromy around ∞") and gives us then the notion of "étale fundamental group based at v," denoted $\pi_1(U, y_\infty)$ (namely: automorphisms of this fiber functor). Deligne [Del89, §15] gives an explicit description of this fiber functor when working over a field k of characteristic 0, as follows:

An étale cover $Y \to U$ induces by normalization a ramified cover $Y' \to \mathbf{P}^1$. Denote by S the preimage of ∞ in $Y'(k)$, and by T_S the disjoint union of all tangent spaces to Y' at points of S; it is therefore a disjoint union of affine lines. By taking "first nonvanishing Taylor coefficient" we obtain a map

$$T_S \to T_\infty(\mathbf{P}^1)$$

that is ramified (possibly) at the origin of each affine component of T_S.

The fiber functor in question, then, associates to the étale cover $Y \to U$ the preimage of $y_\infty \in T_\infty(\mathbf{P}^1)$. Clearly this functor admits an action of the fundamental group of $T_\infty(\mathbf{P}^1)$ with the origin removed, that is to say the fundamental group of $\mathbb{G}_m$, which is isomorphic to $\hat{\mathbf{Z}}(1)$. In other words, there is a canonical morphism $\hat{\mathbf{Z}}(1) \to \pi_1(U, y_\infty)$.

It is not hard to verify that, in this case, the fundamental group $\pi_1(U, y_\infty)$ can be identified with the fundamental group defined in the previous section, taking $A = k$ and π to be a uniformizer in $\mathcal{O}_p$ whose differential $d\pi$ pairs to 1 with y_∞ under the natural pairing of cotangent vectors with tangent vectors.

From now on, we will use the notation y_∞, rather than η_π, to refer to a tangential basepoint based at ∞, since the choice of π can be made once and for all and doesn't intervene in the arguments, and we wish instead to emphasize the connections with the topological situation (where y_∞ is to be thought of as a physical point on the boundary of a disc.)

8.3. Branched G-covers and their invariants. Our preliminaries on tangential basepoints now complete, we are ready to give the algebraic notion of a branched cover, i.e. the algebraic analog of the notion discussed in §5.2. *In all that follows, we suppose fixed a tangential basepoint y_∞ for $\mathbf{P}^1$ at ∞ over* Spec($\mathbf{Z}$)*, i.e., a nowhere vanishing section of the restriction of the tangent bundle of $\mathbf{P}^1$ to the ∞-section of* Spec($\mathbf{Z}$)*. In particular, this y_∞ gives rise also to a tangential vector "over k_0," in the sense that it arises from a uniformizer in $K_{0,\infty}$ as in §8.2.3.*

8.3.1. *Definition of a branched G-cover.* We recall that G is a finite group whose order is invertible in k.

By a *branched G-cover of* $\mathbf{P}^1$ we mean:

- A proper smooth geometrically irreducible curve X over k;
- A morphism $h : X \to \mathbf{P}^1$;
- An embedding $G \hookrightarrow \mathrm{Aut}(h)$ such that G acts simply transitively on each geometric fiber;
- An element $\star$ of $\mathrm{fib}_{y_\infty}(h)$.

This notion is analogous to the notion discussed in §5.2. Such a G-cover f defines a surjection $\pi_1(U, y_\infty) \to G$, as follows: $\pi_1(U, y_\infty)$ acts on $\mathrm{fib}_{y_\infty}(h)$, and to each element $\alpha \in \pi_1(U, y_\infty)$ we associate the unique $g \in G$ such that $\alpha \cdot \star = g \cdot \star$.

We say a G-cover of $\mathbf{P}^1$ is n-branched if it is ramified at exactly n points of $\mathbf{A}^1(k)$ (it may or may not be ramified at ∞, so the morphism $X \to \mathbf{P}^1$ may have either n or $n+1$ branch points.)

8.3.2. *(G, c)-cover. Multidiscriminant $\mathfrak{m}$ and boundary monodromy δ.* Suppose given a branched G-cover h of $\mathbf{P}^1$ as above.

For any branch point $p \in \mathbf{A}^1(k)$, we obtain a conjugacy class of morphisms from $\hat{\mathbf{Z}}(1)$ to G; we denote by

$$\iota_p : \hat{\mathbf{Z}}(1) \to G \tag{8.3.1}$$

any morphism in this class. Similarly, restriction via the map $\hat{\mathbf{Z}}(1) \to \pi_1(U, y_\infty)$ attached to tame inertia at ∞ yields a homomorphism

$$\iota_\infty : \hat{\mathbf{Z}}(1) \to G.$$

Unlike the other morphisms ι_p, the morphism ι_∞ is well-defined, not only defined up to conjugacy.

We say that h is of monodromy type c if $\iota_P(\mu)$ lies in c for all branch points p in $\mathbf{A}^1$ and all $\mu \in \hat{\mathbf{Z}}(1)^\times$. In that case, we will often refer to h as a *branched (G, c)-cover.*

We can now attach a *multidiscriminant*

$$\mathfrak{m} \in \mathbf{Z}^{c/G}\langle -1 \rangle$$

to every G-cover of monodromy type c. The definition is as follows: for each $\mu \in \hat{\mathbf{Z}}(1)^\times$, we define

$$\mathfrak{m}(\mu) = \sum_{p \in \mathbf{A}^1(k), \text{branch point}} e_{\iota_p(\mu)} \in \mathbf{Z}^{c/G} \tag{8.3.2}$$

This function is easily verified to be $\hat{\mathbf{Z}}^\times$-equivariant, thus defining an element of $\mathbf{Z}^{c/G}\langle -1 \rangle$. Note that the sum of coordinates of $\mathfrak{m}(\mu)$ is independent of the choice of $\mu \in \hat{\mathbf{Z}}(1)^\times$; it is simply the number of points in $\mathbf{A}^1$ which are branched in X. This sum is the quantity we have already denoted $|\mathfrak{m}|$.

We have already mentioned that $G\langle -1 \rangle$ is naturally identified with $\mathrm{Hom}(\hat{\mathbf{Z}}(1), G)$; the *boundary monodromy* of h is the element

$$\delta \in G\langle -1 \rangle$$

corresponding to ι_∞ under this identification.

8.3.3. *Galois actions on multidiscriminant and boundary monodromy.* If $f : X \to U$ is a branched G-cover, and $\sigma \in \Gamma_k$, we can construct a Galois conjugate $f^\sigma : X^\sigma \to U^\sigma$, where X^σ is the Cartesian product

$$X^\sigma := X \times_{(k,\sigma)} k, U^\sigma = U \times_{(k,\sigma)} k.$$

In this section we explain how the multidiscriminant and boundary monodromy of f^σ relate to those of f. Again, this material is not original and is included for lack of a precisely applicable reference.

Let $p \in \mathbf{P}^1(k)$ be a branch point of k, let ϕ be a uniformizer in the completed local ring of $\mathbf{P}^1$ at p, let $q \in X(k)$ be a point in the fiber $f^{-1}(p)$, and let N be the ramification degree of f at q. Then the completed local ring of X at q contains Nth roots of ϕ; pick one such and call it $\phi^{1/N}$. Then, for each $\mu \in \mu_N(k)$, we can define $\iota_p(\mu)$ to be the unique element g of G such that $g(\phi^{1/N}) = \mu\phi^{1/N}$. Of course, a different choice of q will modify this element by conjugacy in G.

The cover f^σ is branched at $p^\sigma \in \mathbf{P}^1(k)$; write $\iota^\sigma_{p^\sigma} : \hat{\mathbf{Z}}(1) \to G$ for the corresponding inertia homomorphism. The naturality of the definition above tells us that

$$\iota^\sigma_{p^\sigma}(\mu^\sigma) = \iota_p(\mu)$$

both sides being defined only up to G-conjugacy. This, in turn, implies that the multidiscriminant $\mathfrak{m}(f^\sigma)$ of f^σ is just $\mathfrak{m}(f)^\sigma$.

(Recall that the Γ_k-action on $\mathbf{Z}^{c/G}\langle -1\rangle = \mathrm{Mor}_{\hat{\mathbf{Z}}}(\hat{\mathbf{Z}}(1), \mathbf{Z}^{c/G})$ is specified by the rule

$$f^\sigma(\mu^\sigma) = f(\mu)$$

for $f \in \mathbf{Z}^{c/G}\langle -1\rangle, \mu \in \hat{\mathbf{Z}}(1)$.)

The same argument applies to the boundary monodromy, taking $p = \infty$. In this case, our uniformizer π has been chosen to be fixed by Γ_k, and moreover the choice of a point $\star$ in $f^{-1}(\infty)$ has already been made as part of the definition. Thus the identity

$$\iota^\sigma_\infty(\mu^\sigma) = \iota_\infty(\mu)$$

holds without any ambiguity concerning conjugacy, and shows us that

$$\delta(f^\sigma) = \delta(f)^\sigma$$

in $G\langle -1\rangle$.

8.3.4. *The goal.* Taken together, we have defined in §8.3.2 an invariant $\delta \times \mathfrak{m}$ of a branched G-cover of $\mathbf{P}^1$, which takes values in

$$\left(G \times \mathbf{Z}^{c/G}\right)\langle -1\rangle.$$

It will be more convenient in our setup to deal with $\delta^{-1} \times \mathfrak{m}$ rather than $\delta \times \mathfrak{m}$ (i.e, to invert the boundary monodromy). Our goal in this portion of the paper is to *refine* $\delta^{-1} \times \mathfrak{m}$ to an invariant valued in

$$\mathfrak{z} \in \tilde{G}\langle -1\rangle, \tag{8.3.3}$$

whose projection to $G \times \mathbf{Z}^{c/G}$ under the map (8.1.2) is $\delta^{-1} \times \mathfrak{m}$. Following Fried and Serre, we call $\mathfrak{z}$ the *lifting invariant* of the cover. (Note that to make sense of $\tilde{G}\langle -1\rangle$ one needs an action of $\hat{\mathbf{Z}}^\times$ on $\tilde{G}$, which was specified in §8.1.7).

We have already shown that the invariant $\delta^{-1} \times \mathfrak{m}$ is equivariant for the specified Galois actions on both sides. In § 8.5.3, we show that the lifting invariant $\mathfrak{z}$ is also Galois-equivariant.

The geometric story then develops as follows. As we shall see below, G-covers are parametrized by a moduli scheme called the *Hurwitz scheme*, and we shall show that, when $\mathfrak{m}$ is "large enough" in a suitable sense, two geometric points of the Hurwitz scheme lie in the same geometrically connected component if and only if the corresponding G-covers have the same lifting invariant $\mathfrak{z}$. In other words, $\mathfrak{z}$ is the most informative discrete invariant of a G-cover. The Galois equivariance of $\mathfrak{z}$ is what allows us to determine the Galois action on the set of connected components of the Hurwitz scheme.

8.4. The lifting invariant: preliminaries on generators for π_1. Let U be an open subset of $\mathbf{A}^1$ and let y_∞ be the fixed tangential basepoint at ∞ (see §8.3). Denote by $\pi_1^{\mathrm{t}}(U, y_\infty)$ the tame geometric fundamental group of U based at y_∞(i.e. the group classifying covers of $\mathbf{P}^1$ that are tame along $\mathbf{P}^1 - U$) and by $\pi_1'(U, y_\infty)$ its maximal prime-to-p quotient. As we have discussed in §8.2.4, $\pi_1'(U, y_\infty)$ is endowed with a *canonical* inertia subgroup at ∞. Write $P_1, \ldots, P_r$ for the geometric points of $\mathbf{A}^1 - U$.

Choose elements $\gamma_j \in \pi_1'(U, y_\infty)$ with the property that

$$\gamma_1 \ldots \gamma_n \gamma_\infty = 1 \tag{8.4.1}$$

and $\gamma_1, \ldots, \gamma_n$ (respectively γ_∞) topologically generate inertia groups at $P_1, \ldots, P_n$ (respectively, the canonical inertia group at ∞). Such γ_i *exist* by Grothendieck's comparison of étale and topological π_1; see [SGA03, Corollaire 2.12, Exposé XIII]. That corollary doesn't directly address fundamental groups with a tangential basepoint, but this poses no problem; given a non-tangential basepoint y, we can write down a set of generators γ_i which generate inertia groups at the specified points, then apply an arbitrarily chosen identification of $\pi_1(U, y)$ with $\pi_1(U, y_\infty)$. In the resulting set of generators, γ_∞ generates an inertia group at ∞, but maybe not the canonical one; this can be fixed by conjugating all the γ_j by a suitable element of $\pi_1(U, y_\infty)$.

The procyclic subgroup $\langle \gamma_i \rangle$ in $\pi_1'(U, y_\infty)$ is an inertia group at P_i and, as such, there is a canonical identification (see (8.3.1))

$$r_i : \langle \gamma_i \rangle \longrightarrow \hat{\mathbf{Z}}(1).$$

Moreover, the image of $\underline{\gamma} = \gamma_1 \ldots, \gamma_n, \gamma_\infty$ under $\bigoplus r_i$ in $\bigoplus \hat{\mathbf{Z}}(1)^{n+1}$ lies in the diagonal $\hat{\mathbf{Z}}(1)$. This follows from the exact sequence in étale homology

$$\hat{\mathbf{Z}}(1) \cong H_2(\mathbf{P}^1, \hat{\mathbf{Z}}) \longrightarrow \hat{\mathbf{Z}}(1)^{n+1} \longrightarrow H_1(U, \hat{\mathbf{Z}}).$$

Let $\mathrm{I}(\underline{\gamma}) \in \hat{\mathbf{Z}}(1)$ be the corresponding element of $\hat{\mathbf{Z}}(1)$, that is to say, the common image of the γ_i.

Now take another choice $(\gamma_1', \ldots, \gamma_n', \gamma_\infty')$ satisfying (8.4.1) with corresponding element $\mathrm{I}(\underline{\gamma}') \in \hat{\mathbf{Z}}(1)$. Choose $\alpha \in \hat{\mathbf{Z}}^\times$ so that

$$\mathrm{I}(\underline{\gamma}') = \mathrm{I}(\underline{\gamma}) \cdot \alpha.$$

Such α *exists* because both $\mathrm{I}(\underline{\gamma})$ and $\mathrm{I}(\underline{\gamma}')$ are generators for $\hat{\mathbf{Z}}(1)$. In this case, we claim that γ_r' is conjugate to γ_r^α, for every r.

Indeed, the subgroups $\langle \gamma_r' \rangle$ and $\langle \gamma_r \rangle$ are both inertia subgroups at P_r and thus conjugate inside $\pi_1(U, P_r)$; moreover, this conjugacy preserves the identification of both subgroups with $\hat{\mathbf{Z}}(1)$.

8.5. The lifting invariant: definition. We shall now attach to any branched (G, c)-cover an invariant $\mathfrak{z} \in \tilde{G}\langle -1 \rangle$ whose image in $(G \times \mathbf{Z}^{c/G})\langle -1 \rangle$ under the map (8.1.2) is simply the *inverse of the* boundary monodromy together with the multidiscriminant of the covering (cf. §8.3.4). The inversion of the boundary monodromy is merely an annoyance of our setup; it has no deeper significance.

8.5.1. *Remark.* A similar definition can be made in the number field case: see [VE10].

Let $(\tilde{G}, f, \tilde{c})$ be the prime-to-p profinite completion of a universal c-marked central extension; for $g \in c$ let $[g] \in \tilde{G}$ be the preferred lift of g to $\tilde{G}$. Then the lifting invariant is characterized by the following Proposition:

8.5.2. Proposition. *Let*

$$\varphi : \pi_1(U, y_\infty) \longrightarrow G$$

be the homomorphism associated to a branched (G, c)-cover.

There is a unique element $\mathfrak{z} \in \tilde{G}\langle -1 \rangle$ with the following property:

For any choice of ordering of the branch points $P_1, \ldots, P_n$ and any choice of $\underline{\gamma} = \gamma_1, \ldots, \gamma_n$ (and implicitly γ_∞) as in the previous section, $\mathfrak{z}$ sends $I(\underline{\gamma})$, the common image of the γ_i in $\hat{\mathbf{Z}}(1)$, to

$$Z(\underline{\gamma}) := [\varphi(\gamma_1)] \ldots [\varphi(\gamma_n)] \in \tilde{G}.$$

Note that, for any choice $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$, there is a unique morphism $\mathfrak{z}_{\underline{\gamma}} \in \tilde{G}\langle -1 \rangle$ that sends $I(\underline{\gamma}) \in \hat{\mathbf{Z}}(1)^\times$ to $Z(\underline{\gamma}) \in \tilde{G}$ (see remark after (9.5.1)). The content of Proposition 8.5.2, then, is that this $\mathfrak{z}_{\underline{\gamma}}$ is actually independent of $\underline{\gamma}$.

Assuming the proposition for the moment, let us verify that such an invariant $\mathfrak{z}$ indeed projects, under

$$\tilde{G}\langle -1 \rangle \to (G \times \mathbf{Z}^{c/G})\langle -1 \rangle = G\langle -1 \rangle \times \mathbf{Z}^{c/G}\langle -1 \rangle$$

to the inverse of the boundary monodromy (in the first factor) and the multidiscriminant (in the second factor):

(i) The projection of $\mathfrak{z}$ to $G\langle -1 \rangle$ sends $I(\underline{\gamma}) \in \hat{\mathbf{Z}}(1)^\times$ to

$$\varphi(\gamma_1) \ldots \varphi(\gamma_n) = \varphi(\gamma_\infty)^{-1} \in G;$$

this mapping is, by definition, the boundary monodromy inverted.

(ii) The projection of $\mathfrak{z}$ to $\mathbf{Z}^{c/G}\langle -1 \rangle$ sends $I(\underline{\gamma}) \in \hat{\mathbf{Z}}(1)^\times$ to

$$\sum_{i=1}^{n} e_{\varphi(\gamma_i)} \in \mathbf{Z}^{c/G},$$

which is precisely the multidiscriminant (see §8.3.2).

Proof. (of Proposition 8.5.2) Since π_1' is free as a prime-to-p profinite group, φ lifts to a homomorphism $\tilde{\varphi} : \pi_1'(U, y_\infty) \longrightarrow \tilde{G}^\wedge$, where $\tilde{G}^\wedge$ is the profinite completion of $\tilde{G}$. That lift is unique up to multiplication by a homomorphism $\chi : \pi_1'(U, y_\infty) \longrightarrow H_2'(G, c)^\wedge$, because of the exact sequence

$$H_2'(G, c)^\wedge \to \tilde{G}^\wedge \to G.$$

Thus we may compare $\tilde{\varphi}$ and the specified lift of the conjugacy class c: there exists $z_i \in H_2'(G, c)^\wedge$ so that

$$\tilde{\varphi}(\gamma_i) \cdot z_i = [\varphi(\gamma_i)] \tag{8.5.1}$$

With this convention, we have:

$$Z(\underline{\gamma}) = \tilde{\varphi}(\gamma_1) z_1 \cdot \tilde{\varphi}(\gamma_2) z_2 \cdot \cdots = \tilde{\varphi}(\gamma_\infty^{-1}) z_1 z_2 \ldots z_n, \tag{8.5.2}$$

since $\tilde{\varphi}(\gamma_1)\tilde{\varphi}(\gamma_2) \ldots \tilde{\varphi}(\gamma_n)\tilde{\varphi}(\gamma_\infty) = 1$ and the z_i are central.

Observe that, had we replaced γ_i by a conjugate, the quantity z_i (as defined by the above equation) would be unchanged. We must now show that $\mathfrak{z}$ is independent of our choice of $\underline{\gamma}$. To this end, we choose a new $\underline{\gamma}'$ subject to the same constraints that governed $\underline{\gamma}$; that is,

$$\gamma_1' \dots \gamma_n' \gamma_\infty' = 1.$$

and there exists a permutation $\sigma \in S_n$ such that γ_i' generates a tame inertia group at $P_{\sigma(i)}$. Moreover, γ_∞' is constrained to generate the canonical inertia group at ∞, so that $\gamma_\infty' = \gamma_\infty^\alpha$, for some $\alpha \in \hat{\mathbf{Z}}^\times$. The common image of γ_i' in $\hat{\mathbf{Z}}(1)$ is then given by $I(\underline{\gamma}') = I(\underline{\gamma})\alpha$.

We define $z_i' \in H_2'(G, c)$ by the formula

$$\tilde{\varphi}(\gamma_i') = [\varphi(\gamma_i')] z_i'.$$

Since $\gamma_{\sigma^{-1}(i)}'$ is conjugate to γ_i^α, we have

$$\tilde{\varphi}(\gamma_i^\alpha) = [\varphi(\gamma_i)^\alpha](z_{\sigma^{-1}(i)}')$$

whence

$$\tilde{\varphi}(\gamma_i) = [\varphi(\gamma_i)^\alpha]^{\alpha^{-1}} z_{\sigma^{-1}}'(i)^{\alpha^{-1}}.$$

We compute

$$Z(\underline{\gamma}') = \prod [\varphi(\gamma_i')] = \prod \tilde{\varphi}(\gamma_i') \prod (z_i')^{-1} = \tilde{\varphi}((\gamma_\infty^{-1})^\alpha) \prod (z_i')^{-1}.$$

On the other hand, the image of $Z(\underline{\gamma})$ under the profinite action of α^{-1} is, by (8.1.5)

$$\alpha^{-1} \cdot Z(\underline{\gamma}) = \prod [\varphi(\gamma_i)^\alpha]^{\alpha^{-1}} = \prod \tilde{\varphi}(\gamma_i) \prod ({z_{\sigma^{-1}(i)}'}^{\alpha^{-1}})^{-1} = \tilde{\varphi}(\gamma_\infty^{-1}) \left(\prod (z_i')^{-1}\right)^{\alpha^{-1}}$$

with the last equality obtaining thanks to the fact that $H_2'(G, c)$ is central in $\tilde{G}$.

It is now immediate that

$$Z(\underline{\gamma}')^{\alpha^{-1}} = \alpha^{-1} \cdot Z(\underline{\gamma})$$

which is to say

$$Z(\underline{\gamma}') = \alpha * Z(\underline{\gamma})$$

where $*$ denotes the discrete action of $\hat{\mathbf{Z}}^*$; this was the statement to be proved. $\square$

8.5.3. *Galois action.* Our lifting invariant is compatible with the action of the Galois group, i.e. the mapping

$$\text{branched } (G, c)\text{-covers of } \mathbf{P}^1 \longrightarrow \tilde{G}\langle -1 \rangle$$

is compatible with the Γ_k-actions on both sides; the Γ_k-action on the left-side has been discussed in §8.3.3, whereas the Γ_k-action on the right-hand side was just as discussed after (9.5.1).

Proof. There is a map

$$\sigma^{-1} : \pi_1(U^\sigma, y_\infty^\sigma) \to \pi_1(U, y_\infty)$$

arising from the map of schemes $U^\sigma \to U$ as in §8.3.3. If $X \to U$ is a (G, c)-cover corresponding to a map $\phi : \pi_1(U, y_\infty) \to G$, then $\phi \circ \sigma^{-1} : \pi_1(U^\sigma, y_\infty^\sigma) \to G$ corresponds to the Galois conjugate cover $X^\sigma \to U^\sigma$. Moreover, one can choose the homomorphism

$$\widetilde{\phi \circ \sigma^{-1}} : \pi_1(U^\sigma, y_\infty^\sigma)) \to \tilde{G}^\wedge$$

to be $\tilde{\phi} \circ \sigma^{-1}$. Having done so, and recalling that

$$I(\sigma^{-1}(\underline{\gamma})) = \chi(\sigma)^{-1} I(\underline{\gamma})$$

one argues just as in the proof of Propsition 8.5.2 to see that the lifting invariant of $\varphi \circ \sigma^{-1}$ is obtained from that of φ by allowing $\chi(\sigma)$ to operate on $\tilde{G}\langle -1 \rangle := \mathrm{Hom}(\hat{\mathbf{Z}}(1)^{\times}, \tilde{G})$ through its action on $\hat{\mathbf{Z}}(1)^{\times}$. This is precisely the definition of the Galois action on $\tilde{G}\langle -1 \rangle$.

□

8.6. Connected components of Hurwitz schemes: discussion and examples. We have now defined a lifting invariant $\mathfrak{z}$, which Galois-equivariantly assigns to each branched (G,c)-cover of $\mathbf{P}^1/k$ an element of $\tilde{G}\langle -1 \rangle$. In this section, we work out the relationship between this lifting invariant and the connected components of a moduli space of branched G-covers, the *Hurwitz scheme*. To do this, we will need to expand our attention from G-covers over a k to the relative case of G-covers over a more general base scheme. Our primary source for the definition and properties of the Hurwitz scheme is the paper of Romagny and Wewers [RW06]

8.6.1. *The configuration space of points: algebraic version.* Let $A = \mathrm{Spec}\, \mathbf{Z}[\frac{1}{|G|}]$, and let Conf_n be the configuration space of unordered n-tuples in $\mathbf{A}^1$, considered as a scheme over $\mathrm{Spec}\, A$. Explicitly, Conf_n is obtained from $\mathbf{A}^n$ by deleting the discriminant locus, i.e., the function $f : (a_1, \ldots, a_n) \mapsto \mathrm{disc}(x^n + \sum_{i=1}^n a_i x^{n-i})$.

There is a universal family $U \to \mathsf{Conf}_n$ whose fiber above a configuration is the corresponding *punctured* plane; in terms of the explicit identification of Conf_n with $\mathbf{A}^n$, U is described as

$$(a_1, \ldots, a_n; x) \in \mathbf{A}^n \times \mathbf{A}^1 : x^n + \sum_1^{n-1} a_i x^{n-i} \neq 0.$$

8.6.2. *An algebraic version of* $\mathrm{Hur}_{G,n}$. In this section, we continue to work over the base ring $\mathbf{Z}[\frac{1}{|G|}]$. We also assume that G is center-free, but we anticipate this is unnecessary, see Remark 8.6.4. We explain how to construct a scheme $\mathsf{CHur}_{G,n}$, with the following property:

For any geometric point $\mathrm{Spec}(k) \to \mathsf{CHur}_{G,n}$, there is a natural bijection

$$\mathsf{CHur}_{G,n}(k) \longrightarrow \text{branched } G\text{-covers of } \mathbf{P}^1 \text{ with } n \text{ branch points in } \mathbf{A}^1,$$

and moreover $\mathsf{CHur}_{G,n}(\mathbf{C})$ is homotopy equivalent to the topological space $\mathrm{Hur}_{G,n}$ studied in the first part of this paper.

It is known (for reference, see [RW06]) that there is a scheme $\mathsf{chur}_{G,n}$, equipped with an étale map

$$\pi : \mathsf{chur}_{G,n} \to \mathsf{Conf}_n$$

which is a moduli space for G-covers up to isomorphism, in the following sense:

Write $\tilde{U}$ for the preimage, in $\mathsf{chur}_{G,n} \times \mathbf{A}^1$, of $U \subset \mathsf{Conf}_n \times \mathbf{A}^1$. Let E be the complement of $\tilde{U}$ in $\mathsf{chur}_{G,n} \times \mathbf{A}^1$. We regard $\tilde{U}$ and E as open and closed subsets of $\mathsf{chur}_{G,n} \times \mathbf{P}^1 = \mathbf{P}^1_{\mathsf{chur}_{G,n}}$.

Then there is a relative curve $X \to \mathsf{chur}_{G,n}$ that admits a map to $\mathbf{P}^1_{\mathsf{chur}_{G,n}}$ with the following properties (see [RW06, §2.1, §2.4]):

- It is etale over $\tilde{U} \subset \mathbf{P}^1_{\mathsf{chur}_{G,n}}$;

- Let E_X be its preimage of E in X. Then E_X is finite étale over $\mathrm{Hur}_{G,n}$ and the ramification of X at each geometric point of this locus is of order > 1 and prime to the residue characteristic.
- Over any geometric point $\mathrm{Spec}(k) \to \mathsf{chur}_{G,n}$, the induced map

$$\mathsf{chur}_{G,n}(k) \longrightarrow \{ \text{ } n\text{-branched } G\text{-covers of } \mathbf{P}^1\}/G$$

is a bijection. Here the action of G on the right-hand side is by permuting the specified element of fib_{y_∞}.

8.6.3. *Remark.* When deducing this theorem from the stated theorem of [RW06] one needs to recall the fact that an "n-branched G covers of $\mathbf{P}^1$" is, by definition, a cover branched at n points in $\mathbf{A}^1$; it may or may not be branched at ∞.

8.6.4. *Remark.* For this section, we imposed again the restriction that G be center-free. The restriction to center-free G is simply a technical one, for in the general case $\mathsf{chur}_{G,n}$ is only a Deligne-Mumford stack, not a scheme. It should be the case that the moduli space $\mathsf{CHur}_{G,n}$ defined below is a scheme in general, but since this is ancillary to the main point of this paper we have not carried out such an argument here.

We recall that the space $\mathsf{chur}_{G,n}$ is analogous not to our topological space $\mathrm{Hur}_{G,n}$ but rather to its quotient by G; in other words, it parametrizes objects that are like branched G-covers of $\mathbf{P}^1$ in the sense used here, but without the specified point in the fiber over the tangential basepoint.

We obtain the desired algebraic model $\mathrm{CHur}_{G,n}$ for $\mathrm{CHur}_{G,n}$ as follows. The moduli property of $\mathsf{chur}_{G,n}$ provides us with a branched G-cover $X \to \mathbf{P}^1 \times \mathsf{chur}_{G,n}$. Write V for the formal neighborhood of $\infty \times \mathsf{chur}_{G,n}$. If π is a uniformizer of the completed local ring of ∞ in $\mathbf{P}^1/\operatorname{Spec}\mathbf{Z}$ corresponding to our fixed tangent vector y_∞ (see §8.3.1), we can define as in section §8.2.3 a cover $V_N$of V tamely ramified at $\infty \times \mathsf{chur}_{G,n}$ by adjoining Nth roots of π, and by choosing N sufficiently divisible we can arrange for $X \times_V V_N \to V_N$ to be an etale cover; this is a consequence of Abhyankar's lemma. In particular, the fiber of $X \times_V V_N$ over $\infty \times \mathsf{chur}_{G,n}$ is an étale cover of $\mathsf{chur}_{G,n}$, which we call $\mathsf{CHur}_{G,n}$, and which parametrizes branched G-covers of $\mathbf{P}^1$ (in the sense specified at the start of this section).

8.7. Components of the Hurwitz scheme. Let A be a strictly henselian dvr with residue field k, and quotient field K; let $\bar{K}$ be an algebraic closure of K.

For any morphism $f : \operatorname{Spec} A \to \mathsf{CHur}_{G,n}$, let P be the corresponding point of $\mathsf{CHur}_{G,n}(k)$ and Q the corresponding point of $\mathsf{CHur}_{G,n}(\bar{K})$. We say in this situation that "Q lifts P."

For fixed such A, there is a unique bijection

$$\text{components of } \mathsf{CHur}_{G,n} \text{ over } k \xrightarrow{\sim} \text{components of } \mathsf{CHur}_{G,n} \text{ over } \bar{K} \tag{8.7.1}$$

characterized by the property that if Q lifts P, then Q and P belong to components that correspond under (8.7.1). Compare [RW06, 4.15]. This is a consequence of tame specialization: Conf_n has a smooth proper compactification in which the complement of Conf_n is a relative divisor with normal crossings, and the cover by $\mathsf{CHur}_{G,n}$ is tamely ramified over the boundary (see [EVW09, §7.2]).

8.7.1. Proposition. *Suppose $Q \in \mathsf{CHur}_{G,n}(\bar{K})$ lifts $P \in \mathsf{CHur}_{G,n}(k)$. Let X_Q, X_P be the corresponding branched G-covers. Then X_Q has all monodromy of type c if and only if the same is true for X_P. In that case, $\mathfrak{z}(X_Q) = \mathfrak{z}(X_P)$.*

A word on interpretation is in order. Suppose, for the sake of the discussion, that the characteristic of K is zero, the equal characteristic case being easier. Now we understand the lifting invariant of $\mathfrak{z}(X_Q)$ as being defined according to our previous discussion with k replaced by $\bar{K}$. As such, it is a certain map from $\varprojlim_n \mu_n(\bar{K})$ to $\tilde{G}$. On the other hand, $\mathfrak{z}(X_P)$ is a function from $\varprojlim_{(n,p)=1} \mu_n(k)$. When we say the two are "equal" what we mean, more precisely, is that they are equal upon restriction to

$$\varprojlim_{(n,p)=1} \mu_n(\bar{K}) \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n(k).$$

Proof. Let U_Q be the open subscheme of $\mathbf{A}^1$ over $\mathrm{Spec}(\bar{K})$ corresponding to Q – i.e., the complement of the branch locus of the corresponding cover. Similarly define U_P, an open subscheme of $\mathbf{A}^1$ over $\mathrm{Spec}(k)$.

By virtue of our fixed tangential basepoint y_∞, it makes sense to speak of $\pi_1(U_P, y_\infty)$ and $\pi_1(U_Q, y_\infty)$.

Associated to P, Q there are surjections

$$\pi_1(U_P, y_\infty) \twoheadrightarrow G, \ \pi_1(U_Q, y_\infty) \twoheadrightarrow G.$$

Claim: There is an isomorphism $\pi_1(U_P, y_\infty) \xrightarrow{\sim} \pi_1(U_Q, y_\infty)$ which commutes with the respective maps to G, and which moreover "preserves inertia groups." By this we mean that the $\hat{\mathbf{Z}}(1) \to \pi_1$ associated to y_∞ is preserved, and the $\hat{\mathbf{Z}}(1) \to \pi_1$ associated to each point of $\mathbf{A}^1 - U_P$ is carried to a corresponding morphism for U_Q.

The Proposition follows easily from that claim (cf. remark after (8.5.2)).

This claim is substantially a consequence of the tame specialization theorem. We reprise some elements of this proof, simply because the standard references do not treat tangential basepoints (and we wish to ensure that the isomorphism preserves our inertia group at ∞ not merely up to conjugacy.)

The pull-back of the universal G-cover over $\mathsf{CHur}_{G,n}$ yields an étale covering:

$$X_A \to U_A$$

which is a "G-torsor," i.e. a tame étale cover together with a G-action simply transitive on every geometric fiber. Here, U_A is an open subscheme of $\mathbf{P}^1/A$ whose complement is etale over $\mathrm{Spec}\, A$.

We denote by $X_{\bar{K}}$ the pullback of X to $\mathrm{Spec}\, \bar{K}$, and use similar notation X_k.

Consider the categories and functors

$$\text{prime to } p \text{ étale covers of } U_k \leftarrow \text{prime to } p \text{ étale covers of } U_A \to \text{ prime to } p \text{ étale covers of } U_{\bar{K}}$$

As in [SGA03, XIII Corollaire 2.12] these are equivalences of categories.

The tangent vector y_∞ gives rise, as in §8.2.4, to fiber functors y_∞ on the category of covers of U_k and $U_{\bar{K}}$, *and a canonical identification of those functors* – a *class de chemins* in the notation of SGA 1. In this way we obtain an isomorphism

$$\pi'(U_{\bar{K}}, y_\infty) \to \pi'(U_k, y_\infty)$$

compatible with the inertia homomorphisms at ∞: that homomorphism arises from the action of $\hat{\mathbf{Z}}(1)$ on $A(\pi^{1/\infty})$ and therefore on $\mathrm{fib}_{y_\infty}(f)$. As in the proof of *loc. cit.* this map also preserves the conjugacy class of inertia at the other punctures.

□

8.7.2. *The Hurwitz schemes with restricted monodromy.* We are now ready to define the scheme $\mathsf{CHur}^c_{G,\mathfrak{m}}$ that is the analog of the topological space $\mathrm{Hur}^c_{G,m}$:

The prior result (Proposition 8.7.1) implies, first of all, that there is a union of connected components of $\mathsf{CHur}_{G,n}/k$ which parameterizes (G,c)-covers. We denote this as $\mathsf{CHur}^c_{G,n}$. Now let $\mathfrak{m} \in \mathbf{Z}^{c/G}\langle -1\rangle$ be a multidiscriminant. Since the multidiscriminant is locally constant on CHur_G/k by Proposition 8.7.1, the subscheme

$$\mathsf{CHur}^c_{G,\mathfrak{m}}$$

of $\mathsf{CHur}^c_{G,|\mathfrak{m}|}/k$ parametrizing covers with multidiscriminant $\mathfrak{m}$ is a union of connected components of $\mathsf{CHur}_{G,|\mathfrak{m}|}$. It may not be connected in general. We have now given all the necessary ingredients for the promised explicit description of the set of components of this space, together with the Galois action on this set.

A little more notation for the final theorem: If $\mathfrak{m}$ is a multidiscriminant in $\mathbf{Z}^{c/G}\langle -1\rangle$, we choose a $\mu \in \hat{\mathbf{Z}}(1)$ and write $\mathrm{mindeg}(\mathfrak{m})$ for the minimum multiplicity of any coordinate of $\mathfrak{m}(\mu)$; we note that this is invariant under choice of μ, justifying the notation. In keeping with the notation of Proposition 7.6.1, we use "$\mathfrak{m}$ sufficiently large" to mean "$\mathfrak{m}$ such that $\mathrm{mindeg}(\mathfrak{m})$ is sufficiently large." We denote by $\tilde{G}_{\mathfrak{m}}$ the preimage of $\mathfrak{m}$ under the map $\tilde{G}\langle -1\rangle \to \mathbf{Z}^{c/G}\langle -1\rangle$.

8.7.3. Theorem. *Let k be a separably closed field. Take $X, X' \in \mathsf{CHur}^c_{G,\mathfrak{m}}(k)$. Then $\mathfrak{z}(X) = \mathfrak{z}(X')$ if X, X' are parameterized by points in the same geometric component of $\mathsf{CHur}^c_{G,\mathfrak{m}}$.*

The map

$$\Phi : \pi_0(\mathsf{CHur}^c_{G,\mathfrak{m}}/k) \longrightarrow \tilde{G}_{\mathfrak{m}}\langle -1\rangle$$

induced by $X \mapsto \mathfrak{z}(X)$ satisfies $\Phi(a^\sigma) = \Phi(a)^\sigma \in \tilde{G}_{\mathfrak{m}^\sigma}\langle -1\rangle$ for $a \in \pi_0(\mathsf{CHur}_{\mathfrak{m}})$ and $\sigma \in \Gamma_k$.

Finally, Φ is a bijection for sufficiently large $\mathfrak{m}$.

Proof. The first claim is clear when k has characteristic zero, by reduction to $k = \mathbf{C}$. Otherwise we apply (8.7.1) and Proposition 8.7.1: lift the points P, Q corresponding to X, X' to points $\tilde{P}, \tilde{Q} \in \mathsf{CHur}_{G,n}(K)$ where K has characteristic zero and also lie in the same component of $\mathsf{CHur}_{G,n}$ over K.

That Φ is equivariant for the Γ_k action follows from the analogous statement for $\mathfrak{z}(X)$.

By (8.7.1) and Proposition 8.7.1, it suffices to prove the bijectivity in the case when k has characteristic zero, and then one reduces to $k = \mathbf{C}$; so it suffices to prove that

$$\pi_0(\mathrm{CHur}_{G,\underline{m}}/k^c) \to \tilde{G}_{\underline{m}}$$

is a bijection for all sufficiently large $\underline{m}$. This is Proposition 7.6.1. □

9. User's guide to §7, §8, and examples.

We now give what we hope will be a user-friendly reformulation of the results of the prior two sections, as well as some examples. The key point in our reformulation is a certain canonical cocycle in $H^1(\hat{\mathbf{Z}}^\times, H_2(G,c))$, which we will define below.

For convenience, we return to the situation of the introduction, slightly generalized to allow several conjugacy classes of local monodromy:

- G is a center-free finite group;
- $\underline{\mathcal{C}} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\} \subset G$ ia a set of conjugacy classes in G such that $\bigcup \mathcal{C}_i$ is rational (i.e., if $g \in \bigcup \mathcal{C}_i$ then $g^t \in \bigcup \mathcal{C}_j$ for every t prime to the order of g).

- $\underline{m} = (m_1, \dots, m_k)$ is a k-tuple of integers so that $m_i = m_j$ whenever $\mathcal{C}_i = \mathcal{C}_j^t$ for some t as above. (We call $\underline{m}$ a *rational* multidiscriminant.)

We write $c = \bigcup \mathcal{C}_i$.

9.1. The set of components of interest. Given this data, we are interested in the set $\pi_0 = \pi_0(G, \mathcal{C}_i, m_i)$ of topological types of G-coverings of the Riemann sphere that have exactly m_j branch points of type $\mathcal{C}_j$. More formally, we may describe this set π_0 in several equivalent ways:

(a) The component set of the space whose points are triples (X, f, ι), where:
  - X is a compact Riemann surface,
  - $f : X \to \mathbf{P}^1_{\mathbf{C}}$ is a holomorphic map;
  - $\iota : G \to \mathrm{Aut}(f)$ is an embedding that makes G act simply transitively on a generic fiber, and
  - There are exactly $\sum m_j$ critical values of f on $\mathbf{P}^1_{\mathbf{C}}$, and m_j of them are of "type $\mathcal{C}_j$" with respect to ι.

(b) The component set of the space whose points are tuples $(E_1, E_2, \dots, E_k, f)$, where
  - E_i is a subset of $\mathbf{C}$ of size m_i;
  - f is a conjugacy class of surjections
  $$f : \pi_1(\mathbf{P}^1_{\mathbf{C}} - E, \infty) \twoheadrightarrow G$$
  - f carries a loop around every point of E_j into $\mathcal{C}_j$.

(c) The set of orbits of the Artin braid group on n tuples $(g_1, \dots, g_n)$ modulo G-conjugacy, where $\prod g_i = e$, the g_i generate G, and exactly m_j of the g_i lie in $\mathcal{C}_j$.

Warning: The spaces defined in (a) and (b) are different. The space (b) corresponds to the $\mathrm{CHur}_{G,n}$ that we have been using in the body of this paper. The space (a) refers to the definition of Hurwitz space more customary among algebraic geometers. The natural inclusion of space (b) into space (a) induces a bijection on connected components.

Note that the sets described in (a) and (b) are both naturally the component-sets of complex points of certain moduli spaces defined over $\mathbf{Q}$ (see §8.6.2.) Therefore they carry an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The results of sections §7 and §8 compute both π_0 (which is ultimately a question about topology) and the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on π_0 (which, a priori, is not.) We will summarize these results here.

We have defined (Definition 7.3) a reduced Schur cover as a certain central stem extension
$$H_2(G, c) \to \tilde{G}_c \to G,$$
where $H_2(G, c)$ is the quotient of $H_2(G, \mathbf{Z})$ by all "commutators" $[x, y] \in H_2(G, \mathbf{Z})$ where x, y commute[8] and $x \in c$. Such a cover is not unique; nonetheless our conclusions will be independent of choice of it.

There exists a conjugacy-invariant subset $\tilde{c} \subset \tilde{G}_c$, called the *marking*, that projects isomorphically to $c \subset G$; we denote by $[g]$ the unique element of $\tilde{c}$ above $g \in c$.

Choose $\alpha \in \hat{\mathbf{Z}}^\times$. The action of α permutes $\mathcal{C}_1, \dots, \mathcal{C}_k$. There exists $z_i(\alpha) \in H_2(G, c)$ so that
$$[g^\alpha] = [g]^\alpha z_i(\alpha) \quad \text{for any } g \in \mathcal{C}_i.$$

[8]By $[x, y]$ we mean here the image of a generator for $H_2(\mathbf{Z}^2, \mathbf{Z})$ under $(a, b) \in \mathbf{Z}^2 \to x^a y^b \in G$.

Thus if we take a rational multiplicity $\underline{m}$, the associated function

$$Z_{\underline{m}} : \alpha \mapsto \prod_i z_i(\alpha)^{m_i} \tag{9.1.1}$$

is a cocycle, i.e. its cohomology class defines an element of $H^1(\hat{\mathbf{Z}}^\times, H_2(G,c))$, where the action of $\hat{\mathbf{Z}}^\times$ on $H_2(G,c)$ is the power action: $\alpha \in \hat{\mathbf{Z}}^\times$ sends $h \in H_2(G,c)$ to h^α. Moreover, this cocycle is multiplicative, i.e. $Z_{\underline{m}+\underline{m}'} = Z_{\underline{m}} \times Z_{\underline{m}'}$.

Proof. (that $Z_{\underline{m}}$ obeys the cocycle condition): We denote by τ_α the permutation of $\{1,\dots,k\}$ induced by α. If g is an element of $\mathcal{C}_i$, we denote $z_i(\alpha)$ by $z(g,\alpha)$. Now note that

$$[g^{\alpha\beta}] = [g]^{\alpha\beta} z(g,\alpha\beta)$$

and moreover

$$[g^{\alpha\beta}] = [g^\alpha]^\beta z(g^\alpha,\beta) = [g]^{\alpha\beta} z(g,\alpha)^\beta z(g^\alpha,\beta)$$

and thus we obtain the condition:

$$z_i(\alpha\beta) = z_i(\alpha)^\beta \cdot z_{\tau_\alpha(i)}(\beta). \tag{9.1.2}$$

By the rationality of $\underline{m}$, we can express $Z_{\underline{m}}$ as a product of functions on $\hat{\mathbf{Z}}^\times$ of the form

$$Z_{\mathcal{O}} = \prod_{i\in\mathcal{O}} z_i$$

where $\mathcal{O}$ is an orbit of $\hat{\mathbf{Z}}^\times$ on $\{1,\dots,k\}$. It is clear from (9.1.2) that $Z_{\mathcal{O}}$ satisfies the cocycle condition, whence so does $Z_{\underline{m}}$ for any rational $\underline{m}$. □

9.2. Summary of the theorems. The map

$$(g_1,\dots,g_n) \text{ as in (c) of §9.1} \mapsto [g_1]\dots[g_n] \in H_2(G,c)$$

defines a *bijection*

$$\pi_0 \xrightarrow{\sim} H_2(G,c)$$

if every m_i is sufficiently large.

If the Galois automorphism σ induces $\alpha \in \hat{\mathbf{Z}}^\times$ under the cyclotomic character, the action of σ on π_0 corresponds to the action

$$h \in H_2(G,c) \mapsto h^{\alpha^{-1}} \cdot Z_{\underline{m}}(\alpha^{-1}).$$

In particular:

(i) Suppose that every coordinate m_i of the rational multidiscriminant $\underline{m}$ is sufficiently large. Then the Hurwitz space (i.e., either the algebraic variety underlying the space of §9.1 (a), or the algebraic variety underlying the space of §9.1 (b)) admits a $\mathbf{Q}$-rational component if and only if

$$Z_{\underline{m}} \in H^1(\hat{\mathbf{Z}}^\times, H_2(G,c))$$

is trivial.

(ii) In particular, the Hurwitz space has a $\mathbf{Q}$-rational component whenever each coordinate m_i of $\underline{m}$ is sufficiently large and divisible by the exponent of $H_2(G,c)$.

(iii) (Again, assuming that every coordinate m_i of $\underline{m}$ is sufficiently large): When there is a rational component, the set of all rational components is a torsor under the 2-torsion subgroup $H_2(G,c)[2]$. (In fact, for any finite group, $g \in G$ is fixed by the power action of $\hat{\mathbf{Z}}^\times$ if and only if g is 2-torsion.)

9.2.1. *Remark.* In the previous sections §7, §8 a somewhat more general situation is considered – in the situation of (c) above, it would correspond to removing the constraint that $\prod g_i = e$.

9.3. Some examples.

9.3.1. *A simple condition guaranteeing that Z is trivial.* There is a convenient class of examples, which in practice encompasses many natural examples:

> (+): Every conjugacy class in c is rational, and the order of any element of c is relatively prime to the order of $H_2(G, \mathbf{Z})$.

In this case $H_2(G, c; \mathbf{Z}) = H_2(G, \mathbf{Z})$ *and* the cocycle Z is *trivial*. This is the situation, for instance, in the case first studied by Fried, where $G = A_5$ and c consists of 3-cycles; in this case $H_2(G, c; \mathbf{Z})$ has order 2.

9.3.2. *Example: dihedral groups.* Condition (+) just enunciated applies when G is a group of order congruent to 2 mod 4 and c is the unique class of involutions:

Such a group is necessarily of the form $A \rtimes \{\pm 1\}$, where A is the unique normal subgroup of index 2. In this case, consideration of the spectral sequence shows that the map $H_2(A, \mathbf{Z}) \to H_2(G, \mathbf{Z})$ is a surjection and identifies $H_2(G, \mathbf{Z})$ with the coinvariants of G/A on $H_2(A, \mathbf{Z})$. By virtue of (+) we have $H_2(G, c; \mathbf{Z}) \cong H_2(A, \mathbf{Z})_{G/A}$.

9.3.3. *An example with nontrivial cocycle:* $\mathrm{PSL}_2(\mathbf{F}_7)$. Let $G = \mathrm{PSL}_2(\mathbf{F}_7)$ and let c be the unique conjugacy class of elements of order 4 in G (so that $\mathbf{Z}^{c/G} = \mathbf{Z}$.) One checks that $H_2(G, c; \mathbf{Z}) = H_2(G, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$, and a reduced Schur cover is provided by $\mathrm{SL}_2(\mathbf{F}_7) \to \mathrm{PSL}_2(\mathbf{F}_7)$. But c does not lift to a rational conjugacy class in $\mathrm{SL}_2(\mathbf{F}_7)$. Indeed let $\chi : \hat{\mathbf{Z}}^\times \longrightarrow \{\pm 1\}$ be the nontrivial homomorphism that factors through $(\mathbf{Z}/8\mathbf{Z})^\times$ with kernel $\pm 1 + 8\mathbf{Z}$; then the action of $\hat{\mathbf{Z}}^\times$ on the lifts of c are precisely through χ.

So we have proved:

> The moduli space of geometrically connected G-covers of $\mathbf{P}^1$, unramified at ∞, with m branch points of type c, has two geometric components; they are $\mathbf{Q}$-rational when m is even, and defined over $\mathbf{Q}(\sqrt{2})$ when m is odd.

In particular, *if k_0 is a field not containing a square root of 2, there are no G-covers of $\mathbf{P}^1$, defined over k_0, with an odd number of branch points, all of type c.*

The last fact can be seen directly to be a consequence of a reciprocity law.

9.4. User's guide to the universal marked central extension and Galois twisting. To conclude this section, and as a convenient reference for the reader for Part 3, we also summarize the result on the universal marked central extension and the Galois action it carries.

Maintaining the notation used previously, set

$$\tilde{G} = \tilde{G}_c \times_{G^{\mathrm{ab}}} \mathbf{Z}^{c/G}.$$

The set $\hat{c} = \{([g], e_g) : g \in c\}$ defines a union of conjugacy classes of $\tilde{G}$ which projects bijectively to c. There is a natural projection

$$\tilde{G} \to G \times \mathbf{Z}^{c/G}.$$

We have proved that:

(i) $\tilde{G}$, together with $\hat{c}$, is *universal* amongst central extensions of G endowed with a lifting of c;
(ii) The group $\hat{\mathbf{Z}}^\times$ acts on $\tilde{G}$, by means of the rule:

$$\alpha \in \hat{\mathbf{Z}}^\times : (g \in \tilde{G}_c,\ \underline{m}) \longrightarrow (g^\alpha Z_{\underline{m}}(\alpha),\ \underline{m}^\alpha). \tag{9.4.1}$$

where $\underline{m} \in \mathbf{Z}^{c/G}$, and $Z_{\underline{m}}$ is the cocycle defined in (9.1.1).

Now let us describe the topological space whose components are parameterized by $\tilde{G}$. It is related to what we have discussed previously in §9.1 but not precisely the same, and to avoid confusion we will simply define it from scratch:

Choose, once and for all, a smooth arc $y : [0,1] \to \mathbf{P}^1_{\mathbf{C}}$ with $y(0) = \infty$ and with nonzero derivative at $t = 0$. By the fundamental group $\pi_1(\mathbf{A}^1_{\mathbf{C}}, y)$ we shall mean the fundamental group $\pi_1(\mathbf{A}^1_{\mathbf{C}}, y(t))$ for "small t", i.e. the direct limit

$$\varinjlim \pi_1(\mathbf{A}^1_{\mathbf{C}}, y(t))$$

where we take the limit over $t \to 0^+$, and the transition maps are the canonical isomorphisms induced by the path y. This notion corresponds to the notion of a fundamental group with tangential basepoint, recalled in §8.2.

Now, consider the space $\mathcal{H}$ whose points are tuples $(E_1, E_2, \dots, E_k, f)$, where

- E_i is a subset of $\mathbf{C}$ of size m_i;
- f is a surjection[9]

$$f : \pi_1(\mathbf{A}^1_{\mathbf{C}} - E, y) \twoheadrightarrow G$$

- f carries a loop around every point of E_j into $\mathcal{C}_j$.

We refer to such a $(E_1, E_2, \dots, E_k, f)$ as a *branched G-cover* with multidiscriminant $\underline{m}$.

To any such branched G-cover, i.e. to any point of $\mathcal{H}$, we may assign an invariant

$$\mathfrak{z} \in \tilde{G}$$

with the following properties:

(iii) The image of $\mathfrak{z}$ under

$$\tilde{G} \to G \times \mathbf{Z}^{c/G}$$

is given by $(\delta^{-1}, \underline{m})$, where δ is the boundary monodromy, i.e., δ is that element of G given by the image under f of a small loop around ∞ in $\pi_1(\mathbf{A}^1_{\mathbf{C}} - E, y)$.

(iv) Suppose that each m_i is sufficiently large. Then the association

$$(E_1, \dots, E_k, f) \longrightarrow \mathfrak{z} \in \tilde{G}_{\underline{m}}$$

gives a bijection between the elements of $\tilde{G}$ with multidiscriminant $\underline{m}$, and the set of components of $\mathcal{H}$.

[9] *not* merely a conjugacy class of such, as was the case in §9.1 (a).

9.5. Branched covers over non-algebraically-closed fields. The space $\mathcal{H}$ is in fact naturally identified with the space of complex points of a certain moduli space defined over $\mathbf{Q}$, so that its components carry a corresponding action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The action of an element $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ carries a component with invariant $\mathfrak{z} \in \tilde{G}$ to the component with invariant $\mathfrak{z}^\alpha \in \tilde{G}$. Here $\alpha \in \hat{\mathbf{Z}}^\times$ is the image of σ under the cyclotomic character, and $\mathfrak{z}^\alpha$ is as in (9.4.1).

Let us now summarize some of the difficulties that arise when analyzing the situation over a field of constants that is not algebraically closed. First of all, there is an algebraic counterpart of the notion of branched G-cover; we refer to §8.3.1 for the exact definition. However, one cannot even define the multidiscriminant of a branched G-cover without first fixing a suitable root of unity in k; the problem is that the notion of "a loop around a puncture" which we use in the definition of multidiscriminant for points on $\mathcal{H}$ is not available as such in the algebraic case; there is still an inertia group which is analogous to the cyclic group generated by a loop around a puncture, but this group does not come endowed with a canonical choice of generator.

In order to phrase the situation intrinsically, we have introduced Galois twisting:

Let k_0 be a field and k a separable closure of k_0; let Γ_k be the Galois group of k over k_0. We define $\hat{\mathbf{Z}}(1) = \varprojlim \mu_n(k)$, the limit being taken over n prime to the characteristic of k. In this setting, for any set X endowed with an action of $\hat{\mathbf{Z}}^\times$, we have denoted

$$X\langle -1\rangle := \mathrm{Mor}_{\hat{\mathbf{Z}}^\times}(\hat{\mathbf{Z}}(1)^\times, X) \tag{9.5.1}$$

the set of functions $\hat{\mathbf{Z}}(1)^\times \to X$ that are equivariant for the $\hat{\mathbf{Z}}^\times$-actions. Γ_k acts on $X\langle -1\rangle$ through its action on $\hat{\mathbf{Z}}(1)^\times$.

With this notation, the multidiscriminant of an (algebraic) branched G-cover is not an element of $\mathbf{Z}^{c/G}$ as before, but rather an element $\mathfrak{m}$ of the twisted version $\mathbf{Z}^{c/G}\langle -1\rangle$; this definition of the multidiscriminant is independent of any choice, and we have shown that $\mathfrak{m}$ can be lifted to an invariant

$$\mathfrak{z} \in \tilde{G}\langle -1\rangle,$$

in a Galois-compatible way.

That is to say – if we think of $\mathcal{H}$ as the complex points of a scheme $H/\mathbf{Q}$, the connected components of H are in *Galois-equivariant* bijection with elements of $\tilde{G}\langle -1\rangle$ with prescribed multidiscriminant.

Part 3. Counting and the Cohen–Lenstra heuristics

10. Branched covers and the Hurwitz scheme over $\mathbf{F}_q(t)$

As before, G is a finite group, which we now assume to be centre-free; $c \subset G$ is a conjugacy-invariant subset, and other notations are as before (see §9 for summary).

Let q be a prime power that is relatively prime to the order of G. In this section we take $k = \overline{\mathbf{F}_q} \supset k_0 = \mathbf{F}_q$, and so $\Gamma = \Gamma_k = \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$, where $\overline{\mathbf{F}_q}$ is a fixed algebraic closure; the group Γ is pro-cyclic, topologically generated by the Frobenius morphism $\mathrm{Frob} : x \mapsto x^q$.

Fix a uniformizer z for $\mathbf{P}^1$ at ∞ which pairs to 1 with y_∞. We will usually parametrize $\mathbf{P}^1$ by a rational function $t = z^{-1}$, so that z vanishes simply at $t = \infty$ as claimed.

10.1. Local G-extensions and Puiseaux series. We will need to pay close attention to the local behavior of G-extensions of $\mathbf{F}_q(t)$ at the prime at ∞. To do this, we use Puiseaux series:

Recall that, if t is an indeterminate, the maximal tame extension of $\mathbf{F}_q((t))$ is given by $\overline{\mathbf{F}_q}((t^{1/\infty}))$, and its Galois group Δ is canonically an extension:

$$\hat{\mathbf{Z}}(1) \to \Delta \to \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q).$$

We split the sequence by lifting Frob to that element $\mathrm{F}_\Delta \in \Delta$ which fixes the extension $\mathbf{F}_q((t^{1/\infty}))$ and which projects to $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$.

By a *marking* of a tame extension of $\mathbf{F}_q((t))$ or $\overline{\mathbf{F}_q}((t))$ we mean an embedding of it into $\overline{\mathbf{F}_q}((t^{1/\infty}))$.

By a G-extension of $\mathbf{F}_q((t))$ we shall mean an étale algebra E over $\mathbf{F}_q((t))$, together with a G-action on $E/\mathbf{F}_q((t))$ which acts simply transitively on the set of markings.

To any marked G-extension E we have a unique homomorphism $\varphi : \Delta \to G$ such that the action of $\varphi(\mathfrak{d})$ on E is compatible with the action of $\mathfrak{d} \in \Delta$ on $\overline{\mathbf{F}_q}((t^{1/\infty}))$. We say "the marking is of type φ."

Changing the marking changes this homomorphism by a conjugacy; to give a G-extension of $\mathbf{F}_q((t))$ without specifying a marking is the same as giving a conjugacy class of morphisms $\Delta \to G$.

For a G-extension E of $\mathbf{F}_q((t))$ we denote by $\mathrm{Aut}_G(E)$ the collection of automorphisms of E as a G-algebra. Equivalently, this equals the size of the centralizer of a corresponding morphism $\varphi : \Delta \to G$.

10.2. G-extensions: their discriminant and infinity type. By a G-extension of the field $K := \mathbf{F}_q(t)$ we shall mean an extension L/K equipped with an isomorphism $G \xrightarrow{\sim} \mathrm{Gal}(L/K)$. We say that L is regular if $\mathbf{F}_q$ is algebraically closed in L. *In this paper we will consider the asymptotic count only of regular extensions.* This diverges slightly from the analogous situation for number fields, where one typically wants to count G-extensions L of $\mathbf{Q}$ without excluding those extensions L containing a nontrivial cyclotomic subextension (see [Klü05, Tur08] for a discussion of some of the subtleties related to this.)

We can impose on our G-extensions the further condition that all ramification away from ∞ is of type c; again, this means that each (necessarily cyclic) inertia group is generated by an element of c. In that case, we shall say that L is a (G, c)-extension.

We may attach to a (G, c)-extension L a multidiscriminant, which is a a Γ-fixed element of $\mathbf{Z}^{c/G}\langle -1\rangle$, i.e., a $\hat{\mathbf{Z}}^\times$-equivariant function from $\hat{\mathbf{Z}}(1)$ to $\mathbf{Z}^{c/G}$ (see §8.1).

We construct this function just as we did in §8.3.2 in the case of covers of $\mathbf{P}^1$ over algebraically closed fields. Take $\mu \in \hat{\mathbf{Z}}(1)^\times$. For each $x \in \mathbf{A}^1$ we get a conjugacy class of morphisms $\Delta \to G$ and therefore, by restriction to inertia, functions $\iota_x : \hat{\mathbf{Z}}(1) \to G$. Then the multidiscriminant sends μ to the sum of $\deg(x) e_{\iota_x(\mu)}$ over all x. As before, $e_{\iota_x(\mu)}$ is the element of $\mathbf{Z}^{c/G}$ corresponding to the conjugacy class of $\iota_x(\mu)$.

To be compatible with our earlier usage (cf. §8.3.1) we say that L is m-branched if $|\mathfrak{m}| = m$, that is to say, if the sum of $\deg(x)$ over all ramified x equals m.

For each (G, c) extension L, we denote by L_∞ the étale algebra over $\mathbf{F}_q((z))$ induced by L, via the embedding

$$\mathbf{F}_q(\mathbf{P}^1) \hookrightarrow \mathbf{F}_q((z))$$

where z is, as before, a uniformizer at ∞.

Note that L_∞ has the structure of G-extension of $\mathbf{F}_q((z))$. (We use the slightly inconsistent convention that a (G, c)-extenson of the global field $\mathbf{F}_q(t)$ is understood to be a field, while the local (G, c)-extension L_∞ is not required to be.)

The isomorphism class of L_∞ defines, as discussed previously, a conjugacy class of morphisms $\Delta \to G$. By a *marking* of L we shall mean a marking of L_∞, that is to say, an embedding of L_∞ into $\overline{\mathbf{F}_q}((z^{1/\infty}))$. A marking of L gives rise to a morphism (not just up to conjugacy) $\Delta \to G$, the *infinity type* of L. (See §10.1)

10.3. Geometric points on CHur. It is not the case that (G, c)-extensions of $\mathbf{F}_q(t)$ are the same thing as $\mathbf{F}_q$-rational points of $\mathsf{CHur}^c_{G,\mathfrak{m}}$; recall [10] that the Hurwitz scheme as defined here parametrizes branched covers with a marked point in the fiber over the tangential basepoint y_∞; to say this data is $\mathbf{F}_q$-rational places a constraint on the infinity type of the (G, c)-extension L. For instance, if L is unramified at ∞, the constraint is that L is split at ∞. In order to consider general (G, c)-extensions, we need to consider $\overline{\mathbf{F}_q}$-points of $\mathsf{CHur}^c_{G,\mathfrak{m}}$ satisfying certain descent conditions, which we explain below.

Chasing through the definitions, we see that the points corresponding to (G, c)-extensions are in bijection with diagrams:

$$\overline{\mathbf{F}_q}(t) \overset{\iota}{\hookrightarrow} L \overset{f}{\hookrightarrow} \overline{\mathbf{F}_q}((z^{1/\infty})), \tag{10.3.1}$$

where the composite is the natural embedding induced by $t \mapsto z^{-1}$, and L is a (G, c)-extension of $\overline{\mathbf{F}_q}(t)$ with multidiscriminant $\mathfrak{m}$.

There are three actions on $\mathsf{CHur}^c_{G,\mathfrak{m}}(\overline{\mathbf{F}_q})$ of interest to us:

- The action of Frobenius.

  This arises from the structure of CHur as an $\mathbf{F}_q$-scheme. We often refer to it as Frob.
- The Δ-action:

  Δ acts on the target of f, and on the source of ι, the latter action being through the quotient $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q) \simeq \mathrm{Gal}(\overline{\mathbf{F}_q}(t)/\mathbf{F}_q(t))$; we also twist the linear structure on L compatibly.

  More precisely, given a diagram (10.3.1), the action of $\mathfrak{d} \in \Delta$ sends it to the diagram

  $$\overline{\mathbf{F}_q}(t) \overset{\iota'}{\hookrightarrow} L' \overset{f'}{\hookrightarrow} \overline{\mathbf{F}_q}((z^{1/\infty})),$$

  where $\iota' = \iota \circ \mathfrak{d}^{-1}$, $f' = \mathfrak{d} \circ f$, and L' is L but with the structure of $\overline{\mathbf{F}_q}$-vector space twisted through $\mathfrak{d}$, so that the maps are $\overline{\mathbf{F}_q}$-linear.
- The G-action:

  Via its action on L. Thus, given a diagram (10.3.1), the action of $g \in G$ replaces ι by $\iota' = g \circ \iota$ and f by $f \circ g^{-1}$.

With these notations, the action of Frob and the action of $\mathrm{F}_\Delta \in \Delta$ coincide. Also, the boundary monodromy of $x \in \mathsf{CHur}(\overline{\mathbf{F}_q})$ is the function $\delta : \hat{\mathbf{Z}}(1)^\times \to G$

[10] see §8.6 for definition of general Hurwitz schemes, and §8.7.2 for the subschemes with monodromy type $\mathfrak{m}$.

such that the action of $\alpha \in \hat{\mathbf{Z}}(1)^\times$ (considered in Δ) and $\delta(\alpha)$ (considered in G) on x coincide.

10.4. Galois descent. Let us now fix a morphism $\varphi : \Delta \to G$. In this section we will exhibit a *bijection*:

- Regular marked (G, c)-extensions L of $\mathbf{F}_q(t)$, of infinity type φ;
- points of $\mathsf{CHur}^c_{G,\mathfrak{m}}(\overline{\mathbf{F}_q})$, with the property that

$$(*)\ \varphi(\mathfrak{d}).x = \mathfrak{d}.x \text{ for all } \mathfrak{d} \in \Delta.$$

  where the action on the left is the G-action and the action on the right is the Δ-action.

The property (*) can also be written as follows: $\varphi(\mathrm{F}_\Delta)^{-1}\mathrm{Frob}$ fixes x and the boundary monodromy of x coincides with $\varphi|_{\hat{\mathbf{Z}}(1)}$.

To prove this bijection, we wish to understand the set of marked regular (G, c)-extensions L of $\mathbf{F}_q(t)$ with infinity type φ. Recall that this means that the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\varphi(\mathfrak{d})"'] & \overline{\mathbf{F}_q}((z^{1/\infty})) \arrow[d, "\mathfrak{d}"'] \\ L \arrow[r] & \overline{\mathbf{F}_q}((z^{1/\infty})) \end{tikzcd} \tag{10.4.1}$$

commutes for every $\mathfrak{d} \in \Delta$.

Clearly, such a diagram gives rise to a covering $f : X \to \mathbf{P}^1$, together with a G-action $G \hookrightarrow \mathrm{Aut}(f)$, and a marking, i.e., an element of the fiber functor $\mathrm{fib}_{y_\infty}(f)$. (See §8.2.4 for explanation.) Thus we have a point $x \in \mathsf{CHur}^c_{G,\mathfrak{m}}(\overline{\mathbf{F}_q})$. Moreover, from the G-action on L and the diagram (10.4.1), we see that the corresponding point $x \in \mathsf{CHur}^c_{G,\mathfrak{m}}(\overline{\mathbf{F}_q})$ has the property that

$$\varphi(\mathfrak{d}) \cdot x = \mathfrak{d} \cdot x \tag{10.4.2}$$

i.e., the actions on x of Δ acting directly via the Δ-action and Δ acting through the G-action via φ are compatible.

Conversely, suppose we are given a point with the property (10.4.2); such a point corresponds to a diagram $\overline{\mathbf{F}_q}(t) \overset{\iota}{\hookrightarrow} L' \xrightarrow{f} \overline{\mathbf{F}_q((t^{1/\infty}))}$, where L' is a (G, c)-extension of $\overline{\mathbf{F}_q}(t)$. The validity of (10.4.2) says that, for every $\mathfrak{d} \in \Delta$, there is a map α making this diagram commute:

$$\begin{tikzcd} \overline{\mathbf{F}_q}(t) \arrow[r, "\iota\circ\mathfrak{d}^{-1}"] \arrow[d, "="'] & (L')_{\mathrm{tw}} \arrow[r, "\mathfrak{d}\circ f"] \arrow[d, "\alpha"'] & \overline{\mathbf{F}_q((t^{1/\infty}))} \arrow[d, "="'] \\ \overline{\mathbf{F}_q}(t) \arrow[r, "\varphi(\mathfrak{d})\circ\iota"] & L' \arrow[r, "f\circ\varphi(\mathfrak{d})^{-1}"] & \overline{\mathbf{F}_q((t^{1/\infty}))} \end{tikzcd} \tag{10.4.3}$$

where L'_{tw} is L' but with the $\mathbf{F}_q$-linear structure twisted so that the top line is linear. In particular, $\alpha' = \varphi(\mathfrak{d})^{-1} \circ \alpha$ fits into a diagram

$$
\begin{array}{ccccc}
\overline{\mathbf{F}_q}(t) & \xrightarrow{\iota} & L' & \xrightarrow{f} & \overline{\mathbf{F}_q((t^{1/\infty}))} \\
\mathfrak{d}\downarrow & & \alpha'\downarrow & & \mathfrak{d}\downarrow \\
\overline{\mathbf{F}_q}(t) & \xrightarrow{\iota} & L' & \xrightarrow{\mathfrak{d}} & \overline{\mathbf{F}_q((t^{1/\infty}))}
\end{array}
\tag{10.4.4}
$$

One recovers the (G,c)-extension of $\mathbf{F}_q(t)$ as the subfield of L' fixed by α', taking $\mathfrak{d} = \mathrm{F}_\Delta$.

11. The main counting theorem and the Malle-Bhargava heuristics

We are now ready to state the main arithmetic theorem of the paper. This requires a few more pieces of notation.

Let L_∞ be an étale G-extension of $\mathbf{F}_q((z))$, and let $\varphi : \Delta \to G$ be a representative of the conjugacy class of morphisms corresponding to L_∞, and let δ be the element of $G\langle -1\rangle$ specified by $\varphi|\hat{\mathbf{Z}}(1)$ (see again §8.1 for the notation).

If $\mathfrak{m} \in \mathbf{Z}^{c/G}\langle -1\rangle$ is a multidiscriminant (we will always suppose that $\mathfrak{m}$ is fixed by the action of the Galois group $\Gamma = \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$) and δ an element of $G\langle -1\rangle$, we write $\tilde{G}_{\mathfrak{m}\times\delta^{-1}}$ for the preimage of $\mathfrak{m}\times\delta^{-1}$ inside $\tilde{G}\langle -1\rangle$. (The reason for writing δ^{-1} rather than δ is that, in our notation, a cover with boundary monodromy δ has lifting invariant in $\tilde{G}\langle -1\rangle$ which projects to δ^{-1} in $G\langle -1\rangle$.) Similarly we define $\tilde{G}_{\mathfrak{m}}$ for the preimage of $\mathfrak{m} \in \mathbf{Z}^{c/G}\langle -1\rangle$ inside $\tilde{G}$, i.e. the union of $\tilde{G}_{\mathfrak{m}\times\delta^{-1}}$ over all δ.

In that case, $\tilde{G}_{\mathfrak{m}}$ carries an action of Γ, namely the *discrete action* of §8.1.7. Moreover, $\tilde{G}_{\mathfrak{m}}$ also carries an action of G, by conjugation; more precisely, if g is an element of G, the action of g on $\tilde{G}_{\mathfrak{m}}$ is given by lifting g arbitrarily to $\tilde{G}$ and conjugating. This is independent of the choice of lift, since $\tilde{G} \to G$ is a central extension. It is easy to verify from definitions that the conjugation action of G commutes with the discrete action of Γ.

11.1. Definition. Define $B(L_\infty, \mathfrak{m})$ to be the number of $x \in \tilde{G}_{\mathfrak{m}\times\delta^{-1}}$ such that

$$x^{\mathrm{Frob}} = x^{\varphi(F_\Delta)}.$$

that is to say: the discrete action of $\mathrm{Frob} \in \Gamma$ and the conjugation action of $\varphi(F_\Delta)$ coincide.

In order to justify the notation, we need to check that $B(L_\infty, \mathfrak{m})$ depends only on the isomorphism class of L_∞, and not on the choice of φ; in other words, it doesn't change when φ is conjugated by an element of g. This is easy: x^g satisfies the conditions of the definition with infinity type φ^g if and only if x satisfies the conditions for infinity type φ.

Write $N(\mathfrak{m})$ for the number of regular (G,c)-extensions $L \supset K$ with multidiscriminant $\mathfrak{m}$ and whose completion at infinity is isomorphic to L_∞. Write $C(\mathfrak{m})$ for the number of $\mathbf{F}_q$-points on the configuration space $\mathsf{Conf}_{\underline{m}}$.

Finally, recall that in favorable circumstances, we know (e.g. by the main theorem of [EVW09]) that the Hurwitz spaces of branched G-covers satisfy the following stability condition, for some parameter $\alpha > 0$:

> HS_α: There exists $A > 0$ such that the condition of Corollary 5.8.2 holds whenever $j < \alpha|\underline{m}|$ and $\mathrm{mindeg}(\underline{m}) > A$.

11.1.1. Theorem. *Maintain the notation above, and assume that condition* HS_α *holds.*

Then there is $Q = Q(G, c, \alpha)$ *such that, for* $q \geq Q$,

$$\lim_{\mathrm{mindeg}(\mathfrak{m})\to\infty} \left(\frac{N(\mathfrak{m})}{C(\mathfrak{m})} - \frac{B(L_\infty, \mathfrak{m})}{|\operatorname{Aut}_G(L_\infty)|} \right) = 0. \tag{11.1.1}$$

We note that the factor $|\operatorname{Aut}_G(L_\infty)|^{-1}$ is just as predicted by Bhargava's heuristics. The factor $B(L_\infty, \mathfrak{m})$ constitutes a correction to those heuristics; it is an integer between 0 and $|H_2(G, c; \mathbf{Z})|$ and is, of course, unity when $H_2(G, c, \mathbf{Z}) = 1$.

Our proof also shows that the lifting invariant of such covers is *equidistributed*, giving evidence in favor of a conjecture stated in [VE10]. More precisely: choose an infinity type φ, let $x_\mathfrak{m} \in \tilde{G}_\mathfrak{m}$ be a sequence indexed by multidiscriminants $\mathfrak{m}$, with $x_\mathfrak{m}$ satisfying the conditions of Definition 11.1, and let $N_x(\mathfrak{m})$ be the number of regular (G, c)-extensions L with multidiscriminant $\mathfrak{m}$, infinity type isomorphic to L_∞, and lifting invariant conjugate to $x_\mathfrak{m}$. Then

$$\lim_{\mathrm{mindeg}(\mathfrak{m})\to\infty} \frac{N_x(\mathfrak{m})}{C(\mathfrak{m})} - \frac{1}{|\operatorname{Aut}_G(L_\infty)|} = 0.$$

Proof. (of Theorem 11.1.1)

Choose an infinity type $\varphi : \Delta \to G$ in the conjugacy class associated to L_∞.

Write Y for the subscheme of $\mathsf{CHur}^c_{G,\mathfrak{m}}$ parametrizing branched G-covers with boundary monodromy $\varphi|\hat{\mathbf{Z}}(1)$; this is a union of geometric components of $\mathsf{CHur}^c_{G,\mathfrak{m}}$, and is preserved by $\varphi(F_\Delta)^{-1}\,\mathrm{Frob}$.

We have seen in §10.4 that the *marked* regular G-extensions of $\mathbf{F}_q(t)$ with multidiscriminant $\mathfrak{m}$ and infinity type φ are parametrized by the points of $\mathsf{CHur}^c_{G,\mathfrak{m}}(\overline{\mathbf{F}_q})$ which are fixed by the action of $\varphi(F_\Delta)^{-1}\,\mathrm{Frob}$ and which have boundary monodromy $\varphi|\hat{\mathbf{Z}}(1)$.

The computation of the number of such marked regular G-extensions can thus be computed by applying the Grothendieck-Lefschetz trace formula to the endomorphism $\varphi(F_\Delta)^{-1}\,\mathrm{Frob}$ of Y. (This can also be viewed as a counting formula for $\mathbf{F}_q$-points on a specified $\mathbf{F}_q$-form of Y.)

In order to apply the Lefschetz trace formula, we need to understand the étale cohomology of Y. A crucial point is that the étale Betti numbers of Y agree with the topological Betti numbers of the manifold $Y(\mathbf{C})$; this follows from a comparison theorem for étale covers of complements of relative normal crossing divisors in smooth proper schemes over $\operatorname{Spec}\mathbf{Z}$, see[EVW09, Prop 7.5, 7.6] or [Yu97].

The contribution of H^i for all $i > \alpha|\mathfrak{m}|$ to the alternating sum in the Lefschetz trace formula is negligible compared to $C(\mathfrak{m})$; this follows from the Weil conjectures once we have an priori bound on the Betti numbers, which is given by [EVW09, Prop 2.5]. It is this part of the argument that requires us to place the lower bound $Q(G, c, \alpha)$ on q.

On the other hand, the stability condition HS_α, combined with Theorem 5.8.1, shows that the map from a geometric component of $\mathsf{CHur}^c_{G,\mathfrak{m}}$ to $\mathsf{Conf}_\mathfrak{m}$ induces an isomorphism on H^i for all $i < \alpha|\mathfrak{m}|$ and all sufficiently large $\mathfrak{m}$. Note also that the map $\varphi(F_\Delta)^{-1} \cdot \mathrm{Frob}$ covers the map Frob on $\mathsf{Conf}_\mathfrak{m}$.

It follows immediately that the contribution of cohomology in degrees $i < |\mathfrak{m}|$ to the Lefschetz trace formula is

$$C(\mathfrak{m}) \cdot \#\left\{ \text{ components of } Y \text{ which are fixed by } \varphi(F_\Delta)^{-1}\,\mathrm{Frob} \right\}.$$

But Theorem 8.7.3 gives us a combinatorial description of the geometric components of $\mathsf{CHur}^c_{G,\mathfrak{m}}$ (and thus the geometric components of Y) for sufficiently large $\mathfrak{m}$; from this we see that the number of components fixed by $\varphi(F_\Delta)^{-1}$ Frob equals $B(L_\infty, \mathfrak{m})$.

It follows that the number of marked regular G-extensions with multidiscriminant $\mathfrak{m}$ and infinity type φ is asymptotic to

$$C(\mathfrak{m})B(L_\infty, \mathfrak{m}).$$

The number of infinity types φ associated to the etale G-extension $L_\infty \supset \mathbf{F}_q((z))$ is precisely $\frac{|G|}{|\operatorname{Aut}_G(L_\infty)|}$. Thus the number of marked regular G-extensions L with multidiscriminant $\mathfrak{m}$ satisfying

$$L \otimes_{\mathbf{F}_q(z)} \mathbf{F}_q((z)) \simeq L_\infty$$

is asymptotic to

$$C(\mathfrak{m})B(L_\infty, \mathfrak{m})\frac{|G|}{|\operatorname{Aut}_G(L_\infty)|}.$$

Each (unmarked) regular G-extension has $|G|$ distinct markings, so appears $|G|$ times in the above count. This yields the theorem.

□

This result can be modified or generalized in several ways. In particular, we anticipate that the results of the topological part, and, therefore, the results of this Theorem, generalize to replacing $\mathbf{P}^1$ by an arbitrary base and ∞ by an arbitrary fixed set of punctures, but we will not pursue that generalization here.

11.2. Relationship to Malle–Bhargava heuristics. We observe that Theorem 11.1.1 can be used to show results towards conjectures of Malle and Bhargava about counting G-extensions, under the homological stability hypothesis HS_α.

Suppose that G is provided with a transitive action on set of n elements; then each conjugacy class $\mathcal{C}$ in G has an *index* $i(\mathcal{C})$, which is defined by the condition that $n - i(\mathcal{C})$ is the number of orbits of a representative of $\mathcal{C}$. The *discriminant* of a G-extension is then the linear function of the multidiscriminant $\mathfrak{m}$ which sends $\mathcal{C}$ to $i(\mathcal{C})$; this agrees with the usual notion of discriminant for the (non-Galois) degree-n cover associated to a point stabilizer in G.

Let $N_{G,c}(q^m)$ be the number of (G, c)-extensions of $\mathbf{F}_q(t)$ with discriminant q^m. Malle's conjecture, in its strongest form, asserts that when c is taken to be the set of all non-identity elements, we have an asymptotic expression

$$N_{G,c}(q^m) \sim \gamma m^{b-1} q^{m/a} \tag{11.2.1}$$

where a is the minimal value of $i(\mathcal{C})$ as $\mathcal{C}$ ranges over the nontrivial conjugacy classes, b is the number of orbits of the power action of $\hat{\mathbf{Z}}^\times$ on the set of conjugacy classes with index a, and γ is an unspecified constant depending on q and G.

For example, when $G = S_n$, we have $a = 1$, and also $b = 1$, because the only minimal-index class is the class of a transposition. So in this case Malle's conjecture predicts that

$$N_{S_n,c}(q^m) \sim \gamma q^m$$

In this context, Bhargava makes a precise prediction (in the number field case) for the value of the constant γ, which is a rational multiple of an Euler product.

We can also ask about choices of c smaller than $G - id$. For instance, when $G = S_n$, we can set c to be τ, the conjugacy class of transpositions; then we are counting degree-n extensions with Galois group S_n and *squarefree* discriminant. Neither Malle nor Bhargava explicitly makes conjectures about this situation, but their philosophy clearly entails that one expects the number of such extensions of discriminant at most X to be on the order of X.

For instance, Bhargava's conjectures would suggest that the number of totally real S_n-extensions of $\mathbf{Q}$ with squarefree discriminant between X and $2X$ should be asymptotic to $(1/n!)\zeta(2)^{-1}X$. Given the hypothesis HS_α, Theorem 11.1.1 would supply the analogous function field statement: the number of S_n-extensions of $\mathbf{F}_q(t)$, totally split at ∞, with discriminant exactly q^m is asymptotic to

$$\frac{C(\mathfrak{m})}{|\operatorname{Aut}_{S_n}(\mathbf{F}_q((t))^{\oplus n!})|} = \frac{1}{n!\zeta_{\mathbf{F}_q[t]}(2)}q^m$$

Similarly, if G is a subgroup of S_n with a single minimal-index conjugacy class $\mathcal{C}$, then the discriminant of a cover with m branch points of type $\mathcal{C}$ is q^{am}, and HS_α would imply

$$N_{G,\mathcal{C}}(q^{am}) \sim \gamma q^m$$

for some explicitly computable constant γ, just as Malle's conjecture predicts [11]

Of course, the homological stability condition HS_α is not known in general, and in particular is not known in the case of $G = S_n$ for $n > 3$. Fortunately, the class of groups for which HS_α has been proved is precisely the one which is relevant to the Cohen-Lenstra statistics, as we explain in the section that follows.

12. The Cohen–Lenstra heuristics

Let G be a generalized dihedral group of the form $A \rtimes \mathbf{Z}/2\mathbf{Z}$, where A is a finite abelian group of odd order, on which $\mathbf{Z}/2\mathbf{Z}$ acts as -1. Let c be the conjugacy class of G containing all involutions. In this case, the condition HS_α that is an assumption to Theorem 11.1.1 has been proved in the paper [EVW09] (for some α which could be made explicit if desired.) Moreover, the Schur multiplier of G has odd order. In this case,

$$\tilde{G} = \mathbf{Z} \times_{\mathbf{Z}/2\mathbf{Z}} \tilde{G}_c. \tag{12.0.2}$$

(see Definition 7.3 for the definition of $\tilde{G}_c$), and the discrete action of Γ is the powering action on the right and the trivial action on the left, i.e. $\alpha : (n, g) \mapsto (n, g^\alpha)$. See §9.3.2 for more discussion.

From Theorem 11.1.1 we can now derive results towards the Cohen–Lenstra heuristics over function fields, as promised in §1.1.

12.1. We begin by discussing the Cohen–Lenstra heuristics in the "standard" case, i.e. ℓ-parts of class groups of "imaginary" quadratic extensions of a field with no ℓth roots of unity.

Let q be an odd prime power, and for each odd m let $\mathcal{Q}_m$ be the set of quadratic extensions $F \supset \mathbf{F}_q(t)$ obtained by adjoining the square root of a monic polynomial $f(t)$ of degree m. These are the so-called *imaginary quadratic* extensions of $\mathbf{F}_q(t)$,

[11] Our results are of a slightly different nature to Malle's conjecture, which means that we cannot immediately pass from them to Malle's conjecture in its originally stated form (or vice versa, for that matter). To see why, note that we compute the number of fields of multidiscriminant $\mathfrak{m}$ when the multiplicity of every conjugacy class in $\mathfrak{m}$ grows large. On the other hand, Malle's conjecture asks for an average of this number over $\mathfrak{m}$, including the case where the multiplicity of some conjugacy classes in c stays fixed, while others grow.

named so because they are necessarily ramified at ∞. For any such extension F, we denote by C_F the group $\mathrm{Pic}^0(C)(\mathbf{F}_q)$, where $C/\mathbf{F}_q$ is the unique smooth curve with function field F.

12.1.1. Theorem. *Let A be a finite abelian group of odd order. There exists a real number $Q(A)$ such that, for all odd $q > Q(A)$ with $q-1$ prime to $|A|$,*

$$\lim_{m\to\infty} \frac{\sum_{F\in\mathcal{Q}_m} |\mathrm{Epi}(C_F, A)|}{|\mathcal{Q}_m|} = 1. \tag{12.1.1}$$

where $\mathrm{Epi}(C_F, A)$ *denotes the set of surjective homomorphisms from C_F to A, and m ranges over positive odd integers.*

Prior to giving the proof, let us recall the connection with counting (G, c)-extensions, in the sense of §10.2:

Any epimorphism $\mathrm{Pic}^0(C)(\mathbf{F}_q) \twoheadrightarrow A$ extends uniquely to a surjection $\mathrm{Pic}(C)(\mathbf{F}_q) \twoheadrightarrow A$ which is trivial on the divisor class of the unique point of C above ∞. Equivalently, any epimorphism $\mathrm{Pic}^0(C)(\mathbf{F}_q) \twoheadrightarrow A$ determines an étale cover $Y \to C$, together with an isomorphism $\mathrm{Aut}(Y/X) \xrightarrow{\sim} A$, with the property that Y is totally split above the point of C above ∞. Then Y is also Galois over $\mathbf{P}^1$, and the above isomorphism extends to an isomorphism

$$\mathrm{Aut}(Y/\mathbf{P}^1) \xrightarrow{\sim} G. \tag{12.1.2}$$

This extension is not unique, but it is unique up to A-conjugacy.

Let L be the function field of Y; then, choosing an isomorphism as in (12.1.2), we give L the structure of a (G, c)-extension; any two isomorphisms as in (12.1.2) give isomorphic (G, c)-extensions.

Since m is odd and f is monic, the completion L_∞ of L at ∞ is isomorphic to

$$\mathbf{F}_q((\sqrt{z})) \times \dots \mathbf{F}_q((\sqrt{z})), \quad |A| \text{ copies}, \tag{12.1.3}$$

or, in the terms of the previous section, the infinity type of L is (up to conjugacy) the map $\Delta \to G$ sending F_Δ to the identity and sending $\hat{\mathbf{Z}}(1)$ to a cyclic group $\langle\delta\rangle$ of order 2.

In summary, given an epimorphism $C_F \twoheadrightarrow A$ as in the Theorem, we construct (up to isomorphism) a (G, c)-extension L such that L_∞ is isomorphism to (12.1.3). This construction is two-to-one:

Given a (G, c)-extension L with such that L_∞ is isomorphic to (12.1.3), it is possible to recover (the curve C and) the surjection $\mathrm{Pic}^0(C)(\mathbf{F}_q) \twoheadrightarrow A$ *up to sign.* The issue of sign arises because of the difference between A-conjugacy and G-conjugacy of the isomorphism (12.1.2).

Proof. Our discussion above shows (cf. [EVW09, Prop 8.6]) the sum in the numerator of (12.1.1) is twice the number of (unmarked) m-branched regular (G, c)-extensions $L/\mathbf{F}_q(t)$ with the property that the completion L_∞ of L at ∞ is isomorphic to (12.1.3).

The automorphism group of L_∞ (or, what is the same, the centralizer of δ in G) has order 2. So Theorem 11.1.1 tells us that the number of (G, c)-extensions with infinity type isomorphic to L_∞ and multidiscriminant m is asymptotic to

$$(1/2)B(L_\infty, m)C(m).$$

In the present context, $B(L_\infty, m)$ (as in Definition 11.1) counts the number of elements of $\mathbf{Z} \times_{\mathbf{Z}/2\mathbf{Z}} \tilde{G}_c$ which are fixed by $(n, g) \mapsto (n, g^q)$, and which project to

$m \times \delta^{-1} \in \mathbf{Z} \times G$. (Here, δ is any fixed element of order 2 in G). This amounts to computing the number of elements g of $\tilde{G}_c$ which project to $\delta^{-1} = \delta$ and which satisfy $g^{q-1} = 1$. Since $q-1$ is prime to $|A|$ (and hence also to $|H_2(G, c, \mathbf{Z})|$) this means that $g^2 = 1$. Since δ lifts *uniquely* to an involution in $\tilde{G}_c$, we have that $B(L_\infty, m) = 1$.

The numerator of (12.1.1) is twice the number of (G, c)-extensions, and $C(m)$ is the cardinality of $\mathcal{Q}_m$; this completes the proof. □

From Theorem 12.1.1 it is easy to produce asymptotic statements of the form mentioned in the introduction. For example: for each odd integer d the number of elements of C_F of exact order d is the same as the number of surjections from C_F to $\mathbf{Z}/d\mathbf{Z}$, which we now know to be 1 on average, at least once q is large enough and $q-1$ is prime to n; so, for instance, the average imaginary quadratic extension of $\mathbf{F}_q(t)$ has one point of exact order 15, one of order 5, one of order 3, and one of order 1, for a total of four 15-torsion points, as claimed in the introduction.

The more traditional formulation of the Cohen-Lenstra conjecture says that for any ℓ-group B, the fraction of $L \in \mathcal{Q}_m$ with $C_L[\ell^\infty] \cong B$ is asymptotically

$$\frac{\prod_{i=1}^{\infty}(1 - \ell^{-i})}{|\operatorname{Aut}(B)|}. \tag{12.1.4}$$

That statement would follow if Theorem 12.1.1 were known to hold for all A with a fixed q, i.e. if $Q(A)$ could be chosen independent of A.

It is also straightforward to derive results analogous to Theorem 12.1.1 where one varies the splitting type at infinity, e.g. when F_∞ is an unramified extension of $\mathbf{F}_q(t)$. We discuss instead the more interesting situation where one relaxes the condition that $q-1$ be relatively prime to the order of A.

12.2. Roots of unity. Let us now consider the case when $q-1$ has prime factors in common with the order of A – that is to say, the ground field contains roots of unity which are killed by the order of A.

Malle [Mal10] observed that, in the corresponding situation over number fields, the Cohen–Lenstra conjectures in the form (12.1.4) need modification. For example, he conjectured that if K is a totally real number field satisfying

$$\mu_\ell \subset K, \quad \mu_{\ell^2} \not\subset K$$

the fraction of totally imaginary quadratic extensions L with $\mathbf{C}_L[\ell^\infty]$ trivial should be

$$\prod_{i=1}^{\infty}(1 + \ell^{-i})^{-1} \tag{12.2.1}$$

Our results, in the function field case, cover this situation also. They show that the $\mathbf{F}_q$-rational connected components of the relevant Hurwitz space are identified with the fixed space of $\wedge^2 A$ under the "Frobenius" map $F : x \mapsto x^q$. In particular, we have proven:

Fixing A, there exists $Q(A)$ such that whenever $q > Q(A)$ we have

$$\lim_{X\to\infty} \frac{\sum_{L\in\mathcal{Q}_m} |\operatorname{Epi}(C_L, A)|}{|\mathcal{Q}_m|} = \#(\wedge^2 A)^F, \tag{12.2.2}$$

Garton [Gar12] has shown that the result above is consistent with predictions arrived at via random ℓ-adic matrix heuristics. He shows, futhermore, that if (12.2.2) holds for a fixed q and all A, then the analogy of Malle's conjecture (12.2.1) with $K = \mathbf{F}_q(t)$ holds. Furthermore, Garton describes corresponding results when the base field contains ℓ^kth roots of unity for $k > 1$.

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