Stanford Number Theory Seminar

On the areas of rational triangles.

NOAM ELKIES, HARVARD UNIVERSITY.

By Heron's formula there exists a triangle of area $\sqrt{a}$ all of whose sides are rational if and only if $a > 0$ and $xyz(x + y + z) = a$ for some rationals x, y, z. In a 1749 letter to Goldbach, Euler constructed infinitely many such (x, y, z) for any rational a (positive or not), remarking that it cost him much effort but not explaining his method. We suggest one approach, using only tools available to Euler, that he might have taken, and use it to construct several other infinite families of solutions. Then we reconsider the problem as a question in arithmetic geometry: $xyz(x+y+z) = a$ gives a K3 surface, and each family of solutions is a singular rational curve on that surface defined over $\mathbf{Q}$. The structure of that Neron-Severi group of that K3 surface explains why the problem is unusually hard. Along the way we also encounter the Niemeier lattices (the even unimodular lattices in $\mathbf{R}^{24}$) and the non-Hamiltonian Petersen graph.

12pm Friday October 2, Room 380F.