2. Interlude: The Euler characteristic of $\mathrm{SL}_2(\mathbf{Z})$ is $\zeta(-1)$

As an amusing diversion, we shall compute the Euler characteristic of $\mathrm{SL}_2(\mathbf{Z})$ (it is $\frac{-1}{12}$). We interpret this as meaning: any *torsion-free subgroup* of index N has Euler characteristic $-N/12$. This also shows that any prime dividing the denominator of $\zeta(-1)$ must be torsion in $\mathrm{SL}_2(\mathbf{Z})$.

It suffices to show that $\mathrm{PSL}_2(\mathbf{Z})$ has Euler characteristic $-1/6$. Indeed, consider some torsion-free subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ of index k and Euler characteristic e; it is necessarily a free group, and so its preimage in $\mathrm{SL}_2(\mathbf{Z})$ is a *split* extension of it. In particular, $\mathrm{SL}_2(\mathbf{Z})$ contains an isomorphic subgroup of index $2k$ and Euler characteristic e.

Explicitly: $\mathrm{PSL}_2(\mathbf{Z})$ has a subgroup of index 6, the kernel of the morphism to $\mathrm{PSL}_2(\mathbb{F}_3) \cong S_3$. This subgroup is isomorphic to the free group on two generators; indeed, the corresponding quotient of $\mathbb{H}$ is analytically isomorphic to the moduli space of elliptic curves with marked 2-torsion, which is parameterized by $\mathbb{P}^1 - \{0, 1, \infty\}$.

Exercise. Deduce the formula for the genus of $X_0(N)$.

If it is so easy, what is the point? The method of proof will be very soft and general, and will generalize to many other arithmetic groups.

2.1. In this section, we will begin to consistently work with $X^{(1)}$, the space of lattices of area 1; we shall also use the series $E_f^*(\Lambda) := \sum_{v \in \Lambda_{prim}} f(v)$. Here Λ_{prim} consists of primitive vectors in the lattice Λ.

We often use the word "lattice" to mean "lattice of area 1."

The inclusion $X^{(1)} \hookrightarrow X$ induces a bijection $X^{(1)}/\mathrm{SO}_2 \cong X/\mathbf{C}^\times$. Thus, the quotient $X^{(1)}/\mathrm{SO}_2$ is identified with $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$. Explicitly: given $z \in \mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$, we associate to it the lattice $\frac{1}{\sqrt{y}}\langle 1, z\rangle$; given a lattice $L \in X^{(1)}$, we rotate it so its shortest vector is of length 1, and let z be any other vector which generates L.

Now the measure $\frac{dx\ dy}{y^2}$ on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ normalizes a measure on $X^{(1)}$. It is enough to give a functional on continuous compactly supported functions; given a function on $X^{(1)}$, average it so it is rotation invariant; it is then identified with a function on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$, so integrate it with respect to $\frac{dx\ dy}{y^2}$. The result is – not wholly obviously – an $\mathrm{SL}_2(\mathbf{R})$-invariant measure μ on $X^{(1)}$.

In the actual lecture I droned on and on and on and on about this measure.

2.2. We now claim that $\int E_f^* d\mu = \frac{2}{\pi} \int f$ for any $f \in C_c(\mathbf{R}^2)$.

The map $f \mapsto \int E_f^*$ defines a functional on $C_c(\mathbf{R}^2)$, i.e., a measure on $\mathbf{R}^2$. We claim the only $\mathrm{SL}_2(\mathbf{R})$-invariant measures are linear combinations of δ_0 and Lebesgue. To see this, it suffices to show that any signed measure μ satisfying $\mu\{0\} = 0$ is a multiple of Lebesgue; now, μ is $\mathrm{SL}_2(\mathbf{R})$-invariant, and by a "smoothing" argument it is absolutely continuous with respect to Lebesgue measure; finally, the R-N theorem implies that μ has a density with respect to Lebesgue measure, which is an $\mathrm{SL}_2(\mathbf{R})$-invariant function, thus constant.

Therefore,

$$\int E_f^* \, d\mu = a \int f + bf(0) \tag{2}$$

for certain $a, b \geq 0$.

Now let us take f to be the characteristic function of a disc of radius R (we can easily check that (2) remains valid). For $R \leq 1$, the function E_f^* – thought of as a function on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ – is twice the characteristic function of that part of the standard fundamental domain for $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ where $y \geq R^{-2}$; this implies that $b = 0$ and $a = \frac{2}{\pi}$.

2.3. Continue to let f be the characteristic function of a disc of radius R. Recall that the fraction of pairs of integers with coprime coordinates is $1/\zeta(2)$. For any fixed lattice Λ,

$$E_f^*(\Lambda) \sim \frac{1}{\zeta(2)} \int f + O(R).$$

Note that $E_f(\Lambda) \leq \mathrm{const} \cdot \int f$. The point is that a lattice cannot have too many *primitive* points in a large disc. So, in the formula

$$\int \frac{E_f}{\int f} = \frac{2}{\pi}$$

we may let $R \to \infty$ and switch order of limit and integral, to arrive at

$$\zeta(2)^{-1} \mathrm{vol}(\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}) = \frac{2}{\pi}$$

whence $\mathrm{vol}(\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}) = \frac{2}{\pi}\zeta(2)$.

2.4. Recall the Gauss-Bonnet formula: the total curvature of a Riemannian surface M equals $2\pi\chi(M)$. This follows from the angle defect formula: the angle defect of a geodesic triangle equals the integral of curvature over it. Apply[1] this to $\mathbf{H}/\mathrm{PSL}_2(\mathbf{Z})$. The Euler characteristic is now $\frac{-\zeta(2)}{\pi^2}$; the Euler characteristic of $\mathrm{SL}_2(\mathbf{Z})$ is thus $\frac{-\zeta(2)}{2\pi^2} = \zeta(-1)$.

This *shows* that $\zeta(-1) \in \mathbf{Q}$.

2.5. The same proof works more generally (i.e., to compute Euler characteristics of arithmetic groups). We will see later how to set things up so that there is no difference between $\mathbf{Q}$ and a number field. One of the key ideas in simplifying the general story is the construction of a canonical measure, the *Tamagawa measure*, making our *ad hoc* constructions unnecessary.

Examples.

[1]In fact, this is noncompact; one should therefore consider the manifold with boundary obtained by setting $y \leq T$ on a standard fundamental domain; the Gauss-Bonnet theorem remains true up to an error term which comes measures the "kink" at the boundary of this manifold. This approaches 0 as $T \to \infty$.

(1) See *A Gauss-Bonnet formula for discrete arithmetically defined groups* by Harder, 1971), The Euler characteristic of the symplectic group is

$$\zeta(-1)\zeta(-3)\ldots\zeta(1-2n).$$

It may be interesting to compare this expresion with the formula for the size of $\mathrm{Sp}(\mathbb{F}_p)$, i.e.

$$\#\mathrm{Sp} = q^{\dim}(1-q^{-2})(1-q^{-4})\ldots(1-q^{-2n})$$

So we have obtained yet another proof that $\zeta(\text{negative odd}) \in \mathbf{Q}$.

(2) The Euler characteristic of $\mathrm{SL}_n(\mathbf{Z})$ is zero for $n \geq 3$.

2.6. *Exercise.* Show that space $X_n^{(1)}$ of lattices in $\mathbf{R}^n$ of volume 1 carries an $\mathrm{SL}_n(\mathbf{R})$-invariant finite measure μ; it follows there exists an $\mathrm{SL}_n(\mathbf{R})$-invariant probability measure, i.e., a notion of a *random lattice.*

Moreover, if E_f^* is defined identically, prove that for some $b > 0$,

$$\int E_f^* = b \int f.$$

Deduce that if S is a sphere of volume < 1, there exists a lattice $L \in X_n^{(1)}$ with $L \cap S = \emptyset$. In other words: *there exists a sphere packing in* $\mathbb{R}^n$ *of density* $\geq 2^{-n}$. This theorem – originally due to Minkowski, but this simple proof is due to Siegel– has not been improved except for multiplication by a linear factor in n !