

## 1. HOLOMORPHIC STORY

$X$  is the space of lattices in  $\mathbb{C}$ ;  $E_k$  is the function on  $X$  described by  $\Lambda \mapsto \sum_{\omega \in \Lambda} \omega^{-k}$ .

**1.1.**  $X$  has the structure of a complex analytic variety. It parameterizes pairs  $(E, \omega)$ , where  $E$  is a complex torus and  $\omega$  is a nonvanishing holomorphic 1-form on  $E$ : to each lattice  $\Lambda$ , we associate  $(\mathbb{C}/\Lambda, dz)$ .

The space  $X$  is also naturally the space of complex points of an algebraic variety defined over  $\mathbf{Q}$ , namely, a moduli space parameterizing elliptic curves together with an everywhere nonvanishing differential form. Such pairs are “rigid,” i.e. never have any automorphisms; each such pair is isomorphic to a unique pair  $(y^2 = 4x^3 - g_2x - g_3, dx/y)$  with  $\Delta(g_2, g_3) \neq 0$ .

Thus the underlying affine variety is the complement of a curve in  $\mathbb{A}^2$ , and the ring of functions is the localization of  $\mathbf{Q}[g_2, g_3]$  at the complement of the discriminant divisor.

An interesting class of  $\overline{\mathbf{Q}}$ -points of  $X$  comes from the “complex-multiplication” tori. The elliptic curve  $y^2 = x^3 - x$  admits complex multiplication by  $\mathbf{Z}[i]$ ; take the differential form  $\omega = \frac{dx}{y}$ ; set  $\gamma = \frac{1}{2} \int_0^1 \frac{dx}{y} = \int_0^1 \frac{dt}{\sqrt{1-t^4}}$ . Then the lattice defined by  $\gamma \cdot \mathbf{Z}[i]$  is a rational point of  $X$ . I will return to this later.

*Exercise.* Check it.

**1.2.** Recall that a weight  $k$  holomorphic form on  $\mathbb{H}/\mathrm{SL}_2(\mathbf{Z})$  is a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfying  $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$  together with a growth condition.

Such an object is equivalent to a function on lattices, transforming as  $f(z\Lambda) = z^{-k} f(\Lambda)$ ; given the function on lattices, the map  $z \mapsto f(\langle 1, z \rangle)$  is a weight  $k$  holomorphic form. The vector space of such forms will be called  $M_k$ .

$M_k$  possesses a rational structure, arising from the  $\mathbf{Q}$ -structure on  $X$ . We think of this as a  $\mathbf{Q}$ -vector subspace  $M_{k, \mathbf{Q}} \subset M_k$ . In other terms,  $f \in M_k$  belongs to  $M_{k, \mathbf{Q}}$  if it belongs to the ring of regular functions on the underlying affine  $\mathbf{Q}$ -variety. We speak of a holomorphic form as being *rational* if it belongs to  $M_{k, \mathbf{Q}}$ .

The functions “ $g_2, g_3$ ” belong to  $M_{4, \mathbf{Q}}, M_{6, \mathbf{Q}}$ , by definition. They generate  $\oplus M_k$  as a  $\mathbf{C}$ -algebra, and so also  $\oplus_k M_{k, \mathbf{Q}}$  as a  $\mathbf{Q}$ -algebra. In other terms, a rational modular form is a polynomial in  $g_2, g_3$  with  $\mathbf{Q}$ -coefficients.

The theory of the Weierstrass  $p$ -function gives explicit forms for  $(g_2, g_3)$  as functions of  $z$ : it provides an explicit map

$$(\mathbb{C}/\Lambda, dz) \xrightarrow{(\wp, \wp')} (y^2 = 4x^3 - g_2x - g_3, dx/y),$$

where  $g_2 = 60E_4, g_3 = 140E_6$ . In particular,  $E_4, E_6$  are rational modular forms.

The space of cusp forms – we shall denote it  $S_k$  – is a  $\mathbf{Q}$ -rational subspace. One needs to verify here that “the point at  $\infty$ ” is a  $\mathbf{Q}$ -rational point on a compactification of  $X/\mathbf{C}^\times$  (see below).

*Remark.* It follows by computation of the Fourier expansion of  $E_4, E_6$  – we shall carry this out below – that, for

$$f(z) = \sum_{n \geq 0} a_n q^n \in M_k, \quad q = \exp(2\pi iz).$$

that  $f$  is rational  $\implies a_n \in \pi^k \mathbf{Q}$  for all  $n$ . The converse is also true: Take  $f$  so that  $a_n \in \pi^k \mathbf{Q}$  for all  $n$ ; take a basis  $f_i$  for  $M_{k, \mathbf{Q}}$ , and write  $f = \sum a_i f_i$ ; the  $a_i$  are uniquely determined. Equating Fourier expansions, we see that the  $a_i$  are the unique solution to a system of  $\mathbf{Q}$ -linear equations, and so themselves rational. This type of result is known as “ $q$ -expansion principle.”

*Remark.* Let  $Y = X/\mathbf{C}^\times$ . We think of it as parameterizing complex tori up to homothety. The map

$$z \in \mathbf{H} \rightarrow \langle 1, z \rangle \in Y$$

descends to a set-theoretic bijection between  $\mathbf{H}/\mathrm{SL}_2(\mathbf{Z})$  and  $Y$ . The space  $Y$  is again the space of complex points of an algebraic orbifold (i.e., stack) defined over  $\mathbf{Q}$ , namely, the moduli stack of elliptic curves.  $X \rightarrow Y$  defines a  $\mathbf{C}^\times$ -torsor, and so gives rise to a line bundle  $\mathcal{L}$  over  $Y$ ; we can think of a weight  $k$  holomorphic form as a section of  $\mathcal{L}^{-2k}$ . Note that  $\mathcal{L}^{-2}$  is isomorphic to the canonical bundle, via the association  $f \mapsto f(z)dz$ .

**1.3.** Recall that for every integer  $n \geq 1$  we have an endomorphism  $T(n)$  of  $M_k$ . At the level of lattices, this may be defined as:

$$T(n)f = \frac{1}{n} \sum_{\Lambda' \supset \Lambda, [\Lambda':\Lambda]=n} f(\Lambda').$$

These “Hecke” endomorphisms commute and preserve the rational structure, i.e.  $T(n)M_{k, \mathbf{Q}} \subset M_{k, \mathbf{Q}}$ . The reason is that we can understand  $T(n)$  as arising from a correspondence on  $X$ : Let  $X_0(n)$  be the variety parameterizing pairs  $(E, \omega, E', \omega', f)$ , where  $f : E \rightarrow E'$  is an isogeny of degree  $n$  so that  $f^*\omega' = \omega$ . It is again defined over  $\mathbf{Q}$ , and there is an evident map  $X_0(n)_{\mathbf{Q}} \hookrightarrow X_{\mathbf{Q}}^2$ . Then  $T(n)$  is defined by pulling back functions from  $X$  to  $X_0(n)$  along one projection, and pushing forward along the other. (The second operation is a trace, considered algebraically.)

Moreover, the  $T(n)$  satisfy the Hecke relation

$$T(n)T(m) = \sum_{d|(n,m)} T(nm/d^2)d^{k-1}$$

In particular,  $T(p^a)T(p^b)$  is a sum of  $T(p^s)$  where  $s$  goes from  $|b-a|$  to  $b+a$  in steps of 2. It is no coincidence that this coincides with the Clebsch-Gordon formula for tensor product of  $\mathrm{SL}_2$ -representations!

The Hecke operators were originally introduced to explain multiplicative properties of Fourier coefficients: the first Fourier coefficient of  $T(n)f$  equals the  $n$ th Fourier coefficient of  $f$ . They are also self-adjoint with respect to a

natural inner product, the ‘‘Petersson inner product’’  $\int f \bar{g} y^k \frac{dx dy}{y^2}$ . In particular, the eigenvalues of  $T(n)$  on  $S_k$  are totally real algebraic numbers: real by self-adjointness, closed under conjugation since they preserve a rational structure, and so totally real algebraic.

**1.4.** The function  $\Lambda \mapsto \sum_{0 \neq \omega \in \Lambda} \omega^{-k}$  is the classical holomorphic Eisenstein series. We denote by  $E_k$  the associated weight  $k$  holomorphic form.

A fundamental result is that  $E_k$  is a rational form. Since  $[\gamma \cdot \mathbf{Z}[i]] \in X(\mathbf{Q})$ , we see (Hurwitz)

$$\sum (a + bi)^{-k} \in \gamma^k \mathbf{Q}, \quad 4|k.$$

which generalizes to a result about the Dedekind  $\zeta$ -functions of imaginary quadratic fields.

*Exercise.* Explain how to compute  $\sum (a + bi)^{-k}$  rapidly.

**1.5.** We prepare for the proof by computing the action of the Hecke operator on  $E_k$ :

$$T(n)E_f(\Lambda) = \sum_{v \in \Lambda} z^{-k} a(z),$$

where  $a(z)$  is the number of index- $n$  superlattices that contain  $z$ . For  $n$  prime,  $a(z) = 1$  for  $z \in n^{-1}\Lambda - \Lambda$  and  $n + 1$  for  $z \in \Lambda$ ; in particular,  $T(n)E_f = (1 + n^{k-1})E_f$ .

More generally:  $E_k$  is an eigenvector of  $T(n)$  with eigenvalue  $\sum_{d|n} d^{k-1}$ .

**1.6. First proof.** Fix a prime  $p$ . Then I claim that:

$$(1) \quad \mathbf{C}.E_k = \ker(T_p - p^{k-1} - 1).$$

This implies that the line spanned by  $E_k$  is rational, i.e.  $E_k$  is a multiple of an element of  $M_{k,\mathbf{Q}}$ .

We may proceed in two ways:

- Suppose that  $f$  belongs to the kernel. Taking into account the splitting of  $M_k$  into  $\mathbf{C}.E_k$  and cusp forms, we may suppose that  $f$  is a cusp form. Let us now show that every eigenvalue of  $T_p$  on cusp forms is at most  $p^{k/2} + p^{k/2-1}$  in absolute value. Let  $\Lambda_0$  be a lattice so that  $|f(\Lambda)| \cdot \text{area}(\mathbf{C}/\Lambda)^{k/2}$  is maximized; the existence of such amounts to the fact that, considered as a function on  $\mathbf{H}/\text{SL}_2(\mathbf{Z})$ , the function  $x + iy \mapsto y^k |f(x + iy)|^2$  attains its maximum. Then:

$$|T_n f(\Lambda_0)| \leq (1 + 1/p)p^{k/2} |f(\Lambda_0)|.$$

This establishes the bound on eigenvalues, and therefore also (1).

- Any other solution  $E'_k$  would have Fourier coefficients that were a scalar multiple of those of  $E_k$ ; in particular, some linear combination of  $E_k, E'_k$  would be *constant*, which is an obvious contradiction.

*End of lecture 1.*

It is not hard to deduce the result from this, using the fact  $\zeta(2k) \in \pi^{2k}\mathbf{Q}$ . Observe, first of all, that the constant term of  $E_k$  equals  $\zeta(2k)$ .

Let us suppose that  $E_k = \alpha E'_k$  with  $E'_k$  rational. Since the space of cusp forms and the space of Eisenstein series is defined over  $\mathbf{Q}$ , there exists a rational constant  $\beta$  so that  $E'_k - \beta E_4^a E_6^b$  is a cusp form. Compare constant terms: we see that  $\beta = \frac{\alpha^{-1}\zeta(2k)}{\zeta(4)^a \zeta(6)^b} \in \mathbf{Q} \cdot \alpha$ , so  $\alpha$  is rational.

*Remark.* Although the second proof is more complicated, it is more robust. The fact that Hecke eigenvalues determine the Fourier expansion is actually rather special to the present case. It fails even in the case of half-integral weight.

**1.7. Second proof.** We will make use of the  $q$ -expansion principle (but nothing about  $\zeta$ ).

We compute the Fourier expansion of  $E_k, E_k^*$  and use the  $q$ -expansion principle. By the Hecke relations,

$$E_k = a_0 + a_1 \sum_n d_{k-1}(n)q^n.$$

Let  $z = iy$  and take  $y \rightarrow 0$ . Then

$$E_k(z) = (iy)^{-k} E_k(i/y) = \pm a_0 y^{-k} + \text{exponential decay.}$$

In other terms,  $\sum_{i \geq 1} a_i \exp(-2\pi n y) = -a_0 + \text{const} y^{-k} + o(1)$ . This is one definition of the “regularized summation” of a sequence  $a_i$ : remove the most singular term. We shall see in a moment that this coincides with the  $\zeta$ -regularization of computing the value at  $s = 0$  of  $\sum a_n n^{-s}$ . In our case, the latter function is  $\zeta(s)\zeta(s+1-k)$ . We conclude:

$$a_0 = -a_1 \sum d_{k-1}(n) = -a_1 \zeta(0)\zeta(1-k).$$

Thus we conclude that

$$E_k = a_1 \left( -\zeta(0)\zeta(1-k) + \sum d_{k-1}(n)q^n \right).$$

We may explicitly compute  $a_1$ ; we start by differentiating

$$\pi \cot(\pi z) = \sum \frac{1}{z+n} = -i\pi(1 + 2q + 2q^2 + \dots).$$

and from this deduce that  $a_1 = 2(2\pi i)^k / (k-1)!$ .

Since  $-a_1 \zeta(0)\zeta(1-k) = \zeta(2k)$ , we deduce the functional equation for  $\zeta$  (known to Euler); from the  $q$ -expansion principle we also get  $\zeta(0)\zeta(1-k) \in \mathbf{Q}$  and that  $E_k$  is rational.

**1.8.** The fact that the constant terms of  $E_k$  is related to Eisenstein series is related to  $\zeta$ -functions is of great arithmetic interest and importance. For example, the “Ramanujan congruence”

$$\tau(n) \equiv \sigma_{11}(n) \text{ modulo } 691$$

arises from the fact that  $\zeta(-11) = \frac{691}{32760}$ . In turn, this forces the “Galois representation” associated to  $\Delta$  modulo 691 to have the shape  $\begin{pmatrix} \omega^{11} & 0 \\ \star & 1 \end{pmatrix}$ . The restriction of  $\star$  to the Galois group of  $\mathbf{Q}(\zeta_{691})$  defines a homomorphism into  $\mathbf{Z}/691$ . It is in fact *unramified*.

The resulting connection between divisibility of  $\zeta$ -values and the class groups of cyclotomic fields was established by Ribet: the *converse to Herbrand’s theorem*.

**1.9.** Now let us return to the issue of making  $\sum d_{k-1}(n) = \zeta(0)\zeta(1-k)$  rigorous.

Given any “reasonable” Dirichlet series  $L(s) = \sum a_n n^{-s}$ , absolutely convergent when  $\Re(s)$  is large, holomorphic at  $s = 0$ , and any smooth compactly supported function  $f$  with  $f(0) = 1$ ,

$$\sum a_n f(n/X) = \sum_i a_i X^{\alpha_i} (\log X)^{n_i} + L(0).$$

The  $a_i$ s depend on the function  $f$ , but the lowest order term does not (or rather it depends only on  $f(0)$ .)

In particular,  $\zeta$ -regularization agrees with the regularization we previously employed.

The formula  $\zeta(s) = \lim_N \left( \sum_{n=1}^N n^{-s} - \frac{N^{1-s}}{1-s} \right)$  ( $0 < \Re(s)$ ) was well-known to Hardy; the point is that by using a smooth function one can easily get a corresponding formula valid for all  $s$ .

*Proof.* Mellin inversion; exercise. Express  $f(x)$  as a continuous linear combination of functions  $x^{-s}$  and shift contours. Describe some conditions for the validity of the above.

*Example.* Taking  $f(n) = e^{-x}$ . In that case, we arrive at the usual expression of negative  $\zeta$ -values in terms of Bernoulli numbers. For instance,

$$\zeta(0) = \lim_{q \rightarrow 0} \left( \frac{1}{e^q - 1} - \frac{1}{q} \right) = \frac{-1}{2}.$$

*Exercise.* Compute  $\zeta(-1)$  by hand.

**1.10. Summary.** Let us summarize our proof: The Eisenstein series was uniquely determined by its Hecke eigenvalues, so it inherited the property of rationality. From this we deduced corresponding results for  $\zeta$ -functions.

One enormous advantage of this proof is that it generalizes to the case where we replace  $\mathbf{Q}$  by a totally real field  $K$ .

We may also try to “inherit” more subtle features from the Hecke eigenvalues. For instance, the map  $k \mapsto d_{k-1}(n)$  is even  $p$ -adic analytic in  $k$  if we fix the congruence class of  $k$  modulo  $p-1$ . One can then (try to) “transfer” this to the  $p$ -adic analytic continuation of  $E_k$  and thereby to  $\zeta(1-k)$ , or, for that matter, to the zeta-functions of imaginary quadratic fields. This is

the *p*-adic interpolation of *Eisenstein series*, a topic which I hope to discuss more later.