Do any four questions.

(a) Let V_n be the $(n+1)$-dimensional representation of SU_2 on homogeneous polynomials of degree n.
   (i) Describe an invariant inner product on V_n.
   (ii) Because V_n is the unique irreducible representation of dimension n, it is self-dual. Does SU_2 preserve a symmetric or a skew-symmetric form on V_n?
   (iii) Decompose $V_n \otimes V_m$ into irreducible SU_2-representations.

(b) (Schur orthogonality relations, general case). Let G be a compact Lie group, U, V two irreducible representations of V, and $u, u' \in U, v, v' \in V$. Fix invariant inner products on U and V. Show that $\int_G \langle gu, u' \rangle_U \overline{\langle gv, v' \rangle} = 0$ if U is not isomorphic to V, and evaluate it when U is isomorphic to V.

(c) Prove the fact stated in class: the space of finite functions on $G = \mathrm{SU}_2$ consists precisely of polynomials in the coefficients $a, b, \overline{a}, \overline{b}$ of $g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$.

(d) Let P_n be the space of homogeneous polynomials on $\mathbf{R}^3$ of degree n. Decompose P_n into irreducible representations for the action of SO_3.

Hint: One explicit way to do this is as follows: let $r : P_{n-2} \to P_n$ be defined by "multiplication by $x^2 + y^2 + z^2$" and $l : P_n \to P_{n-2}$ be the Laplacian operator $\partial_{xx} + \partial_{yy} + \partial_{zz}$. Prove that $P_n = \mathrm{image}(r) \oplus \mathrm{kernel}(l)$.

(e) Let $\rho : \mathrm{SU}_2 \to \mathrm{U}_n(\mathbf{C})$ be a *measurable* homomorphism (i.e., ρ is measurable and $\rho(x)\rho(y) = \rho(xy)$ for almost all (x, y)). Prove that ρ is actually continuous (that is to say, there exists a continuous homomorphism P so that $\rho = P$ almost everywhere).

(f) Let $P : U_n \to \mathbf{C}$ be a polynomial in the matrix entries and their complex conjugates, with rational coefficients. Prove that $\int_{U_n} P(u)du \in \mathbf{Q}$, the integral being taken with respect to the Haar measure of mass 1.
Hint. Let W be the space of $\mathrm{GL}_n(\mathbf{C})$-invariants on $(\mathbf{C}^n)^k \otimes ((\mathbf{C}^n)^*)^{\otimes \ell}$. Explain why $W_{\mathbf{Q}}$ spans W, where $W_{\mathbf{Q}}$ is defined similarly but replacing $\mathbf{C}$ by $\mathbf{Q}$. Now deduce the statement from this.