This is a version with some typos in Problem 1 corrected, the word "irreducible" added in part (b) of problem 2, and hints added. I apologize, this problem set is quite hard; please don't despair if you can't do some of the questions.

(1) Let τ be the two-dimensional irreducible representation of S_4, so that S_4 acts on $V = \{(x, y, z) \in \mathbf{C}^3 : x + y + z = 0\}$ via the surjection $S_4 \to S_3$.

Prove that there exists a vector $v_0 \in V$ so that $(12)v_0 = (34)v_0 = -v_0$ and $(13)(24)v_0 = v_0$ and $v_0 + (234)v_0 + (324)v_0 = 0$.

Now let S_τ the corresponding Schur functor, so $S_\tau(V) = \mathrm{Hom}_{S_3}(\tau, V^{\otimes 4})$. Prove that the rule $\phi \in S_\tau(v) \mapsto \phi(v_0)$ defines an isomorphism from $S_\tau(V)$ to the 4-tensors $x \in V^{\otimes 4}$ that satisfy

$$\{(12)x = (34)x = -x, (13)(24)x = x, x + (234)x + (342)x = 0\}.$$

(Note that the right-hand side expresses the symmetries of the Riemann tensor, which is why I chose this ugly presentation of $S_\tau(V)$; a nicer presentation was described in class.)

(2) Let G be a finite group.

Prove that, for any representation ρ of G over the complex numbers, the representation ρ can be realized over a number field K, i.e. there exists a basis such that G acts by matrices with K entries.

Let ρ be irreducible. Prove that the representation $\rho \otimes \tilde{\rho}$ of $G \times G$ can be realized over the field $\mathbf{Q}(\chi)$ generated by all values of characters of ρ; here $\tilde{\rho}$ denotes the dual representation to ρ.

Hint: Let $G \times G$ act on $\mathbf{C}G$ by left and right multiplication, and consider the character χ as an element of $\mathbf{C}G$ (namely the element $\sum_g \chi(g) \cdot g$). The key point is to show that $\{X\chi Y : X, Y \in \mathbf{C}G\}$ is isomorphic as a $G \times G$-representation to $\rho \otimes \tilde{\rho}$.

(3) Describe all the irreducible representations of S_5 over an algebraically closed field F of characteristic 2.

Hint: Let V_1, V_4, V_5 be irreducible one, four and five-dimensional representations of S_5. By what we proved in class, all the three irreducibles of S_5 over F show up in the reductions $\overline{V_1}, \overline{V_4}, \overline{V_5}$ of V_1, V_4, V_5. Show that $\overline{V_1}, \overline{V_4}$ are irreducible. Recall that $\overline{V_5}$ is realized as the representation of S_5 on $\{x_i \in F^6 : \sum x_i = 0\}$ under the homomorphism $S_5 \to S_6$ arising from the action of S_5 on its Sylow 5-subgroups. Note that this contains an invariant line ℓ spanned by $(1, 1, 1, 1, 1, 1)$. Show that $\overline{V_5}/\ell$ is irreducible. It may be helpful here to compute the characteristic polynomial of (12345) acting on this quotient.

(4) Let F be a field of characteristic 2 and V a two-dimensional F vector space; prove that $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^2 V^*$ are not necessarily dual as $\mathrm{GL}(V)$-representations. (Here, recall that $\mathrm{Sym}^p V$ means the quotient of $V^{\otimes p}$ by all expressions $X - \sigma(X)$ with $X \in V^{\otimes p}$.)

Hint: Suppose V is two-dimensional, with basis x, y over a field of characteristic 2. Show that x^2, y^2 span a two-dimensional subrepresentation of $\mathrm{Sym}^2 V$. It now suffices to show that $\mathrm{Sym}^2 V^*$ need not have a one-dimensional subrepresentation!

(5) Prove the *Awesome Lemma* from class: If P and Q are monic integral polynomials, all of whose roots satisfy $x^n = 1$ for some integer with $(n, p) = 1$, and $P \equiv Q \bmod p$, then also $P = Q$.