Exercises, 2

As usual problems after *** are not for assessment. Hand in any five questions; if you are a (post-qualifying exam) graduate student, hand in any two questions.

The first five problems relate to character tables. Problems 6–8 relate to representations over fields besides \(	ext{C}\). Problem 10 studies the importance of “rigidity” in group representations.

1. Let \(D\) (resp. \(Q\)) be the dihedral group of order 8 and the quaternion group of order 8. Write down the character table for \(D\) and \(Q\) and deduce that the group algebras \(\text{C}D\) and \(\text{C}Q\) are isomorphic. Prove that the two-dimensional irreducible representation of \(D\) preserves a symmetric bilinear form, whereas the two-dimensional irreducible representation of \(Q\) preserves a skew-symmetric bilinear form.

2. Compute the character table of \(S_5\).

3. Let \(N\) be the group of upper triangular matrices in \(\text{GL}_3(\mathbb{F}_p)\). Let \(Z\) be the center of \(N\); it is a cyclic \(p\)-group. Prove that, for each nontrivial irreducible character \(Z \to \text{C}^\times\), there exists a unique irreducible representation \((\rho, V)\) of \(N\), up to isomorphism, in which \(Z\) acts by \(\omega\). Describe the character table of \(N\).

4. Compute – in terms of the character table of \(S_d\) – the number of transpositions \(x_1, \ldots, x_g, y_1, \ldots, y_g \in S_d^2\) satisfying

\[
[x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = \text{id},
\]

where \([x, y] = xyx^{-1}y^{-1}\).

This is closely related to counting the number of \(d\)-fold coverings of a compact Riemann surface of genus \(g\). Apparently, the resulting formula was already known to Frobenius.

5. Suppose \(G \subset \text{GL}(V)\) is finite. Prove that every irreducible representation of \(G\) occurs in \(V^{\otimes k}\) for some \(k \geq 1\).

6. Let \(G\) be a finite group. Prove that, for any representation \(\rho\) of \(G\), the representation \(\rho \otimes \bar{\rho}\) of \(G \times G\) can be realized over the field \(\text{Q}(\chi)\) generated by all values of characters of \(\rho\).

Hint: look inside the group algebra. It is not always true that the representation \(\rho\) can be realized over the field \(\text{Q}(\chi)\).

7. Let \(G\) be a finite group. Prove that the number of isomorphism classes \(h\) of irreducible representations of \(G\) over \(\mathbb{F}_p\) is less than or equal to the number \(h'\) of conjugacy classes of \(G\) whose order is prime to \(p\).

(In fact, \(h = h'\) always, as noted in class. One generates “enough” representations by reducing mod \(p\), a process described in the previous exercise set.)

***

8. Let \(F\) be a field of characteristic \(p\) and \(V\) a finite-dimensional \(F\) vector space. Prove that \((V^{\otimes p})_{S_p}\) and \((V^{\otimes p})_{S_p}\) are not necessarily isomorphic as \(\text{GL}(V)\)-representations. In other words, \(\text{Sym}^p V\) and \(\text{Sym}^p V^*\) are not dual as \(\text{GL}(V)\)-representations.

Notation: we have used \(\text{W}^G\) for the \(G\)-invariant vectors in \(W\) and \(\text{W}^G\) for the \(G\)-coinvariants, i.e. the quotient \(W/[gw - w : w \in W]\), i.e. the
largest quotient of $W$ on which $G$ acts trivially. Note that $(V^\otimes p)_{S_p}$ is the usual definition of symmetric power.

(9) Construct a group $\Gamma$ and two element-wise conjugate homomorphisms $\phi_1, \phi_2 : \Gamma \to \text{GL}(V)$ so that $\phi_1, \phi_2$ are not conjugate.

(10) Let $\Gamma$ be a finitely generated group and let $\rho : G \to \text{GL}(n, \mathbb{C})$ be a representation. We say that $\Gamma$ is rigid if "any homomorphism $\rho' : \Gamma \to \text{GL}(n, \mathbb{C})$ near $\rho$ is conjugate to $\rho."$ Give a precise definition, and show that this is true whenever $\Gamma$ is finite. Prove that, under this assumption, $\rho$ can be conjugated so it takes values in $\text{GL}(n, K)$ for some algebraic number field $K$.

(An interesting example of a rigid representation is the defining representation of the fundamental group of a hyperbolic 3-manifold.)