Exercises. Problems 1–3 are some examples that use (not much beyond) the definition of representation. Problem 4 considers the question of classifying representations that admit a vector fixed under a prescribed subgroup. Problems 5–7 use ideas related to Maschke’s theorem and composition series to establish a notion of “reduction modulo $p$” for representations. Problems after *** are more difficult, and not for assessment.

1. Let $k$ be a field. Find, with proof, all subrepresentations of $k^n$ as an $S_n$-representation.

2. Find all the (isomorphism classes of) irreducible representations of the additive group $\mathbb{Z}/23\mathbb{Z}$ over the field $\mathbb{F}_2$. (Hint: the group algebra decomposes as a sum of fields.)

3. Suppose $G$ is a $p$-group and $K$ a field of characteristic $p$.
   (a) Prove that an irreducible representation of $G$ over $K$ is trivial.
   (b) Prove that $G$ embeds in the subgroup of upper triangular matrices in $\text{GL}_n(K)$, for some $n$.

4. Let $G$ be a finite group and $H$ a subgroup. Let $A \subset CG$ be the subalgebra of elements $\{\alpha : h\alpha h' = \alpha\}$ for all $h, h'$. For any complex representation $V$ of $G$, we denote by $V^H$ the elements of $V$ that are fixed by $H$.
   (a) Prove that for $v \in V^H$ and $a \in A$, we have also $a \cdot v \in V^H$.
   (b) Prove that if $V$ is irreducible as a $G$-representation, then $V^H$ is irreducible (i.e., simple) as an $A$-module.
   (c) Prove that if $V, W$ are two irreducible $G$-representations for which $V^H$ and $W^H$ are isomorphic nontrivial $A$-representations, then $V$ is isomorphic to $W$.

*Hint* for part (c): Let $\theta : V^H \to W^H$ an $A$-module isomorphism. Let $S$ be the smallest $G$-invariant subspace of $V \oplus W$ containing the graph of $\theta$. Then $S \neq V \oplus W$. Now the projection maps $S \to V, S \to W$ are both isomorphisms; thus $S$ defines the graph of an isomorphism between $V$ and $W$.

*Remark.* In fact, the association $V \mapsto V^H$ gives a bijection between irreducible representations of $G$ with an $H$-fixed vector, and simple modules for $A$: can you describe the inverse?

5. Let $G$ be a group and $0 \to F_1 \to F_2 \to \cdots \to F_k \to 0$ an exact complex of $G$-representations over a field $k$. Let $F_j^{ss}$ be the semisimplification of $F_j$. Prove that $\bigoplus_{j \text{ odd}} F_j^{ss}$ and $\bigoplus_{j \text{ even}} F_j^{ss}$ are isomorphic.

6. Let $G$ be a finite group and $\rho : G \to \text{GL}(V)$ a representation of $G$ on the finite-dimensional $\mathbb{Q}$-vector space $V$. Prove that we can choose a basis for $V$ so that every $\rho(g)$ acts by a matrix with *integral entries*.

*Hint.* Choose any basis $e_1, \ldots, e_n$, let $L = \sum \mathbb{Z}e_i$, and “average” $L$ to make it $G$-invariant.

7. We continue with the setup of the prior problem. Suppose $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are two different bases for $V$ so that $\rho(G)$ has integral entries; let $\sigma_1, \sigma_2 : G \to \text{GL}_n(\mathbb{Z})$ be the corresponding homomorphisms. Let $\sigma_1, \sigma_2$ be the corresponding representations

$$
\sigma_j : G \overset{\sigma_j}{\to} \text{GL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{Z}/p\mathbb{Z}).
$$

Prove that $\sigma_1$ and $\sigma_2$—considered as representations of $G$ over the field $\mathbb{Z}/p\mathbb{Z}$—have isomorphic semisimplifications (in other words, they have the
same composition factors with the same multiplicity; they need not, in
general, be actually isomorphic).

Hint. Let $L, L'$ be the $\mathbb{Z}$-modules spanned by $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$; it suffices to check in the case when $L' \subseteq L$ (why?); now let $X$ be the finite
abelian group $L/L'$, on which $G$ acts, so that one has an exact sequence
\[
\{x \in X : px = 0\} \to L'/pL' \to L/pL \to X/pX;
\]
use (a slight variant of) 5.

***

(8) Let $k, k'$ be two algebraically closed fields whose characteristic doesn’t di-
vide $|G|$. Does $G$ have “the same” irreducible representations over $k$ and
$k'$? Discuss (e.g., formulate a precise statement).

(9) Let $G = \text{PSL}_2(\mathbb{F}_{23})$ act on $X = \mathbb{P}^1(\mathbb{F}_{23}) = \mathbb{F}_{23} \cup \infty$. Let $V = \mathbb{F}_2X$, the permutation module with coefficients in the field of size 2. Prove that there exist exactly two invariant subspaces of dimension 12. [Hint: Use the
subgroup \[
\begin{pmatrix}
a & x \\ 0 & a^{-1}
\end{pmatrix}
\]
of order $11 \times 23$.]

If $U$ is either one of these subspaces, then $U$ contains no vector $(x_i)$ with
fewer than 8 nonzero entries. Up to automorphism, there is a unique 12-
dimensional subspace of $\mathbb{F}_2^{24}$ with this property; it is the so-called extended Golay code, and its automorphism group is the simple group $M_{24}$. 