Exercises, 2

(1) Compute the character table of A_4.

(2) Prove that all entries in the character table of S_n are real, for any n.

(3) Let D (resp. Q) be the dihedral group of order 8 and the quaternion group of order 8. Write down the character table for D and Q and deduce that the group algebras $\mathbf{C}D$ and $\mathbf{C}Q$ are isomorphic. Prove that the two-dimensional irreducible representation of D preserves a symmetric bilinear form, whereas the two-dimensional irreducible representation of Q preserves a skew-symmetric bilinear form.

(4) Let N be the group of upper triangular matrices in $\mathrm{GL}_3(\mathbf{F}_p)$ with all diagonal entries equal to 1. Let Z be the center of N; it is a cyclic p-group. Prove that, for each nontrivial irreducible character $Z \to \mathbf{C}^\times$, there exists a unique irreducible representation (ρ, V) of N, up to isomorphism, in which Z acts by ω. Describe the character table of N.

(5) Suppose $G \subset \mathrm{GL}(V)$ is finite. Prove that every irreducible representation of G occurs in $V^{\otimes k}$ for some $k \geq 1$.