Homework 3

Hand in questions 2, 3, 4, 6, 7.

1. Prove that $\mathrm{SL}_2(\mathbf{R})$ has no invariant measure in its action on $\mathbf{P}^1_{\mathbf{R}}$ (i.e. the space of lines in $\mathbf{R}^2$).

2. (a) Let G be a connected Lie group and let $\omega(g)$ be the determinant of $\mathrm{Ad}(g)$ acting on the Lie algebra. If dg is a left Haar measure show that $\omega(g)dg$ is a right Haar measure.
   (b) Write down a left and right Haar measure on the subgroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ of real matrices.

3. Suppose that G is a Lie group, A and B are Lie subgroups, such that B is compact and $AB = G$ (i.e. every element of g can be written as ab with $a \in A, b \in B$).
   (a) Verify that these conditions apply with $G = \mathrm{SL}_n(\mathbf{R})$, A the upper triangular matrices with positive diagonal entries, and $B = \mathrm{SO}_n$.
   (a) Assume that the multiplication map $A \times B \to G$ is a diffeomorphism. [3] Let $d_l a$ be a left Haar measure for A. Show that $d_l a\ db$ is a Haar measure for G, i.e. the rule $\int f(ab) da db$ defines a left- and right- G invariant functional on $C_c(G)$.

4. Weyl's unitary trick: Consider the following categories
   a) The category of continuous representations of $\mathrm{SL}_n(\mathbf{R})$.
   (b) The category of holomorphic representations of $\mathrm{SL}_n(\mathbf{C})$.
   (c) The category of continuous representations of SU_n.

   A slight variant of our arguments for GL_n show that restriction gives a equivalence of categories of (b) and (c); you may use this without proof. Verify that restriction also gives an equivalence of categories for (b) and (a). In other words, you should check the following two statements:

   (i) Any continuous representation of $\mathrm{SL}_n(\mathbf{R})$ extends to a holomorphic representation of $\mathrm{SL}_n(\mathbf{C})$.
   (ii) If V, V' are two holomorphic representations of $\mathrm{SL}_n(\mathbf{C})$ then any $\mathrm{SL}_n(\mathbf{R})$-homomorphism $f : V \to V'$ is also a $\mathrm{SL}_n(\mathbf{C})$-homomorphism.

   Hint: For (i) use the simple connectedness of SU_n.

5. By arguments as in the prior question, verify that any *continuous* representation of $\mathrm{SL}_n(\mathbf{C})$ is a direct sum of irreducible representations, and the representations are of the form $g \mapsto \rho(g) \otimes \overline{\sigma(g)}$ where ρ, σ are holomorphic representations.

6. Let G be a connected Lie group and $U \subset G$ an open neighbourhood of 1 with compact closure. Let U^n be the set of elements of G which can be expressed as an n-fold product of elements from U. We say that G has *exponential growth* (resp *polynomial growth*) if the right Haar measure of U^n is $\geq ae^{bn}$ (resp. $\leq an^b$) for some $a, b > 0$
   (a) Show that this notion is independent of the choice of U.

[3] This is not necessary, but it simplifies the situation. It also happens to be true in case (a), but you don't need to prove that.

(b) Show also that if it is true for "right" Haar measure it is also true for "left."

(c) Consider $SL_2(\mathbf{R}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its subgroups A, B, N of diagonal matrices, of upper-triangular matrices, and of upper-triangular matrices with diagonal entries 1, respectively.

Find which have exponential and which have polynomial growth.

7. Estimate as best you can the Haar measure of the set of $g \in SU_3$ such that $|g_{ij} - e_{ij}| \leq \frac{4}{3}$ for each (i, j). You do not need to compute the answer exactly, and you do not to give a detailed justification: Just explain how you estimated it and how accurate you expect your estimate to be (for full marks, you should get within 1 % of the correct answer).