Homework 2

Hand in problems 1, 2, 3.

(1) Consider $G = \mathrm{GL}_n(\mathbf{R})$.

  (a) Show that there is a small neighbourhood V of the identity in G such that, for $g \in V$, $\log(g) := -\sum(1-g)^n/n$ converges and defines a diffeomorphism to a neighbourhood of 0 in $M_n(\mathbf{R})$ which inverts the exponential map.

  (b) Compute the Jacobian (i.e. determinant of the derivative) of this map in terms of the eigenvalues of $g \in G$. (Here, we regard $\mathrm{GL}_n(\mathbf{R})$ as a subset of $M_n(\mathbf{R})$; thus we identify log with a map $M_n(\mathbf{R}) \to M_n(\mathbf{R})$ and can talk about its Jacobian, i.e. the factor by which it distorts volume.)

(2) (a) Prove that the exponential map for the Lie group of upper triangular matrices $\begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbf{R})$, with all diagonal entries equal to 1, is a diffeomorphism. (Use the logarithm map from the first questio to construct an inverse.)

  (b) Prove that the exponential map for the Lie group $\mathrm{GL}_n(\mathbf{C})$ is actually surjective. ((Use Jordan decomposition.)

  (b) Prove that the exponential map for SO_3 is surjective, and describe it as explicitly as you can its geometry. For example, what does the preimage of a point look like?

  (d) Prove that the exponential map for the Lie group $\mathrm{SL}_2(\mathbf{R})$ is *not* surjective.

(3) Let $G \leqslant \mathrm{GL}_n(\mathbf{C})$ be finite. Then there is a constant $A = A(n)$ such that G admits an abelian subgroup G' with index $\leq A$. This is a theorem of Jordan and you can prove it thus:

  (a) Explain why we can assume $G \subset \mathrm{U}_n$.

  (b) Pick a reasonable metric on U_n and let $G(\delta)$ be the subgroup generated by all elements with $d(g,e) < \delta$. Prove that the index $[G : G(\delta)]$ is bounded by a function $f(\delta, n)$ of δ, n.

  (c) Let $z \neq e$ be an element of G which minimizes $d(z,e)$. Explain why there exists a value $\delta = \delta(n)$ such that z commutes with any element of $G(\delta)$.

  (d) Explain why the centralizer of z is of the form $\mathrm{U}_a \times \mathrm{U}_b$ for some $a, b < n$. Thus $G(\delta) \subset U_a \times U_b$. By induction on n conclude.

(4) Verify the following fact due to Dynkin: Suppose given an iterated Lie bracket e.g. $[X,[Y,X]]$ in two variables X, Y. Expand formally via $[X,Y] = XY - YX$, e.g.

$$[X,[Y,X]] = X(YX - XY) - (YX - XY)X = 2XYX - YXX - XXY.$$

Now substitute each term on the right by the corresponding iterated Lie bracket:

$$2[X,[Y,X]] - [Y,[X,X]] - [X,[X,Y]] = 3[X,[X,Y]].$$

Show that the result of this process is always m times the starting expression, where m is the depth of the iterated bracket (e.g. the number of X, Ys). This fact can be used to give a very explicit form of the Baker–Campbell-Hausdorff formula.