## Homework 1

If you are taking the class for credit, you should hand in questions 1, 2,4.
*Notation:* for $\mathbf{m} = (m_1, \dots, m_d)$, where $m_1 \le m_2 \le \cdots \le m_d$, we set

$$s_{\mathbf{m}}(x_1, \dots, x_d) = \frac{\det(x_i^{m_j+j-1})}{\det(x_i^{j-1})}$$

which is sometime called a "Schur polynomial." For example, if $d = 3$ and $\mathbf{m} = (0, 1, 2)$ then $s_{\mathbf{m}}(x, y, z)$ equals

$$\frac{\begin{vmatrix} x^4 & y^4 & z^4 \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}} = x^2(y+z) + z^2(y+x) + y^2(x+z) + 2xyz. \tag{1}$$

As in class, there exists an irreducible algebraic representation $V_{\mathbf{m}}$ of $\mathrm{GL}_d(\mathbf{C})$ such that the trace of $V_{\mathbf{m}}$ on an element $g \in \mathrm{GL}_d(\mathbf{C})$ equals $s_{\mathbf{m}}(\lambda_1, \dots, \lambda_d)$, where the $\lambda_i$ are the eigenvalues of $g$. For example, if $\mathbf{m} = (0, 1, 2)$ as above, the representation $V_{\mathbf{m}}$ of $\mathrm{GL}_3(\mathbf{C})$ must be 8-dimensional (compute trace at the identity element).

1. Prove that $\mathrm{U}_d \subset \mathrm{GL}(d, \mathbf{C})$ is Zariski-dense as follows:
   (a) Let $\mathfrak{u}_d$ be the tangent space to $\mathrm{U}_d$ at the identity. Prove that $\mathfrak{u}_d$ is a $d^2$-dimensional real vector space and it spans the tangent space $T$ to $\mathrm{GL}_d(\mathbf{C})$ (as a complex vector space) at the identity. In other words, you are showing that the natural map $\mathfrak{u}_d \otimes_{\mathbf{R}} \mathbf{C} \to T$ is an isomorphism of complex vector spaces.
   (b) Deduce that any holomorphic function on $\mathrm{GL}(d, \mathbf{C})$ which vanishes on $\mathrm{U}_d$ must actually vanish identically on $\mathrm{GL}(d, \mathbf{C})$. (Compute derivatives.)

2. (a) Explain why an algebraic representation of $\mathrm{GL}_d(\mathbf{C})$ is determined, up to isomorphism, by its character. You may use the corresponding fact for compact groups without proof.
   (b) Suppose $d = 2$ and that $a \ge b$ are integers. Compute explicitly $s_{(a,b)}(x, y)$. Deduce that the representation $V_{(a,b)}$ of $\mathrm{GL}_2(\mathbf{C})$ is isomorphic to the representation of $\mathrm{GL}_2(\mathbf{C})$ on the space $\mathrm{Sym}^{(a-b)}\mathbf{C}^2 \otimes (\wedge^2\mathbf{C}^2)^{\otimes b}$. (The group $\mathrm{GL}_2(\mathbf{C}) = \mathrm{GL}(\mathbf{C}^2)$ acts on this space because $\mathrm{Sym}^{a-b}, \wedge^2$ and $\otimes$ are all functors.)

3. Write out a proof that "restriction from $\mathrm{GL}_d(\mathbf{C})$ to $\mathrm{U}(d)$" defines an equivalence of categories between algebraic representations of $\mathrm{GL}_d(\mathbf{C})$ and continuous representations of $\mathrm{U}(d)$. (Most of this has been done in class. The main point is to check that the homomorphisms are the same.)

4. We say that $(k_1, \dots, k_d)$ is a "weight" of the representation $V_{\mathbf{m}}$ if the character $(z_1, \dots, z_d) \mapsto \prod z_i^{k_i}$ of the diagonal subgroup appears inside $V_{\mathbf{m}}$; equivalently, if the monomial $x_1^{k_1} \dots x_d^{k_d}$ appears inside the Schur function $s_{\mathbf{m}}(x_1, \dots, x_d)$. For example, the weights of the representation $V_{2,1,0}$ (see

(1) for the character) are

$$(1,1,1),(2,1,0),(1,2,0),(0,1,2),(0,2,1),(2,0,1),(1,0,2).$$

(a) Show that all the weights satisfy $\sum k_i = \sum m_i$. (Look at the action of scalar matrices. )

(b) Show that if $(k_1, \dots, k_d)$ is a weight, then any permutation $(k_{\sigma(1)}, \dots, k_{\sigma(d)})$ is also a weight (for $\sigma \in S_d$). (Look at the action of permutation matrices.)

(c) Show there is a unique weight $(k_1, \dots, k_d)$ that maximizes $\sum ik_i$ and it is given by $(m_1, \dots, m_d)$. (Work directly with the formula for the character.)

This is very convenient: It allows us to read off $\mathbf{m}$ from the representation $V_{\mathbf{m}}$ as the unique weight maximizing $\sum ik_i$. This is the start of "highest weight theory," which we will develop more systematically later in the course.

(d) Using (c) (or any other way you like) verify the claims made in class: the irreducible representation of $\mathrm{GL}_d(\mathbf{C})$ indexed by $(0,0,\dots,0,m)$ is the representation on $\mathrm{Sym}^m \mathbf{C}^d$ and the irreducible representation indexed by $(0,0,0,\underbrace{1,1,\dots,1}_{q})$ is the representation on $\wedge^q \mathbf{C}^d$.

5. Let $A \in \mathrm{GL}_d(\mathbf{C})$. For any integer $t \geq 0$, we may write $A^t$ as a linear combination of $1, A^{-1}, \dots, A^{-(d-1)}$ (why?) Prove that the coefficient of 1 equals the trace of $A$ on $\mathrm{Sym}^t \mathbf{C}^d$. Give similar interpretations for the other coefficients.

6. Prove that the Schur functions $s_m$, where we restrict $m_1$ to be non-negative, actually form a $\mathbf{Z}$-basis for the ring of polynomial symmetric functions $\mathbf{Z}[x_1, \dots, x_n]^{S_n}$. (Use 5(c) and an inductive argument.)