Midterm

Same deal as the homeworks, but work by yourself (you are welcome to the course book or other references that you identify clearly, or talk to me or the CA). Note that questions 3 and 4 are similar to exercises in II.5 of the textbook, and some hints are given there.

In this homework, G denotes a compact Lie group.

(1) Let G be a compact group and V a representation with inner product.
   (a) Suppose $v \in V$ is a vector that is almost invariant under G: $\|gv - v\| \leq \varepsilon$ for all $v \in V$. Prove that there exists an invariant vector $u \in V$ (i.e. $gu = u$ for all $g \in G$) which is "close" to v, and quantify "close."
   (b) Suppose that ℓ is a *line* that is almost invariant under G: the angle between ℓ and $g\ell$ is $\leq \varepsilon$ for all $v \in V$. Prove there exists an invariant line ℓ' which is "close to" ℓ, and quantify "close."

(2) (a) Suppose V is a representation, and $\langle v, w \rangle$ and $[v, w]$ are two G-invariant positive Hermitian forms on V.
   Show that in fact there exists $E : V \to V$ which commutes with G and $\langle v, w \rangle = [Ev, w]$, and explain why E is self-adjoint with respect to $[-, -]$. By taking a square root, deduce that there is $F : V \to V$ which commutes with G such that $\langle v, w \rangle = [Fv, Fw]$.
   (c) Suppose $f_1, f_2 : G \to U_n$ are pointwise conjugate (i.e. $f_1(x)$ is conjugate to $f_2(x)$ for every $x \in G$). Prove that f_1 is globally conjugate to f_2, i.e. there exists $u \in U_n$ such that $f_1(x) = u f_2(x) u^{-1}$ for all $x \in G$ at once.

Let V_n denote the irreducible representation of SU_2 of dimension $n + 1$, which we recall is realized on the space of homogeneous polynomials of degree n; let $v_i = x^i y^{n-i}$, so that $v_0, \ldots, v_n$ form a basis. We refer to v_0 as the "highest weight vector."

(3) (a) Describe as explicitly as you can a G-invariant inner product on V_n, and compute the norms $\|v_i\| / \|v_0\|$.
   (b) Prove that V_n is isomorphic to its dual representation V_n^*. This implies there is a G-invariant bilinear form $V_n \times V_n \to \mathbf{C}$; write one down for $n = 1$.

(4) (a) Decompose $V_3 \otimes V_3, \mathrm{Sym}^2 V_3$ and $\wedge^2 V_3$ into irreducibles.
   (b) For each irreducible constituent of $V_3 \otimes V_3$, write down an explicit highest weight vector. (For example, $v_0 \otimes v_0$ corresponds to the highest weight vector in a constituent isomorphic to V_6.)