Homework 5

In this homework, G is a compact connected Lie group with maximal torus T and roots $\Phi \subset X^*(T)$, we fix a positive chamber $C \subset X^*(T) \otimes \mathbf{R}$ and corresponding decomposition $\Phi = \Phi^+ \cup \Phi^-$, we fix an invariant inner product $\langle -, - \rangle$ "on everything", W_λ denotes the representation parameterized by the W-orbit of $\lambda \in X^*(T)$, $\mathfrak{g} = \mathrm{Lie}(G) \otimes \mathbf{C}$ is the complexified Lie algebra, $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ are the root spaces.

You should hand in problems 2, 3, 4, 5.

(1) For the root systems of $\mathrm{U}_n, \mathrm{SO}_{2m}, \mathrm{SO}_{2m+1}, \mathrm{Sp}_{2m}$ (with the usual identification of $X^*(T) \simeq \mathbf{Z}^n$ that we have been using) prove that the following subsets of $\mathbf{R}^n$ is a chamber

$$\begin{aligned} (U_n) &: a_1 > \cdots > a_n \\ (SO_{2m}) &: a_1 > \cdots > a_{n-1} > |a_n| \\ (SO_{2m+1}, Sp_{2m}) &: a_1 > \cdots > a_n > 0. \end{aligned}$$

In each case, describe the positive roots, the associated basis of simple roots, and the half-sum ρ of all positive roots.

(2) (a) Prove that the dual of V_λ is given by $V_{-\lambda}$.
(b) Prove that every representation of G is isomorphic to its own dual if and only if there exists w in the Weyl group such that $w(t) = 1/t$ for every $t \in T$. Deduce that every representation of SO_m and Sp_m is self-dual.
(c) Describe a representation of U_m that is not self-dual for each m.

(3) Suppose that λ, μ both lie inside the closure of the fixed chamber C. Show that $V_{\lambda+\mu}$ occurs as an irreducible subrepresentation of $V_\lambda \otimes V_\mu$ with multiplicity 1.

(4) Here $G = U_n$ and V denotes the standard representation of G on $\mathbf{C}^n$ and (e.g) $W_{(t,0,\dots,0)}$ denotes the representation parameterized by the W-orbit of $(t, 0, 0, \dots, 0) \in X^*(T)$.
(a) Prove that $W_{(t,0,0,\dots,0)}$ is isomorphic to $\mathrm{Sym}^t V$.
(b) Prove that $W_{(1,1,\dots,1,0,\dots,0)}$ is isomorphic to $\wedge^k V$ (where k is the number of 1s).
(c) Prove that $W_{(2,2,0,\dots,0)}$ is isomorphic to the kernel of $\mathrm{Sym}^2(\wedge^2 V) \to \wedge^4 V$ where the map sends $a \otimes b \ \ (a, b \in \wedge^2 V)$ to $a \wedge b$. (This representation has dimension $n^2(n^2-1)/12$ and is essentially where the Riemann curvature tensor lies.)
(d) Prove that there's an exact sequence of U_n-representations

$$W_{(t,0,0,\dots,0,-1)} \to \mathrm{Sym}^t V \otimes V^* \to \mathrm{Sym}^{t-1} V.$$

(5) (a) Suppose that a representation V of G has the property that the subspace of $v \in V$ such that $\mathfrak{g}_\alpha v = 0$ for all $\alpha \in \Phi^+$ is one-dimensional. Prove that V is irreducible. (The converse is also true.)
(b) Prove that Sp_m acts irreducibly on $\mathrm{Sym}^k \mathbf{C}^m$ for every $k \geq 1$; show this is false in general for SO_m. Extra credit: what happens for $\wedge^k \mathbf{C}^m$?

(6) Let $G = \mathrm{SU}_m, \mathrm{SO}_m\ (m > 4), \mathrm{Sp}_m$. Prove that the adjoint action of G on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) \otimes \mathbf{C}$ is irreducible, and determine the highest weight.

(7) Prove that G_2 has an irreducible 7-dimensional representation V and that it fixes an element of $\wedge^3 V$. (This leads to the definition of G_2 as the subgroup of GL_7 fixing an appropriate trilinear form.)

(8) (a) Let $\mathbf{x} = (x_1, x_2, x_3)$ be any weight of the diagonal torus that occurs in the irreducible representation of $\mathrm{GL}_3(\mathbf{C})$ with highest weight $\mathbf{r} = (r_1 \geq r_2 \geq r_3)$. Prove that in fact $(x_1, x_2, x_3) \in \mathbf{R}^3$ belongs to the *convex hull* of points $(r_{\sigma(1)}, r_{\sigma(2)}, r_{\sigma(3)}) : \sigma \in S_3$.

(b) Let M be any Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Show that the diagonal entries $(m_1, m_2, m_3) \in \mathbf{R}^3$ of M belong to the *convex hull* of points $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)})$.

(9) Let $P : U_n \to \mathbf{C}$ be a polynomial in the matrix entries and their complex conjugates, with rational coefficients. Prove that $\int_{U_n} P(u)du \in \mathbf{Q}$, the integral being taken with respect to the Haar measure of mass 1.