Homework 4

In this homeowrk, G is a compact connected Lie group, and BTD is the textbook. Hand in questions 2,3,4,5.

0. (Not for handing in but I very strongly recommend doing it): Verify all the claims made about the root systems of the classical groups. (Check that the tori described in class are maximal, and compute all the roots, compute the Weyl group, compute a chamber, etc.)
1. For each of the classical root systems (V, Φ, W) let L be the $\mathbf{Z}$-span of all roots $\alpha \in \Phi$, and let $L' = \{\lambda \in V : \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi.\}$. Note that $L \subset L'$. Compute the quotient L'/L for each of the classical root systems (it turns out that this gives the center of the corresponding simply connected group in all cases).
2. In class we constructed a morphism $\iota_\alpha : \mathrm{SU}(2) \to G$ for each root α. Explicate this morphism for each root of $G = \mathrm{SO}(4)$ and $G = \mathrm{SO}(5)$.
3. BTD exercises (V.2.18), questions 7,8.
4. BTD exercises (V. 3.15), questions 3,9,10.
5. The *Killing form* is the biliner form on $\mathrm{Lie}(G)$ defined by

$$\langle X, Y \rangle = \mathrm{trace}(\mathrm{ad}X \ \mathrm{ad}Y \text{ acting on } \mathrm{Lie}(G)).$$

   Recall that $\mathrm{ad}(X)$ is the linear endomorphism of $\mathrm{Lie}(G)$ defined by $Z \mapsto [X, Z]$. Compute the Killing form for $G = \mathrm{SU}_n$ as an explicit multiple of $\mathrm{trace}(XY)$, where we regard X, Y as skew-Hermitian $n \times n$ matrices. Prove that the Killing form is negative semidefinite for any compact group.