Take home final exam

Notations:

- Let G be a compact connected Lie group.
- If $\mathfrak{g}$ is a Lie algebra, a representation of $\mathfrak{g}$ on a vector space W is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(W)$, where $\mathrm{End}(W)$ has Lie bracket $[A, B] = AB - BA$. Equivalently, it is an "action" $\mathfrak{g} \times W \to W$ with the property that $X(Yv) - Y(Xv) = [X, Y]v$ for $X, Y \in \mathfrak{g}$.
- $\mathfrak{sl}_2$ is th Lie algebra of $\mathrm{SL}_2(\mathbf{C})$, i.e. the complex 2×2 matrices. We denote

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = [e, f].$$

1. Let V be an irreducible finite-dimensional representation of G, i.e. a continuous (automatically smooth) map $G \to \mathrm{End}(V)$. By differentiating we get a map $\mathrm{Lie}(G) \to \mathrm{End}(V)$, i.e. a representation of $\mathrm{Lie}(G)$ on V.
   (a) Explain why the action of $\mathrm{Lie}(G)$ on V is also irreducible (i.e. no proper nontrivial subspace W satisfies $XW \subset W$ for all $X \in \mathrm{Lie}(G)$).
   (b) Let $\langle -, - \rangle$ denote a G-invariant inner product on $\mathrm{Lie}(G)$ and let X_i be an orthonormal basis. Let $\mathcal{C} : V \to V$ be the "Casimir operator"
   $$v \mapsto \sum X_i^2 v$$
   where $X_i^2 v$ means $X_i(X_i v)$. Show that $\mathcal{C}$ commutes with every element of G and deduce that $\mathcal{C}$ is a scalar multiple of the identity.
   (c) For $G = \mathrm{SU}_2$ compute the scalar by which $\mathcal{C}$ acts on the n-dimensional irreducible representation (to define $\mathcal{C}$ we need a G-invariant inner product on $\mathrm{Lie}(G)$; just be clear about which one you use).
2. Let $G = \mathrm{SO}_n$. Let V_k be the vector space of homogeneous polynomials of degree k on $\mathbf{R}^n$. Construct a morphism $f : V_{k-2} \to V_k$ commuting with the G-action, and show that the cokernel $V_k / f(V_{k-2})$ is an irreducible representation of SO_n with highest weight $(k, 0, 0, \ldots, 0)$
   (We discussed the $n = 3$ case of this result in class.)
3. Have you ever wanted to diagonalize the Fourier transform? The following question gives one very nice answer, related to the quantum harmonic oscillator. Let V be the space of functions on $\mathbf{R}$ of the form $P(x)e^{-x^2/2}$, where P is a polynomial.
   (a) Show that the rules
   $$eF = \frac{ix^2}{2}F, \quad fF = \frac{i}{2}\frac{d^2F}{dx^2} \quad hF = x\frac{dF}{dx} + F/2$$
   define an action of $\mathfrak{sl}_2$ on V.
   (b) Set $K = i(f - e) \in \mathfrak{sl}_2$. Find elements $P, Q \in \mathfrak{sl}_2$ such that $[K, P] = 2P, [K, Q] = -2Q, [P, Q] = K$. Use P and Q as "raising" and "lowering" operators to prove that there is a basis $v_0, v_1, \ldots,$ for V with the properties that $Kv_i = (i + 1/2)v_i$.
   (c) Define the Fourier transform $\mathcal{F}$ by $\mathcal{F}f(k) = \frac{1}{\sqrt{2\pi}} \int f(x)e^{ikx}dx$. Verify that $\mathcal{F}$ preserves V and check the rules $\mathcal{F}e = -f\mathcal{F}, \mathcal{F}f = -e\mathcal{F}, \mathcal{F}h = -h\mathcal{F}$. Deduce that $\mathcal{F}v_k = i^k v_k$.