

The starred problem in 4(b) is trickier and so you should attempt it only if you have time.

- (1) Definitions.
 - (a) Define the following terms: “submodule,” “module isomorphism”, “quotient module.”
 - (b) Suppose that I try to define a “basis” of a module as “a minimal set $A \subset M$ generating M .” Give an example to show that this does not behave as nicely as the vector space case. (For example, you could give two bases of the same module of different sizes, or give $N \subset M$ where a basis for N is larger than a basis for M , etc.)
- (2) Calculations. No justification is needed, just the answer. Please be careful to include trivial cases in your counts (the zero module is always a submodule, for instance).
 - (a) Let $R = \mathbf{C}[t]$ and let M be the module \mathbf{C}^3 where t acts by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$
 How many submodules does M have?
 - (b) How many \mathbf{Z} -module homomorphisms from $\mathbf{Z}/15\mathbf{Z}$ to \mathbf{Q}/\mathbf{Z} ?
 - (c) What is $(\mathbf{Z}/15\mathbf{Z} \oplus \mathbf{R}) \otimes (\mathbf{Z}/6\mathbf{Z} \oplus \mathbf{Q}) \otimes (\mathbf{Z}/3\mathbf{Z})$, where the tensor product is of \mathbf{Z} -modules?
- (3)
 - (a) State the universal property of tensor products, and prove that this property characterizes the tensor product uniquely.
 - (b) Prove that if I, J are ideals of the commutative ring R , and we regard $R/I, R/J$ as R -modules, then the tensor product $R/I \otimes R/J$ is isomorphic to R/K , where K is the ideal generated by I and J .
- (4)
 - (a) State the classification of modules over a PID, in a form that also guarantees uniqueness.
 - (b)* Let R be a PID. Let $a_1, a_2, a_3 \in R$ be relatively prime. Prove that there exists an invertible matrix A with entries in R and with bottom row a_1, a_2, a_3 . (Hint: look at the submodule N of R^3 spanned by (a_1, a_2, a_3) and now try to choose a better basis for R^3).