(1) Book problems: 11.3.1, 11.3.2, 18.1.4, 18.2.4.

(2) (Converse to Schur's lemma:) Let $G$ be a finite group and $V$ a $G$-representation over $\mathbf{C}$ such that any $G$-homomorphism $V \to V$ is a multiple of the identity. Prove that $V$ is irreducible. Give a counterexample when $G$ is infinite.

(3) Consider the representation of $S_4$ on $\{(x, y, z, w) \in \mathbf{C}^4 : (x+y+z+w) = 0\}$. Choose a basis and write out matrices for an element of each conjugacy class in $S_4$. Therefore compute the character of this representation.

(4) Let $G$ be a group and $A$ an abelian subgroup of index $\leq N$. Prove that every irreducible representation of $G$ has dimension $\leq N$.

(5) Let $V, W$ be finite-dimensional vector spaces over a field $F$.
   (a) Construct a natural isomorphism (i.e., without choosing bases) between $(V \otimes W)^*$ and $V^* \otimes W^*$.
   (b) Let $M$ be the vector space of all linear transformations from $V$ to $W$. Define a bilinear map $V^* \times W \to M$ sending $(v^*, w)$ to the linear transformation
   $$v \mapsto w\langle v^*, v\rangle.$$
   Show that this defines (via the universal property) an *isomorphism* $\Phi : V^* \otimes W \to M$.
   (c) Define the *rank* of an element of $V^* \otimes W$ be the least integer $r$ such that it can be written as $\sum_{i=1}^{r} v_i^* \otimes w_i$. Show that the rank of $x \in V^* \otimes W$ coincides with the matrix rank of $\Phi(x) \in M$.
   (d) Now suppose that $V = W$. In this case, there is a linear transformation $\ell : V^* \otimes V \to F$ which sends $v^* \otimes v$ to $\langle v^*, v\rangle$. Prove that $\ell(x) = \operatorname{trace} \Phi(x)$.

(6) Let $F$ be a field and $P \in F[x_1, \dots, x_n]$ a polynomial of degree $d$ in $n$ variables. Write $V = F^n$ and consider $P$ as a function $V \to F$.
   (a) Suppose that the characteristic of $F$ is zero. Show that $P(v) = B(v, v, \dots, v)$ where
   $$B : \underbrace{V \times V \times \cdots \times V}_{d} \to F$$
   is a symmetric[1] multilinear form form on $d$ copies of $V$. Show also that such a $B$ is unique.
   (b) Show that (a) can be false when the characteristic of $F$ is nonzero.

[1]Symmetric means that $B(v_1, \dots, v_d)$ is unchanged by any permutation of $(v_1, \dots, v_d)$.