(1) A *direct product* of two modules $X_1, X_2$ is a module $M$ together with homomorphisms $f_i : M \to X_i$ (for $i = 1, 2$) such that, for any other module $N$ and homomorphisms $f'_i : N \to X_i$, there is a *unique homomorphism* $F : N \to M$ such that $f'_i = f'_i \circ F$. Write out the proof that the set $X_1 \times X_2$, with coordinatewise addition and scalar multiplication, together with the homomorphisms $f_1 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_1 \in X_1$ and $f_2 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_2 \in X_2$ is a direct product.

(2) Take the universal property of "direct sum," but replace modules by "groups", and module homomorphisms by "group homomorphisms".

Prove that *in this context* a direct sum of $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$ is given by the infinite group of matrices $\begin{pmatrix} \pm 1 & x \\ 0 & 1 \end{pmatrix}$ where $f_1$ sends the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $f_2$ sends the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ to to $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ (In particular, the result is not even abelian.)

(3) (a) Let $A$ be a finite abelian group of order $p$ and $B$ a finite abelian group of order $q$. Prove that $A \otimes_{\mathbf{Z}} B = 0$ if $p$ and $q$ are relatively prime.
(b) Prove also that $\mathbf{Q} \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z})$ is zero.
(c) (Harder:) Prove that $\mathbf{R} \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z})$ is *not* zero.

(4) Let $M, N$ be $R$-modules, and let $K$ be a submodule of $N$.
(a) Show that there is a unique homomorphism $M \otimes K \to M \otimes N$ which sends $m \otimes k$ to $m \otimes k$ for every $m \in M, k \in K$.
(b) Show this homomorphism need not be injective. (Consider the case of $R = \mathbf{Z}$, $M = \mathbf{Z}/2\mathbf{Z}$, $N = \mathbf{Z}$, and $K$ the even integers inside $N$).
(c) Prove that the quotient of $M \otimes N$ by the image of $M \otimes K$ is isomorphic to $M \otimes (N/K)$. (Hint: call $Q$ this quotient. Construct a bilinear map $M \times N \to Q$ and show it has the correct universal property).

(5) Now take $M = \mathbf{C}^2, N = \mathbf{C}^3$ as $\mathbf{C}$-modules. Give an element of $\mathbf{C}^2 \otimes \mathbf{C}^3$ which *isn't* a pure tensor. Prove, however, that any element of $\mathbf{C}^2 \otimes \mathbf{C}^3$ is the sum of two pure tensors. (A "pure tensor" is an element of the form $m \otimes n$).

(6) Book problems: 10.4.15, 10.4.17.

(Bonus problem over page).

(7) Bonus problem: An "axis-parallel rectangle" will be a rectangle in the $(x, y)$-plane whose sides are parallel to $x$- and $y$-axes. If $R$ is an axis-parallel rectangle, with side lengths $a$ and $b$ parallel to the $x$ and $y$-axis respectively, set $s(R) := a \otimes b \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}$.

(a) Show that if $R_1, \ldots, R_k$ tile the rectangle $R$ then $\sum s(R_i) = \sum s(R)$.

(b) Let $L$ be a $\mathbf{Q}$-vector space and $a_1, \ldots, a_n \in L$. Show that if $\sum a_i \otimes a_i = 0$ then actually all $a_i = 0$. Show also that this statement is false if we replace $\mathbf{Q}$ by $\mathbf{C}$.

(c) Deduce that in any tiling of a rectangle with rational side lengths by squares, all the squares have rational side lengths too (hint: project $s(R)$ to $L \otimes_{\mathbf{Q}} L$ where $L = \mathbf{R}/\mathbf{Q}$ and use (b)).

You can solve part (c) in many other ways too, but the tensor method is very powerful and handles many such questions at once. Google "Dehn invariant" for more.