We could have alternatively defined direct sums and direct products through their universal properties, as follows:

A *direct sum* of two modules X_1, X_2 is a module M *together with* homomorphisms $f_i : X_i \to M$ (for $i = 1, 2$) such that, for any other module N and homomorphisms $f_i' : X_i \to N$ there is a *unique homomorphism* $F : M \to N$ so that $f_i' = F \circ f_i$.

A *direct product* of two modules X_1, X_2 is a module M together with homomorphisms $f_i : M \to X_i$ (for $i = 1, 2$) such that, for any other module N and homomorphisms $f_i' : N \to X_i$, there is a *unique homomorphism* $F : N \to M$ such that $f_i' = f_i' \circ F$.

Hand in *one of* problems 1,2,3, as well as all of problems 4–8.

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(1) Let X_1, X_2 be two R-modules. Write out the proof that $X_1 \times X_2$, with coordinate-wise addition and scalar multiplication, together with the homomorphisms

$$f_1 : x_1 \in X_1 \mapsto (x_1, 0) \in X_1 \times X_2 \text{ and } f_2 : x_2 \in X_2 \mapsto (0, x_2) \in X_1 \times X_2$$

is a direct sum.

(2) Let X_1, X_2 be two R-modules. Write out the proof that the set $X_1 \times X_2$, with coordinatewise addition and scalar multiplication, together with the homomorphisms

$$f_1 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_1 \in X_1 \text{ and } f_2 : (x_1, x_2) \in X_1 \times X_2 \mapsto x_2 \in X_2.$$

is a direct product.

(3) Let X_1, X_2 be two R-modules. Write out a proof that a direct sum is unique up to unique isomorphism.

(4) Take the universal property of "direct sum," but replace modules by "groups", and module homomorphisms by "group homomorphisms".

Prove that *in this context* a direct sum of $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$ is given by the infinite group of matrices $\begin{pmatrix} \pm 1 & x \\ 0 & 1 \end{pmatrix}$ where f_1 sends the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and f_2 sends the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ to to $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

(In particular, the result is neither free nor abelian.)

(5) Now replace modules by "sets" and homomorphisms by "functions."

Prove that the direct sum of sets X_1, X_2 is now given by the *disjoint union* $X_1 \cup X_2$, where the functions f_1, f_2 are given by the inclusions of X_1 and X_2.

For the next two problems: Recall that a tensor product $M \otimes_R N$ of two R-modules M and N is equipped with a bilinear form $M \times N \to M \otimes_R N$, which we denote by $(m, n) \mapsto m \otimes n$. An element of $M \otimes_R N$ in the image of this map is called a *pure tensor*.

(6) Let A be a finite abelian group of order p and B a finite abelian group of order q. Prove that $A \otimes_{\mathbf{Z}} B = 0$ if p and q are relatively prime. Prove also that $\mathbf{Q} \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z})$ is zero. (Tricky:) prove that $\mathbf{R} \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z})$ is *not* zero.

(7) Let M, N be R-modules, and let K be a submodule of N.

(i) Show that there is a unique homomorphism $M \otimes K \to M \otimes N$ which sends $m \otimes k$ to $m \otimes k$ for every $m \in M, k \in K$.

(ii) Show this homomorphism need not be injective. (Consider the case of $R = \mathbf{Z}$, $M = \mathbf{Z}/2\mathbf{Z}$, $N = \mathbf{Z}$, and K the even integers inside N).

(iii) Prove that the quotient of $M \otimes N$ by the image of $M \otimes K$ is isomorphic to $M \otimes (N/K)$. (Hint: call Q this quotient. Construct a bilinear map $M \times N \to Q$ and show it has the correct universal property).

(8) Now take $M = \mathbf{C}^2, N = \mathbf{C}^3$ as $\mathbf{C}$-modules. Give an element of $\mathbf{C}^2 \otimes \mathbf{C}^3$ which *isn't* a pure tensor. Prove, however, that any element of $\mathbf{C}^2 \otimes \mathbf{C}^3$ is the sum of two pure tensors.