You can consult your book and class notes and homeworks, but not your fellow students, nor any other sources. State any result from these sources that you use clearly. You are free to talk to me or the CA about the questions.

(1) Suppose that M is a module over a ring R (commutative and with 1) and let N be a submodule.

(i) Suppose that R is a PID, π is an irreducible, and N and M/N are *both* isomorphic to $(R/\pi) \times (R/\pi)$. Prove that $\pi^2 m = 0$ for every $m \in M$. What are the possibilities for M, up to isomorphism?

(ii) The prior question showed that M need not be isomorphic to $N \oplus M/N$. Now suppose that R is arbitrary but M/N is a *free* module. Show that M is isomorphic to $N \oplus M/N$.

(2) You might recall that $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} = 0$. In this question, we will show this doesn't happen for finitely generated modules by "reduction to fields." Let M be a finitely generated nonzero module over a ring R (commutative and with 1). Fix a generating set $m_1, \ldots, m_n$ for M.

(a) Suppose that J is a maximal ideal of R such that $M/JM = 0$. Prove that there exists $\delta \notin J$ such that $\delta M = 0$.

Hint: Since $JM = M$, we can write every m_i as a combination $\sum \alpha_{ij} m_j$ where $\alpha_{ij} \in J$. Let A be this square matrix with entries α_{ij}; thus we have a matrix equation

$$\begin{pmatrix} m_1 \\ m_2 \\ \vdots \end{pmatrix} = A \begin{pmatrix} m_1 \\ m_2 \\ \vdots \end{pmatrix}.$$

Now you can use without proof Cramer's rule, which says that, given any square matrix A with entries in R, we can always find a matrix B (the *adjugate* matrix) such that $AB = BA = \det(A) \cdot \mathrm{Identity}_n$. Now take $\delta = \det(\mathrm{Id}_n - A)$.

(b) Let $0 \neq x \in M$ and let $I = \{r \in R : rx = 0\}$. Prove that I is an ideal. Deduce from (a) that, if J is a maximal ideal containing I, then M/JM cannot be the zero module.

(c) Deduce from (b) that $M \otimes_R M$ is not the zero module.

(3) Let X, Y be finite-dimensional complex vector spaces. Let $x, x' \in X$ and $y, y' \in Y$, all nonzero. Prove that $x \otimes y = x' \otimes y'$ in $X \otimes Y$ if and only if $x' = x\alpha$ and $y = y'\alpha$ for some nonzero $\alpha \in \mathbf{C}$.

(4) Suppose V is an n-dimensional complex vector space and T is a linear transformation of V such that $T^n = 0$ but $T^{n-1} \neq 0$. Find, with proof, the Jordan canonical form of T. *Bonus question:* What is the Jordan normal form of $T \otimes T$, considered as a linear transformation $V \otimes V \to V \otimes V$?

(5) Let G be a finite group.

(i) Let $b \in G$, and set $C_b = \sum_{a \in G} aba^{-1} \in \mathbf{C}G$. Prove that C_b acts on any irreducible representation V as $\frac{|G|\chi_V(b)}{\dim V} \mathrm{Id}_V$; here Id_V is the identity transformation on V.

(ii) Let $x \in G$. Prove that the number of pairs $(a, b) \in G$ that satisfy $aba^{-1}b^{-1} = x$ is given by

$$|G| \sum_V \frac{\chi_V(x)}{\dim V},$$

where the sum is taken over irreducible representations V.

Hint: Use the inversion formula in Wedderburn's theorem, i.e. for an element $Y \in \mathbf{C}G$, the coefficient of e in Y is given by $\sum_V \frac{(\dim V)}{|G|}($ trace of Y on $V)$. Apply this to $Y = x^{-1} \sum_{a,b \in G} aba^{-1}b^{-1}$.