1. Solutions to some tricky homework questions

(1) (HW 2.3)

Let \( G = S_n \) and let \( H \) be the stabilizer of 1 in \( G \) (i.e., the permutations \( \sigma \) so that \( \sigma(1) = 1 \)); thus \( H \) is isomorphic to \( S_{n-1} \). We show that any \( G \)-representations with an \( H \)-fixed vector is either isomorphic to the trivial representation or to the representation on \( \{(x_1, \ldots, x_n) : \sum x_i = 0\} \).

Let \( C^n \) be the permutation module for \( G \) and \( e_i \) \( (1 \leq i \leq n) \) the usual basis for \( C^n \). Therefore, the action of \( G \) is via \( \sigma e_i = e_{\sigma(i)} \).

Suppose \( V \) is a representation and \( v \in V \) an \( H \)-fixed vector. If \( g \in G \), then \( gv \in V \) only depends on \( g(1) \): if \( g(1) = g'(1) \), then \( g \in g'H \) and so \( gv = g'v \).

For \( 1 \leq i \leq n \), let \( v_i = gv \), where \( g \in G \) is any element with \( g(1) = i \). Note that \( v_{g(i)} = xv_i \) for any \( x \in G \).

Define

\[
\phi : C^n \rightarrow V, \quad \phi(\sum a_i e_i) = \sum a_i v_i.
\]

It is a \( G \)-homomorphism. This follows from the fact that \( v_{g(i)} = xv_i \) for \( x \in G \). Since \( V \) is irreducible, \( \phi \) is surjective. But we have proven that \( C^n = \text{span}(1, 1, \ldots, 1) \oplus P \) (with \( P = \{x_i : \sum x_i = 0\} \)), with each factor irreducible; so \( \phi \) restricted to each factor is either zero or an isomorphism (by Schur’s lemma).

Thus \( V \) is isomorphic to either the trivial representation or \( P \).

(2) (HW 2.4) [This solution looks quite intimidating, but the core of it is the irreducible case; study that first.]

Consider first of all a nonzero irreducible subrepresentation \( W \subset V^\oplus n \).

Each projection map \( \pi_j : W \rightarrow V \) is either zero or an isomorphism, by Schur’s lemma. Choose \( j_0 \) so that \( \pi_{j_0} \) is an isomorphism. For every \( 1 \leq k \leq n \), the map

\[
\pi_k \circ \pi_{j_0}^{-1} : V \rightarrow V
\]

is, by Schur’s lemma, a scalar; call this scalar \( \lambda_k \in \mathbb{C} \). Then, for every \( w \in W \),

\[
w = (\pi_1(w), \pi_2(w), \ldots, \pi_n(w)) = (\lambda_1 \pi_{j_0}(w), \lambda_2 \pi_{j_0}(w), \ldots, \lambda_n \pi_{j_0}(w)).
\]

Since \( \pi_{j_0} \) is surjective, we conclude that

\[
(1) \quad W = \{(\lambda_1 v, \lambda_2 v, \ldots, \lambda_n v) : v \in V\}.
\]

This classifies the irreducible subrepresentations of \( V^\oplus n \). We now consider the general case; for this, we need some notation to handle “systems of equations.”

For any subspace \( Y \subset C^n \), let

\[
W(Y) = \{(v_1, \ldots, v_n) \in V^\oplus n : \sum y_i v_i = 0 \text{ for all } (y_1, \ldots, y_n) \in Y\}.
\]

Then \( W(Y) \subset V^\oplus n \) is a subrepresentation; we shall show all subrepresentations arise thus. One has the dimension formula (check it!)

\[
(2) \quad \dim W(S) = (n - \dim S) \dim V.
\]

Let \( W \subset V^\oplus n \) be an arbitrary subrepresentation. We may write \( W \) as an internal direct sum \( W_1 \oplus \cdots \oplus W_k \) where each \( W_i \) is irreducible. Each
\(W_i\) is of the form (1) for some vector \(\Delta_i = (\lambda_{i1}, \ldots, \lambda_{in})\). Let \(S = \{ y \in \mathbb{C}^n : \Delta_i \cdot y = 0 \text{ for all } 1 \leq i \leq k \}\) (here we write \(x \cdot y = \sum x_j y_j\)).

Then \(S\) is a subspace of dimension \(\geq n - k\), because it is defined by \(k\) linear equations. On the other hand, \(W_i \subset W(S)\) for each \(i\), so also \(W \subset W(S)\). But \(\dim(W) = k \dim(V)\) whereas (2) shows \(\dim W(S) \leq k \dim(V)\). So \(W = W(S)\).

We have shown that every subrepresentation is of the form \(W(S)\) for some \(S \subset \mathbb{C}^n\), as required.

(3) (HW 3.5) Write \(G = S_n\).

Recall that \(\mathbb{C}^n = P \oplus \text{span}(1, 1, \ldots, 1)\). Let \(\phi : \mathbb{C}^n \to P\) be the projection and \(e_1, \ldots, e_n\) the standard basis for \(\mathbb{C}^n\). (Therefore, \(\phi\) is given explicitly as \(\phi((x_1, \ldots, x_n)) = (x_1 - \bar{x}, \ldots, x_n - \bar{x})\) where \(\bar{x} = (\sum x_i)/n\).) Let \(f_i = \phi(e_i)\).

The map \(\Psi : \mathbb{C}^n \to P \otimes P\) defined by

\[
\Psi : \sum a_i e_i \mapsto \sum a_i (f_i \otimes f_i)
\]

is a \(G\)-homomorphism. Its image lies in \(\text{Sym}^2 P\) and it is \textit{injective}.

Here is how to prove injectivity (you might want to skip this): Being a \(G\)-homomorphism, if \(\Psi\) were not injective, its kernel would be a subrepresentation of \(\mathbb{C}^n\); the only possibilities are \(\text{span}(1, 1, \ldots, 1)\) and \(P\). In the first case, \(\Psi(\sum e_i) = 0 \implies \sum f_i \otimes f_i = 0\) and, in the second case, \(\Psi(e_1 - e_2) = 0 \implies f_1 \otimes f_1 = f_2 \otimes f_2\). However, \(f_1,\ldots,f_{n-1}\) is a basis for \(P\), and thus \(f_i \otimes f_j\) (\(1 \leq i, j \leq n - 1\)) is a basis for \(P \otimes P\); this together with the fact that \(f_n = -f_1 - f_2 - \cdots - f_{n-1}\) shows that neither possibility occurs.

Note also that \(\dim \text{Sym}^2 P = \frac{n(n-1)}{2} > n\), and we have just shown that \(\mathbb{C}^n\) is isomorphic to a subrepresentation of \(\text{Sym}^2 P\). Therefore, \(\text{Sym}^2 P\) decomposes as a direct sum of \(1, P\) and another representation \(X\) (not necessarily irreducible). Consequently,

\[P \otimes P \cong 1 \oplus P \oplus X \oplus \wedge^2 P.\]

Note now that \(\langle \chi_{P \otimes P}, \chi_{P \otimes P} \rangle \geq 4\) with equality if and only if \(X, \wedge^2 P\) are irreducible. We shall show that equality holds; this implies that \(X\) and \(\wedge^2 P\) are both irreducible, as desired.

Let \(\chi_P\) be the character of \(P\), so that \(\chi_P(g) = \text{fix}(g) - 1\), where \(\text{fix}(g)\) denotes the number of fixed points of \(g\) on \(\{1, 2, \ldots, n\}\). Then

\[
\langle \chi_{P \otimes P}, \chi_{P \otimes P} \rangle = \frac{1}{|G|} \sum_{g \in G} (\text{fix}(g) - 1)^4
\]

\[= \frac{1}{|G|} \sum (\text{fix}(g)^4 - 4\text{fix}(g)^3 + 6\text{fix}(g)^2 - 4\text{fix}(g) + 1).\]
Now\footnote{Indeed if $G$ is any finite group acting on the set $X$ then (Burnside’s lemma) \( \frac{1}{|G|} \sum_{g \in G} \text{fix}(g) \) is the number of orbits of $G$ on $X$; apply this with $X = \{1, \ldots, n\}^j$. To prove Burnside’s lemma, note that \( \frac{1}{|G|} \sum_{g \in G} \text{fix}(g) = (\chi_{\mathbf{C}X}, 1) \), where $\mathbf{C}X$ is the permutation representation of $G$ associated to $X$. As we proved in class, this inner product equals the dimension of the space of $G$-invariant vectors on $\mathbf{C}X$. But a function $f : X \to \mathbf{C}$ is $G$-invariant if and only if it is constant on each $G$-orbit; thus, the dimension of $G$-invariants equals the number of $G$-orbits.}

\[ \frac{1}{|G|} \sum_{g \in G} \text{fix}(g)^j = \#G\text{-orbits on } \{1, \ldots, n\}^j. \]

It remains to count the number of $S_n$-orbits on $\{1, \ldots, n\}^j$ for each $1 \leq j \leq 4$. For example, with $j = 3$, there are five orbits: \( \{(x, y, z) : x, y, z \text{ all distinct}\}, \{(x, x, y) : x \neq y\}, \{(x, y, x) : x \neq y\}, \{(y, x, x) : x \neq y\}, \text{and finally } (x, x, x) \). In all cases the variables $x, y, z$ are understood to take values in $1, \ldots, n$.

We shall abbreviate this reasoning by saying

*There is one orbit of type $(x, y, z)$, three orbits of type $(x, x, y)$, and one orbit of type $(x, x, x)$.\]

Reasoning similarly:

- $j = 1$. One orbit.
- $j = 2$. Two orbits: one of type $(x, y)$, one of type $(x, x)$.
- $j = 3$. Five orbits (as above).
- $j = 4$. Fifteen orbits: one of type $(x, y, z, w)$, six of type $(x, x, y, z)$, four of type $(x, x, x, y)$, three of type $(x, x, y, y)$, one of type $(x, x, x, x)$.

Thus \( \langle \chi_{P \otimes P}, \chi_{P \otimes P} \rangle = (15 - 20 + 12 - 4 + 1) = 4 \), concluding the proof.

(4) (Midterm, problem 1(b)).

The **single most common error** was on problem 1(b): to assert that any subrepresentation of $V \oplus V$ is of the form $\{0\} \oplus V$ or $V \oplus \{0\}$. This is false; a counterexample is given by $\{(v, v) : v \in V\}$. Make sure you understand this point; see also solution to HW 2.4.

Here is a solution to part (b) of the question.

Write $V = \bigoplus_i U_i$ with $U_i$ irreducible. We claim there exists $Y \subset V$ with $V = X \oplus Y$ (internal direct sum).

Proceed by contradiction, and choose $X \subset V$ of *maximal dimension* so that the claimed result is false. Clearly $X \neq V$ (otherwise we could take $Y = 0$). There exists $i$ so that $U_i$ is not contained in $X$; by irreducibility of $U_i$, the intersection $U_i \cap X = \{0\}$. Write $X' = X + U_i$; it is the internal direct sum of $X$ and $U_i$:

\[ X' = X \oplus U_i \text{ (internal direct sum)}. \]

By assumption ($X$ is a counterexample of maximal dimension) there is $Y'$ so that $V = X' \oplus Y'$ (internal direct sum).

Then $V$ is the internal direct sum of $X, U_i$ and $Y'$, in particular,

\[ V = X \oplus (U_i \oplus Y'), \text{ (internal direct sum)}. \]
contradiction.

Thus there exists $Y \subset V$ with $V = X \oplus Y$ (internal direct sum). The map $\pi : V \to V$ defined by $x + y \mapsto x$ ($x \in X, y \in Y$) is a $G$-homomorphism with image equal to $X$, as desired.