1. Math 122, homework 2

(1) Let $G$ be a finite group and $V$ be an irreducible $G$-representation over $\mathbb{C}$. We proved that there is an inner product on $V$ that is invariant under $G$, i.e.

\[ \langle x, y \rangle = \langle gx, gy \rangle. \]

We proved this over the field of real numbers, but this case is identical. Prove that any two such inner products are multiples of one another. Hint: if $[x, y]$ and $\langle x, y \rangle$ are two inner products, there exists a linear transformation $A : V \to V$ so that $[x, y] = \langle Ax, y \rangle$.

(2) Suppose $G$ is a group possessing an abelian subgroup of index $\leq N$. Prove that every irreducible representation of $G$ has dimension $\leq N$.

(3) Find with proof all irreducible representations $V$ for $S_n$ over $\mathbb{C}$ that admit a nonzero vector fixed by $S_{n-1}$.

(4) Let $V$ be an irreducible $G$-representation over $\mathbb{C}$. Let $W = V \oplus V \oplus V$. Show that all submodules of $W$ are given by “imposing linear constraints,” e.g. \{(x, y, z) \in V \oplus V \oplus V : 2x + 3y + z = 0, x - y - z = 0\} is an example of a submodule obtained thus.

(5) Let $D$ be the dihedral group of order 8, the group of rotations and reflections in the plane that preserve a square. Thus $|D| = 8$. Describe all the irreducible representations of $D$ over $\mathbb{C}$ and compute the character table of $D$.

(6) Book problems: §18.3 problems 7, 10, 11, 12, 23.