In checking irreducibility you may wish to use Gauss’ Lemma, page 303 of textbook. Do all book problems and three of the remaining problems.

1. Text problems: §§13.1, problems 1, 2, 3; §§13.2, problems 2, 3, 4, 7, 8, 10.

2. Let \( \alpha \) be a root of \( \alpha^3 - \sqrt{2} \alpha + 1 \). Write down a polynomial \( P \) with rational coefficients so that \( P(\alpha) = 0 \). Express \( \alpha^{-1} \) as a \( \mathbb{Q} \)-linear combination of \( 1, \alpha, \alpha^2, \ldots \).

3. Let \( \alpha \) be a root of \( x^3 + x + 1 = 0 \). Compute the minimum polynomial \( m \) of \( 1 + \alpha + \alpha^2 \) and prove that \( \mathbb{Q}[\ell]/(m) \) is isomorphic to \( \mathbb{Q}[x]/(x^3 + x + 1) \).

4. Write down all the irreducible polynomials of degree 5 over \( \mathbb{Z}/2\mathbb{Z} \).

5. In lecture we discussed various results involving “irreducible” polynomials. Suppose that \( f \in \mathbb{Q}[x] \). Explain how to test in a finite time whether or not \( f \) is irreducible. Your procedure need not be particularly efficient; it should just be clear that it always terminates in finite time.