Math 121 Homework 6: Notes on Selected Problems

12.3.5. Compute the Jordan canonical form for the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Solution. Write A for the given matrix. The characteristic polynomial of A is $(\lambda-1)^2(\lambda-2)$ so the two possible minimal polynomials are $(\lambda-1)(\lambda-2)$ or the characteristic polynomial itself. We find that $(A-I)(A-2I)=0$ so the minimal polynomial is $(\lambda-1)(\lambda-2)$, and hence the invariant factors are $\lambda-1$, $(\lambda-1)(\lambda-2)$. The prime power factors of the invariant factors are the elementary divisors: $\lambda-1, \lambda-1$, $\lambda-2$. Finally the Jordan canonical form of A is diagonal with diagonal entries 1, 1, 2. ☐

Note. After determining that the minimal polynomial has all roots in the ground field and no repeated roots, we can immediately conclude that the matrix is diagonalizable and therefore the Jordan canonical form is diagonal.

12.3.12. Determine the Jordan canonical form for the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution 1. By inspection the change of basis $e_i \mapsto 2^i e_i$ transforms the given matrix into a Jordan block of type $(1,4)$. ☐

Solution 2. Write A for the given matrix. The characteristic polynomial of A is $(\lambda-1)^4$ and one can check that $(A-I)^3$ is nonzero so the minimal polynomial is $(\lambda-1)^4$. Therefore the single invariant factor of A is $(\lambda-1)^4$, which implies that A has a single Jordan block of type $(1,4)$. ☐

12.3.17. Prove that any matrix A is similar to its transpose A^{t}.

Solution. If A is the companion matrix to a polynomial $f(x)$, then A has minimal polynomial $f(x)$ (consider a "cyclic basis"). Since taking the matrix transpose is a homomorphism of the matrix subalgebra generated by any single matrix, A^{t} also has minimal polynomial $f(x)$.

Therefore A and A^{t} both have $f(x)$ as their only invariant factor. Consequently the $F[x]$-modules associated to A and A^{t} are isomorphic and so A and A^{t} are similar. The result follows in general by first writing A in rational canonical form and then applying the above result to each block.

Alternatively, one may use the Jordan canonical form, being careful to justify the assertions. First one must produce a field extension K of F in which the characteristic polynomial of A splits completely. Then A is similar over K to its Jordan canonical form, but each block in the Jordan canonical is similar to its transpose via the change of basis that reverses the order of the basis. Therefore A is similar to A^{t} over K, but this implies that A and A^{t} are similar over any field containing the coefficients of A and in particular over F. □

12.3.22. Prove that an $n \times n$ matrix A with entries from $\mathbf{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over *any* field F?

Solution. This statement is true over any field F. A matrix can be diagonalized over F if and only if the minimal polynomial splits completely over F and has no repeated roots. (This follows by considering the Jordan canonical form, recalling that the roots of the minimal polynomial coincide with the roots of the characteristic polynomial.) The minimal polynomial of A must divide $x^3 - x$, which has no repeated roots, so the minimal polynomial for A does not have any repeated roots. The minimal polynomial $x^3 - x = x(x-1)(x+1)$ also splits completely over any field, and in particular over F, so A can be diagonalized over F. □

12.3.37. Let J be a Jordan block of size n with eigenvalue λ over $\mathbf{C}$.

- **(a)** Prove that the Jordan canonical form for the matrix J^2 is the Jordan block of size n with eigenvalue λ^2 if $\lambda \neq 0$.
- **(b)** If $\lambda = 0$ prove that the Jordan canonical form for J^2 has two blocks (with eigenvalues 0) of size $\frac{n}{2}, \frac{n}{2}$ if n is even and of size $\frac{n-1}{2}, \frac{n+1}{2}$ if n is odd.

Solution. Write N for the Jordan block of type $(0, n)$. Then $J = \lambda I + N$ where I is the $n \times n$ identity matrix.

- **(a)** Assume $\lambda \neq 0$. Then $J^2 - \lambda^2 I = \lambda N + N^2$ has rank $n - 1$ so $J^2 - \lambda I^2$ has a single Jordan block. Finally, J^2 is (conjugate to) a Jordan block of type (λ^2, n).

- **(b)** Assume $\lambda = 0$ so $J = N$ and $J^2 = N^2$. Then for a nonnegative integer i, $(J^2)^i = J^{2i}$ has rank $n-2i$ for $2i \le n$ and 0 otherwise. For a positive integer i with $i \le n/2 - 1$, the number of Jordan blocks of J^2 of size i (see note to Problem 3 below) is

  $$n - (2(i-1)) - 2(n-2i) + n - 2(i+1) = 0.$$

  If n is even, the number of blocks of size $n/2$ is

  $$n - (2(n/2-1)) - 2(n-2(n/2)) = 2$$

  and if n is odd, the number of blocks of size $\frac{n-1}{2}$ is

  $$n - (2((n-1)/2-1)) - 2(n-2((n-1)/2)) = 1$$

  and the number of blocks of size $\frac{n+1}{2}$ is

  $$n - (2((n+1)/2-1)) = 1. \qquad \square$$

12.3.38. Determine necessary and sufficient conditions for a matrix $A \in M_n(\mathbf{C})$ to have a square root, i.e., for there to exist another matrix $B \in M_n(\mathbf{C})$ such that $A = B^2$.

Solution. Since A has a squareroot if and only if a conjugate of A has a square root, it suffices to answer the question when A is in the Jordan canonical form.

Assume first that A is a Jordan block with a nonzero eigenvalue λ. Choose a square root μ for λ and let B' be the Jordan block of type (μ, n) so that $(B')^2$ has Jordan form A and hence is similar to A, say $C(B')^2C^{-1} = A$. Then $B = CB'C^{-1}$ is a square root of A.

Therefore the only obstruction to finding a square root of A lies in the structure of the nilpotent blocks of A. Let $n_1 \ge n_2 \ge \cdots \ge n_r \ge 1$ be the sizes of the nilpotent blocks in the Jordan form of A. In view of the computation of the square of a Jordan block, a necessary and sufficient condition for A to have a square root is that $n_{2i} - n_{2i-1}$ lies in $\{0, 1\}$ for each i with $2i \le r$, and, if r is odd, $n_r = 1$. This is equivalent to the sequence $\operatorname{rank} A^k - \operatorname{rank} A^{k+1}$ for k varying in nonnegative integers not containing successive occurrences of the same odd integer and, if $\operatorname{rank} A^0 - \operatorname{rank} A$ is odd, $\operatorname{rank} A^0 - 2\operatorname{rank} A + \operatorname{rank} A^2$ being at least 1. $\square$

Problem 2. Assume T is an $n \times n$ complex matrix. Prove that the following are equivalent:

- **(a)** $T^n = 0$;
- **(b)** There exists a sequence g_i of invertible complex matrices so that $g_i T g_i^{-1} \to 0$ (in the natural topology on $n \times n$ complex matrices).

Solution. Note that the first condition is equivalent to the only eigenvalue of T being 0 (with multiplicity n), or equivalently that the spectral radius (maximum of absolute values of eigenvalues) of T is 0.

Consider a matrix norm $\|\cdot\|$ compatible with a norm $|\cdot|$ on $\mathbf{C}^n$ in the sense that $|Tx| \le \|T\||x|$. For any eigenvector v of T with eigenvalue λ,

$$|\lambda||v| = |\lambda v| = |Tv| \le \|T\||v|$$

and $|v| \neq 0$ so $|\lambda| \le \|T\|$. Thus if T has positive spectral radius r_0, then for any invertible $n \times n$ complex matrix g, gTg^{-1} also has positive spectral radius r_0 so $\|gTg^{-1}\| \ge r_0 > 0$. Therefore **(b)** implies **(a)**.

For the converse, assume that the only eigenvalue of T is 0 (with multiplicity n). Then there exists a basis

$$\{v_k^j \mid 1 \le j \le r, 1 \le k \le r_j\}$$

for $\mathbf{C}^n$ so that

$$Tv_k^j = \begin{cases} v_{k-1}^j & \text{if } k > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let g_i be the change of basis matrix defined by

$$g_i v_k^j = 2^{ki} v_k^j.$$

Then one can check that[1] $g_i T g_i^{-1} = 2^{-i} T$ so $g_i T g_i^{-1} \to 0$. □

Note. An alternative proof of **(b)** implies **(a)** is as follows. Assume that some eigenvalue of T is nonzero. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, listed with multiplicity. Over an infinite field, distinct polynomials cannot define the same polynomial function. Since the characteristic polynomial $(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ of T is not λ^n, there exists c such that

$$(c - \lambda_1) \cdots (c - \lambda_n) \neq c^n.$$

The eigenvalues of $cI - T$ are $c - \lambda_1, \ldots, c - \lambda_n$ by Jordan form, for example, so for any invertible matrix g, $g(cI - T)g^{-1}$ has determinant $(c - \lambda_1) \cdots (c - \lambda_n)$, which by choice of c is not equal to c^n, the determinant of cI. The determinant is a continuous map $\mathbf{C}^{n \times n} \to \mathbf{C}$ so for any sequence g_i of invertible $n \times n$ matrices, $g_i T g_i^{-1} + cI = g_i(cI - T)g_i^{-1}$ does not converge to cI, and so, since addition is also continuous, $g_i T g_i^{-1}$ does not converge to 0.

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[1] Both sides evaluate to zero on v_k^j for $k = 1$ and if $k > 1$,

$$g_i T g_i^{-1} v_k^j = g_i T 2^{-ki} v_k^j = 2^{-ki} g_i v_{k-1}^j = 2^{-ki} 2^{(k-1)i} v_{k-1}^j = 2^{-i} T v_k^j.$$

Problem 3. Assume $T, S\colon V \to V$ both have a single Jordan block of type $(\lambda = 0, n)$. Describe the Jordan blocks of $T \otimes S\colon V \otimes V \to V \otimes V$.

Solution. For i in $\{0, \dots, n\}$, T^i and S^i each have rank $n - i$ so $(T \otimes S)^i$ has rank $(n - i)^2$. Thus for j in $\{1, \dots, n - 1\}$, the number of Jordan blocks of $T \otimes S$ of type $(0, j)$ is

$$\begin{aligned}&\operatorname{rank}(T \otimes S)^{j-1} - 2\operatorname{rank}(T \otimes S)^{j} + \operatorname{rank}(T \otimes S)^{j+1}\\ &= (n - (j - 1))^2 - 2(n - j)^2 + (n - (j + 1))^2\\ &= (2n - 2j + 1) - (2n - 2j - 1) = 2\end{aligned}$$

and the number of Jordan blocks of $T \otimes S$ of type $(0, n)$ is

$$\operatorname{rank}(T \otimes S)^{n-1} = 1$$

since $\operatorname{rank}(T \otimes S)^n = \operatorname{rank}(T \otimes S)^{n+1} = 0$. □

Note. If W is an m-dimensional vector space, R is a nilpotent linear transformation $R\colon W \to W$, and j is a positive integer, then the number of Jordan blocks of R of type $(0, i)$ for some $i \geq j$ is

$$\operatorname{rank} R^{j-1} - \operatorname{rank} R^{j}$$

so the number of Jordan blocks of R of type $(0, j)$ is

$$\begin{aligned}&\operatorname{rank} R^{j-1} - \operatorname{rank} R^{j} - (\operatorname{rank} R^{j} - \operatorname{rank} R^{j+1})\\ &\qquad = \operatorname{rank} R^{j-1} - 2\operatorname{rank} R^{j} + \operatorname{rank} R^{j+1}.\end{aligned}$$

Problem 4. A subspace $W \subseteq V$ is T-*stable* if $T(W) \subseteq W$.

- **(a)** Suppose that T has a single Jordan block with eigenvalue λ. Prove there are only finitely many T-stable subspaces, and describe them all.
- **(b)** Prove that the number of T-stable subspaces is finite if and only if all Jordan blocks of T have distinct eigenvalues.

Solution.
(a). The $\mathbf{C}[x]$-module V is isomorphic to $\mathbf{C}[x]/(x - \lambda)^n$. The T-stable subspaces of V correspond to the $\mathbf{C}[x]$-submodules of $\mathbf{C}[x]/(x - \lambda)^n$, that is the ideals of $\mathbf{C}[x]/(x - \lambda)^n$. The ideals of the principal ideal domain $\mathbf{C}[x]$ containing the ideal generated by $(x - \lambda)^n$ are those generated by $(x - \lambda)^i$ for $i \in \{0, \dots, n\}$. We have described the $n + 1$ subspaces of V that are T-stable. More explicitly, if a surjective $\mathbf{C}[x]$-linear map $\mathbf{C}[x] \to V$ with kernel the submodule generated by $(x - \lambda)^n$

sends $(x-\lambda)^i$ to v_i, then the T-stable subspaces are the $n+1$ subspaces appearing in the complete flag

$$\begin{aligned} 0 \subset \mathbf{C}v_{n-1} \subset \mathbf{C}v_{n-2} + \mathbf{C}v_{n-1} \\ \subset \cdots \subset \mathbf{C}v_1 + \cdots + \mathbf{C}v_{n-2} + \mathbf{C}v_{n-1} \subset V. \end{aligned}$$

(b). Assume that all Jordan blocks of T have distinct eigenvalues, and let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues. Let n_i be the size of the Jordan block with eigenvalue λ_i so the $\mathbf{C}[x]$-module V is isomorphic to $\bigoplus \mathbf{C}[x]/(x-\lambda_i)^{n_i}$. Since $(x-\lambda_i)^{n_i}$ and $(x-\lambda_i)^{n_i}$ are relatively prime for $i \neq j$, the natural $\mathbf{C}[x]$-linear map

$$\mathbf{C}[x] \to \bigoplus \mathbf{C}[x]/(x-\lambda_i)^{n_i}$$

with kernel generated by $\prod(x-\lambda_i)^{n_i}$ is surjective. Thus the $\mathbf{C}[x]$-submodules of $\bigoplus \mathbf{C}[x]/(x-\lambda_i)^{n_i}$ correspond to $\mathbf{C}[x]$-submodules of $\mathbf{C}[x]$ containing $\prod(x-\lambda_i)^{n_i}$, which are the principal submodules generated by a divisor of $\prod(x-\lambda_i)^{n_i}$. Since $\bigoplus \mathbf{C}[x]/(x-\lambda_i)^{n_i}$ has finitely many $\mathbf{C}[x]$-submodules, V has finitely many T-stable subspaces.

Conversely, assume that T has two Jordan blocks with the same eigenvalue λ. Then T has two linearly independent eigenvectors with eigenvalue λ, say v_1 and v_2. The span of $v_1 + av_2$ is T-stable, and the span of $v_1 + av_2$ equals the span of $v_1 + bv_2$ if and only if $a = b$ by linear independence. Therefore there are infinitely many T-stable subspaces of V. □