Math 121 Homework 4: Notes on Selected Problems

11.2.9. If W is a subspace of the vector space V stable under the linear transformation φ (i.e., $\varphi(W) \subseteq W$), show that φ induces linear transformations $\varphi|_W$ on W and $\overline{\varphi}$ on the quotient vector space V/W. If $\varphi|_W$ and $\overline{\varphi}$ are nonsingular prove that φ is nonsingular. Prove the converse holds if V has finite dimension and give a counterexample with V infinite dimensional.

Solution. There is at most one map completing the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, dashed] & V \arrow[d, "\varphi"] \\ W \arrow[r] & V \end{tikzcd}$$

since the bottom map is an inclusion, and in this case there exists such a map $\varphi|_W$ since the composition of the top and right maps factors as

$$W \to \mathrm{Im}(W \to V \xrightarrow{\varphi} V) \hookrightarrow V$$

and by assumption the inclusion on the right factors through the inclusion $W \hookrightarrow V$.

Similarly, there exists at most one map completing the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\varphi"'] & V/W \arrow[d, dashed] \\ V \arrow[r] & V/W \end{tikzcd}$$

since the top map is surjective, and in this case there exists such a map $\overline{\varphi}$ by the universal property of the quotient $V \to V/W$ since precomposing with $W \hookrightarrow V$ at the top left of the above diagram gives the zero map, which can be seen by adjoining the earlier commutative square involving $\varphi|_W$.

We now have the commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\varphi|_W"'] & V \arrow[r] \arrow[d, "\varphi"'] & V/W \arrow[d, "\overline{\varphi}"] \\ W \arrow[r] & V \arrow[r] & V/W. \end{tikzcd}$$

Assume that $\varphi|_W$ and $\overline{\varphi}$ are nonsingular. Let v be an element of the kernel of φ. By commutativity of the right square, the image $v + W$ of v in V/W, is killed by $\overline{\varphi}$, but $\overline{\varphi}$ is injective so $v + W = 0 + W$, that is, v is in W, or more precisely there is w in W whose image under the inclusion $W \hookrightarrow V$ is v. The commutativity of the left square and injectivity of $W \hookrightarrow V$ on the bottom then imply that $\varphi|_W(w) = 0$, but $\varphi|_W(w)$ is injective so $w = 0$. Finally, the image v of w in V is also 0. This proves that φ is nonsingular.

Note that a partial converse holds even without restricting the dimension of V. If φ is nonsingular, then $\varphi|_W$ must also be nonsingular since the first map in any factorization of an injective map must also be injective.

Assume now that V is finite dimensional and that φ is nonsingular. The dimension of the image of φ equals that of V so φ is an isomorphism. Then $\overline{\varphi}$ is surjective since the last map in any factorization of a surjective map must also be surjective. Any surjective linear map of vector spaces of the same dimension has trivial kernel (since it induces an isomorphism from the domain modulo the kernel and isomorphisms preserve dimension). Therefore $\overline{\varphi}$ is nonsingular.

For an example where φ is nonsingular, but $\overline{\varphi}$ is singular (by the above necessarily V is infinite dimensional), let V be the free vector space on the nonnegative integers $\{0\} \cup \mathbf{N}$ and let W be the subspace generated by the basis vectors corresponding to $\mathbf{N}$. If φ is the linear map of V that takes the basis element corresponding to i to the basis element corresponding to $i + 1$, then it is invariant on W, but the induced map $\overline{\varphi}$ is singular because the image in V/W of the basis element corresponding to 0 is nonzero, but killed by $\overline{\varphi}$. $\square$

11.2.38. Let A and B be square matrices. Prove that the trace of their Kronecker product is the product of the traces: $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$.

Solution. The corresponding (and equivalent, given the relation between Kronecker product and tensor product) basis-free statement is that if for $i = 1, 2$, $\phi_i \colon V_i \to V_i$ is a linear map of vector spaces, then the trace of $\phi_1 \otimes \phi_2 \colon V_1 \otimes V_2 \to V_1 \otimes V_2$ is the product of the trace of ϕ_1 and the trace of ϕ_2. It suffices to prove this in the case that for each i, ϕ_i is of the form $v \mapsto \alpha_i(v)v_{i,0}$ for α_i in V_i^* and $v_{i,0}$ in V_i. Then $\phi_1 \otimes \phi_2$ is given by $v_1 \otimes v_2 \mapsto \alpha_1(v_1)v_{1,0} \otimes \alpha_2(v_2)v_{2,0}$. Note that for $i = 1, 2$, ϕ_i corresponds to $\alpha_i \otimes v_{i,0}$ in $V_i^* \otimes V_i$ and $\phi_1 \otimes \phi_2$ corresponds to $(\alpha_1 \otimes \alpha_2) \otimes v_{1,0} \otimes v_{2,0}$ in $(V_1 \otimes V_2)^* \otimes V_1 \otimes V_2$. (We write $\alpha_1 \otimes \alpha_2$ for the element of $(V_1 \otimes V_2)^*$ given by $v_1 \otimes v_2 \mapsto \alpha_1(v_1)\alpha_2(v_2)$. This abuse of notation is justified because identification of the base field with the tensor product of the base field with itself over itself gives an inclusion $V_1^* \otimes V_2^* \hookrightarrow (V_1 \otimes V_2)^*$ that takes $\alpha_1 \otimes \alpha_2$ in $V_1^* \otimes V_2^*$ to precisely the above functional.) Hence the trace of ϕ_i is $\alpha_i(v_{i,0})$ and the trace of $\phi_1 \otimes \phi_2$ is $(\alpha_1 \otimes \alpha_2)(v_{1,0} \otimes v_{2,0}) = \alpha_1(v_{1,0})\alpha_2(v_{1,0})$. This completes the proof. $\square$

11.3.3(b). Let S be any subset of V^* for some finite dimensional space V. Define $\mathrm{Ann}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}$. ($\mathrm{Ann}(S)$ is called the *annihilator of S in V*.) Let W_1 and W_2 be subspaces of V^*. Prove that $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.

Solution. For each i, $W_1 \cap W_2 \subseteq W_i$ so

$$\{v \in V \mid f(v) = 0 \text{ for all } f \in W_i\} \subseteq \{v \in V \mid f(v) = 0 \text{ for all } f \in W_1 \cap W_2\},$$

that is $\mathrm{Ann}(W_i) \subseteq \mathrm{Ann}(W_1 \cap W_2)$. Thus

$$\mathrm{Ann}(W_1) + \mathrm{Ann}(W_2) \subseteq \mathrm{Ann}(W_1 \cap W_2).$$

For the reverse inclusion, we first prove a lemma about extending bases.

Lemma. Let V be a vector space with subspaces V_1 and V_2. Then there exists a basis B for V so that $B \cap V_i$ is a basis for V_i for $i = 1, 2$ and $B \cap V_1 \cap V_2$ is a basis for $V_1 \cap V_2$.

Proof. Let B_3 be an ordered basis for $V_3 = V_1 \cap V_2$. Since any linearly independent set is contained in a basis, there exists B_i for $i = 1, 2$ such that the concatenation of the sets B_i and B_3 is a basis for V_i. In particular the concatenation of B_1 and B_2 and B_3 spans the subspace $V_1 + V_2$ of V, and we now show that it is in fact independent and hence a basis for the subspace. Assume that for each i, there is a finite set of distinct elements v^i_j for $1 \le j \le r_i$ of B_i and scalars a^i_j such that

$$\sum a^i_j v^i_j = 0$$

where the sum is over all i and j. Then

$$\sum a^1_j v^1_j = -\sum a^2_j v^2_j - \sum a^3_j v^3_j,$$

but the right side is in V_2 so $\sum a^i_j v^1_j$ is V_2 and hence in $V_1 \cap V_2$. Since the concatenation of B_1 and B_3 is an independent set, there is a unique way to write zero as a linear combination and hence $a^1_j = 0$ for all j. Similarly $a^2_j = 0$ for all j. Then by independence of B_3, $a^3_j = 0$ for all j. This proves independence of the concatenation of B_1 and B_2 and B_3, and we may extend this ordered set to a basis B of V. Then B is the desired basis for V. $\square$

Apply the lemma to choose a basis $\mathscr{B}$ for V^* so that for $i = 1, 2$, $\mathscr{B} \cap W_i$ is a basis for W_i and $\mathscr{B} \cap W_1 \cap W_2$ is a basis for $W_1 \cap W_2$. Let v be an element of $\mathrm{Ann}(W_1 \cap W_2)$. In particular $f(v) = 0$ when f is in the intersection of $\mathscr{B} \cap W_1$ and $\mathscr{B} \cap W_2$ so we may define functionals ϕ_1 and ϕ_2 on V^*, that is elements of V^{**}, by the requirements that

$$\begin{aligned}
\phi_1(f) &= 0 && \text{for } f \text{ in } \mathscr{B} \cap W_1 \\
\phi_1(f) &= f(v) && \text{for } f \text{ in } \mathscr{B} \cap W_2 \\
\phi_1(f) &= f(v) && \text{for } f \text{ in } \mathscr{B} \setminus (W_1 \cup W_2)
\end{aligned}$$

and

$$\begin{aligned}
\phi_2(f) &= f(v) && \text{for } f \text{ in } \mathscr{B} \cap W_1 \\
\phi_2(f) &= 0 && \text{for } f \text{ in } \mathscr{B} \cap W_2 \\
\phi_2(f) &= 0 && \text{for } f \text{ in } \mathscr{B} \setminus (W_1 \cup W_2).
\end{aligned}$$

The natural injection $V \hookrightarrow V^{**}$ is surjective because V is finite dimensional so for each i we may choose v_i in V whose image in V^{**} is ϕ_i. This means that, for each i and all f in $\mathscr{B} \cap W_i$,

$$f(v_i) = \phi_i(f) = 0,$$

and the same will be true for any f in W_i so v_i is in $\operatorname{Ann}(W_i)$. For any f in $\mathscr{B}$,

$$f(v_1 + v_2) = f(v_1) + f(v_2) = \phi_1(f) + \phi_2(f) = f(v)$$

so the same is true for any f in V^*, which implies that $v_1 + v_2 = v$. This shows that v is in $\operatorname{Ann}(W_1) + \operatorname{Ann}(W_2)$, but v was an arbitrary element of $\operatorname{Ann}(W_1 \cap W_2)$ so the proof is complete. □

Problem 2. Let V be a finite-dimensional vector space over the field $\mathbf{Q}$. Prove that there do not exist linear transformations $X, Y \colon V \to V$ so that $X \circ Y - Y \circ X$ is the identity map. Show, however, that such transformations *can* exist for vector spaces over a finite field, or for infinite-dimensional vector spaces.

Solution. Since $X \circ Y - Y \circ X$ has zero trace while the identity of V has trace $\dim V$ (we assume V is of finite nonzero dimension), which is not zero in $\mathbf{Q}$, $X \circ Y - Y \circ X$ can never be the identity of V for a $\mathbf{Q}$-vector space V.

Let k be a field and consider the polynomial ring $V = k[t]$, which is a k-vector space. Differentiation ($t^i \mapsto it^{i-1}$) defines a k-linear endomorphism X of V. The k-algebra structure on V allows us to define another k-linear endomorphism Y of V by multiplication by t. By the product rule for differentiation, $X \circ Y = Y \circ X + \mathbb{1}_V$. Hence $X \circ Y - Y \circ X$ is the identity of V.

Now assume that k has positive characteristic p. Let W be the ideal of the ring $V = k[t]$ generated by t^p. Then W is invariant under Y, and, because the derivative of t^p is zero in characteristic p, W is also invariant under X. Therefore X and Y induce k-linear endomorphisms of the quotient V/W, say $\overline{X}$ and $\overline{Y}$, respectively, such that $\overline{X} \circ \overline{Y} - \overline{Y} \circ \overline{X}$ is the identity of V/W, which is a finite dimensional vector space. $\square$

Note. An element of the endomorphism ring of a finite dimensional vector space (over any field) is a commutator (element of the form $X \circ Y - Y \circ X$) if and only if it has zero trace.

Problem 3. Let V,W be finite-dimensional spaces, and

$$\varphi\colon V^* \otimes W \to \operatorname{Hom}(V, W)$$

the unique homomorphism sending $v^* \otimes w$ to the functional $x \in V \mapsto w\langle x, v^*\rangle$. Prove (as asserted in class) that if $X \in V^* \otimes W$ is so that the rank of $\varphi(X)$ is r, then X can be written as the sum of at most r pure tensors.

Solution. Let X be an element of $V^* \otimes W$, and let W' be the image of $\varphi(X)$, which is a subspace of W. Write $\iota\colon W' \to W$ for the inclusion and $\rho\colon W \to W/W'$ for the projection. Consider the commutative diagram

$$\begin{tikzcd}
V^* \otimes W' \arrow[r, "V^*\otimes\iota"] \arrow[d, "\psi"'] & V^* \otimes W \arrow[d, "\varphi"] & \\
\operatorname{Hom}(V, W') \arrow[r, "\operatorname{Hom}(V,\iota)"'] & \operatorname{Hom}(V, W) \arrow[r, "\operatorname{Hom}(V,\rho)"'] & \operatorname{Hom}(V, W/W')
\end{tikzcd}$$

where the vertical maps are the natural isomorphisms and the bottom row has the property that the image of the left map equals the kernel of the right map. Since $\operatorname{Hom}(V,\rho)\varphi(X) = 0$, there exists g in $\operatorname{Hom}(V, W')$ such that $\operatorname{Hom}(V,\iota)g = \varphi(X)$. By commutativity, $(V^* \otimes \iota)(\psi^{-1}(g))$ and X both have the same image under the isomorphism φ. Therefore $X = (V^* \otimes \iota)(\psi^{-1}(g))$ is in the image of $V^* \otimes \iota$. Every element of $V^* \otimes W'$ may be written as a linear combination of at most $r = \dim W'$ pure tensors, and such a representation for $\psi^{-1}(g)$ gives the desired representation for X because $V^* \otimes \iota$ maps pure tensors to pure tensors. $\square$

Problem 4. Let V be a finite dimensional vector space over a field F. Let T be a linear transformation from V to V^*. We say T is symmetric if, for some basis B of V with dual basis B^*, the matrix of T with respect to B, B^* is symmetric. Prove that this definition is independent of choice of B.

Solution. Consider v and w in B. The w^* component of Tv may be obtained by evaluating at w to obtain $(Tv)w$. Similarly the v^* component of Tw is $(Tw)v$. The condition that the matrix of T with respect to B, B^* be symmetric is then that $(Tv)w = (Tw)v$ for all v and w in B. By linearity this implies, and hence is equivalent to, the stronger statement that $(Tv)w = (Tw)v$ for all v and w in V. The latter condition is independent of B so the definition of symmetric is independent of the choice of B. (The condition that T be symmetric can be restated by saying that the pairing $\langle -, - \rangle \colon V \times V \to F$ given by $\langle v, w \rangle = (Tv)w$ is symmetric: $\langle v, w \rangle = \langle w, v \rangle$.) $\square$