Midterm, Math 113, Fall 2008.

- All **five** questions are worth 5 marks each. If you use any theorem or result from class, you should state that result precisely. (For example, you might say *By a result from class, a minimal spanning list is a basis.*)
- Allowed only pens and pencils. No books, notes, calculators, wikipedia etc.
- **Important:** Some problems are hard, so don't despair if you cannot solve some of them! I will give generous partial credit on hard problems, so long as you explain whatever you write clearly and correctly. If you get stuck on something that you can't prove, please say "I don't know".

Problem 1. Let $V = \mathbb{R}^7$ as vector space over $\mathbb{R}$. True/false? No proof needed.

(1) Any spanning list in V has length at least 7.
True.

(2) If T is a linear map from V to V, then $V = \text{null}(T) \oplus_{int} \text{image}(T)$.
False: Take T to be the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $(x, y) \mapsto (0, x)$. Then the image and the null-space are both the span of $(0, 1)$.

(3) Any four-dimensional subspaces U, W have a nonzero vector in common.
True. We saw that $\dim(U \cap W) + \dim(U + W) = \dim(U) + \dim(W)$. Since $\dim(U + W) \leq 7$, the dimension of $U \cap W$ is ≥ 1.

(4) There exist four-dimensional subspaces U, W with $U + W = V$.
True.

(5) Let ℓ be a nonzero linear functional (i.e., $\ell \in V^*$). Then the set $\{\ell' \in V^* : \text{null}(\ell') = \text{null}(\ell)\}$ is a subspace of V^*.
False – a trick question: $\ell' = 0$ does not belong to the set.

Problem 2. Let V be the vector space over $\mathbb{R}$ whose vectors are real-valued sequences $(x_1, x_2, \ldots,)$ where *all but finitely many x_i are zero.*[1] The addition and scalar multiplication are given by:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots), \quad \lambda(x_1, x_2, \ldots,) = (\lambda x_1, \lambda x_2, \ldots),$$

for $x_i, y_i, \lambda \in \mathbb{R}$.

(1) For each $w \in V$, define $l_w \in V^*$ by $l_w(x_1, x_2, \ldots) = \sum x_i y_i$. Prove that $w \mapsto l_w$ defines a linear map from V to V^*.

(2) Prove that it is *not* an isomorphism. (*Hint:* It is not surjective!)

Set $w = (y_1, y_2, \ldots), w' = (y_1', y_2', \ldots)$. Then

$$l_{w+w'}(x) = \sum (y_i + y_i')x_i = \sum y_i x_i + \sum y_i x_i' = l_w(x) + l_{w'}(x).$$

Since this equality holds for all x, we have $l_{w+w'} = l_w + l_{w'}$. Similarly, for $\lambda \in \mathbb{R}$, $l_{\lambda w}(x) = \sum(\lambda y_i)x_i = \lambda \sum y_i x_i = \lambda l_w(x)$. Thus $w \mapsto l_w$ preserves addition and scalar multiplication, and is thus linear.

(**Remark.** Many people proved that $l_w(x+x') = l_w(x) + l_w(x')$. This shows that l_w is linear, but that is not what the question asked for. Make sure you understand this point!)

Define the functional $\Lambda \in V^*$ by $\Lambda(x_1, \ldots, x_n, \ldots) = \sum x_i$. Since all but finitely many of the x_is are zero, the sum on the right-hand side involves only finitely many nonzero terms, and thus is well-defined. Then Λ is a linear functional. Suppose that there exists $w = (y_1, y_2, \ldots) \in V$ so that $l_w = \Lambda$. Let N be so large that

[1] In other words, for every $(x_1, x_2, \ldots) \in V$ there exists N so that $x_i = 0$ for all $i \geq N$.

$y_i = 0$ for $i \geq N$. Let $x \in V$ be defined by $x_i = \begin{cases} 1, i = N \\ 0, i \neq N \end{cases}$. Then $\Lambda(x) = 1$, but $l_w(x) = 0$, contradiction. Therefore, Λ is not in the image of $w \mapsto \ell_w$. Thus, the map $w \mapsto l_w$ is not an isomorphism.

Problem 3. Let P be the space of cubic polynomials with real coefficients; thus, for example, $1 - x + x^2 - x^3 \in P$. We consider P as a vector space over $\mathbb{R}$. Consider the linear map $T : P \to \mathbb{R}^3$ given by $f \mapsto (f(0), f(1), f(2))$.

(1) What is the matrix of T with respect to the basis $(1, x, x^2, x^3)$ of $\mathcal{P}$, and the coordinate basis $(1, 0, 0), (0, 1, 0), (1, 0, 0)$ of $\mathbb{R}^3$?

Note that $T(1) = (1, 1, 1), T(x) = (0, 1, 2), T(x^2) = (0, 1, 4), T(x^3) = (0, 1, 8)$. Accordingly, the matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix}$.

(2) What is the dimension of the image of T? Prove your answer.

The image has dimension 3, i.e., is all of $\mathbb{R}^3$. By the rank-nullity theorem, $\dim \text{image}(T) = \dim P - \dim \text{null}(T)$. Now, $\dim(P) = 4$. I claim that, also, $\dim \text{null}(T) = 1$; this will show the claim. If $p \in \text{null}(T)$, then $p(0) = p(1) = p(2) = 0$. So p is divisible by $x, x - 1, (x - 2)$; so p is a scalar multiple of $x(x - 1)(x - 2)$. Thus, $\text{null}(T)$ is spanned by the single vector $x(x - 1)(x - 2)$, and thus has dimension 1.

Problem 4. Let V be a finite dimensional vector space over a field F and $U \subset V$ a subspace.

(1) Prove there exists a finite-dimensional vector space W and a linear map $T : V \to W$ so that U is the null-space of T.

Let $u_1, \ldots, u_r$ be a basis for U. Extend to a basis $u_1, \ldots, u_r, v_1, \ldots, v_k$ for V. Define $T : V \to F^k$ by

$$\sum \alpha_i u_i + \sum \beta_j v_k \mapsto (\beta_1, \ldots, \beta_k) \in F^k.$$

By the definition of T, its null-space consists of those vectors of the form $\sum \alpha_i u_i$, i.e. the span of u_i. Thus $\text{null}(T) = U$, as required.

(2) Let W be any finite-dimensional vector space. What is the dimension of the following subspace X of $\mathcal{L}(V, W)$?

$$X = \{S \in \mathcal{L}(V, W) : U \subset \text{null}(S)\}.$$

Express your answer in terms of $\dim(V), \dim(W), \dim(U)$. **No proof is necessary**.[2]

Answer: $(\dim V - \dim U) \dim W$. A very similar problem was on the practice midterm; see it for the reasoning.

Problem 5 is on the other side.

[2]although if you indicate your reasoning, you may get partial credit even for a wrong answer.

Problem 5. Let V, W be finite-dimensional vector spaces over a field F.

(1) Suppose that $T_1, \dots, T_k \in \mathcal{L}(V, W)$ each satisfy $\dim \operatorname{image}(T_i) \leq 1$. Let $T = \sum_{i=1}^k T_i \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{image}(T) \leq k$.

For $1 \leq i \leq k$, let u_i span the image of T_i. (If the image of T_i is zero, we can take $u_i = 0$.) For any $v \in V$, then, there exists some scalar $\alpha_i \in F$ so that $T_i(v) = \alpha_i v$. Then

$$T(v) = \sum_i \alpha_i u_i \in \operatorname{span}(u_1, \dots, u_k)$$

This is true for every $v \in V$. So the image of T is a subspace of $\operatorname{span}(u_1, \dots, u_k)$. The latter has a basis of size $\leq k$ (any spanning set can be shrunk to a basis), and so the image of T also has dimension $\leq k$.

(2) (**Hard;** attempt only if you have time.) Suppose that $T \in \mathcal{L}(V, W)$ is so that $\dim \operatorname{image}(T) \leq k$. Prove that exists $T_1, \dots, T_k$ so that $T = \sum_{i=1}^k T_i$ and $\dim \operatorname{image}(T_i) \leq 1$.

Choose a basis $w_1, \dots, w_r$ for the image of T. Here $r \leq k$. For every $v \in V$, there exist unique scalars $\lambda_i \in F$ (for $1 \leq i \leq r$) so that $T(v) = \sum \lambda_i w_i$. Set $T_i(v) = \lambda_i w_i$. Then T_i is linear (why? prove this! In the exam it was fine to state without proof.) Also, $\operatorname{image}(T_i)$ is contained in the span of w_i, so $\dim \operatorname{image}(T_i) \leq 1$.

By definition, $T(v) = \sum_{i=1}^r T_i(v)$, for every $v \in V$, so $T = \sum_{i=1}^r T_i$.