Homework 9

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December 6, 2008

Problem 1 (Ch7 - Problem 30)**.** Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.

Proof. S^*S is self-adjoint operator with non-negative eigenvalues, ie a positive operator. By theorem 7.27-Axler, S^*S has a unique positive square root denoted by $\sqrt{S^*S}$.

- If S is an isometry then $S^*S = \text{Id}$ by Theorem 7.36 - Axler. Hence we have $\sqrt{S^*S} = I$, all of whose eigenvalues are then exactly those of Id, whis comprises of only 1.

  By definition, the singular values of S are the eigenvalues of $\sqrt{S^*S}$, thus are equal 1.

- On the other hand, suppose all the singular values of S equal to 1, ie all the eigenvalues of $\sqrt{S^*S}$ are 1.

  $\sqrt{S^*S}$ is self-adjoint thus diagonalizable . This together with the fact it only has eigenvalues 1 forces $\sqrt{S^*S} = \text{Id} \Rightarrow S^*S = \text{Id}$. By theorem 7.36-Axler, this is equivalent to S being an isometry.

□

Problem 2 (Ch 7 - Ex 31)**.** Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values if and only if there exist isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1 T_2 S_2$.

Proof. T_1, T_2 have the same singular values decomposition $s_1, \ldots, s_n$. By The Singular-Value Decomposition theorem (thm 7.46 - Axler), there exists orthonormal bases $(e_1, \ldots, e_n)$ and $(\tilde{e}_i, \ldots, \tilde{e}_n)$ and isometries S and $\tilde{S}$ such that for all $v \in V$, we have

$$T_1 v = s_1 \langle v, e_1 \rangle S e_1 + \ldots + s_n \langle v, e_n \rangle S e_n$$

$$T_2 v = s_1 \langle v, \tilde{e}_1 \rangle \tilde{S} \tilde{e}_1 + \ldots + s_n \langle v, \tilde{e}_n \rangle \tilde{S} \tilde{e}_n$$

Guesswork : from the requirement $T_1 = S_1 T_2 S_2$, we would like to find isometries S_1, S_2 so that

$$\langle S_2 v, \tilde{e}_i \rangle S_1 \tilde{S} \tilde{e}_i = \langle v, e_i \rangle S e_i (*)$$

- Hence it is reasonable to define S_1 as the linear transformation

$$S_1\tilde{S}\tilde{e}_i = Se_i$$

Note that $(\tilde{e}_1, \ldots, \tilde{e_n})$ is an orthonormal basis and $\tilde{S}$ is an isometry so $(\tilde{S}\tilde{e}_1, \ldots, \tilde{S}\tilde{e}_n)$ is also an orthonormal basis. Similarly for $(Se_1, \ldots, Se_n)$. Hence S_1 maps an orthonormal basis to an orthonormal basis. By theorem 7.36 in Axler, this shows that S_1 is an isometry.

- For S_2, we define S_2 as the linear transformation such that

$$\langle S_2\tilde{e}_i, \tilde{e}_j\rangle = \langle \tilde{e}_i, e_j\rangle$$

This gives the values of $S_2\tilde{e}_i$ for each i since we have specified the coefficients when $S_2\tilde{e}_i$ is written as a linear combination of the basis elements $(\tilde{e}_1, \ldots, \tilde{e}_n)$.

It remains to verify that S_2 is also an isometry. We need to check

$$\langle S_2\tilde{e}_i, S_2\tilde{e}_j\rangle = \langle \tilde{e}_j, \tilde{e}_j\rangle = \delta_{ij}$$

LHS is equal to

$$\begin{aligned}\sum_{k,l}\langle S_2\tilde{e}_i, \tilde{e}_k\rangle\langle S_2\tilde{e}_j, \tilde{e}_l\rangle\delta_{kl} &= \sum_k \langle \tilde{e}_i, e_k\rangle\langle \tilde{e}_j, e_k\rangle \\ &= \langle \tilde{e}_i, \tilde{e}_j\rangle \\ &= \delta_{ij}\end{aligned}$$

Hence we have found isometries S_1, S_2 with the required property :

$$\begin{aligned}S_1T_2S_2\tilde{e}_i &= \sum_j s_j\langle S_2\tilde{e}_i, \tilde{e}_j\rangle S_1\tilde{S}\tilde{e}_j \\ &= \sum_j s_j\langle \tilde{e}_i, e_j\rangle Se_j \\ &= T_1\tilde{e}_i\end{aligned}$$

Since this true for each basis elements , this equality holds for all $v \in V$ hence $T_1 = S_1T_2S_2$
□

Problem 3 (Ch 7 - Ex 32)**.** Suppose $T \in \mathcal{L}(V)$ has singular-value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \ldots + s_n\langle v, e_n\rangle f_n$$

1. Prove that for every $v \in V$

$$T^* v = s_1 \langle v, f_1 \rangle e_1 + \ldots + s_n \langle v, f_n \rangle e_n$$

2. Prove that if T is invertible, then for every $v \in V$

$$T^{-1} v = \frac{\langle v, f_1 \rangle}{s_1} e_1 + \ldots + \frac{\langle v, f_n \rangle}{s_n} e_n$$

Proof.

1. $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are orthonormal basis. Hence from the expression

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \ldots + s_n \langle v, e_n \rangle f_n$$

   the matrix representation of $T : (V, (e_1, \ldots, e_n)) \to (V, (f_1, \ldots, f_n))$ is the diagonal matrix with entries $s_1, \ldots, s_n$

$$M = \begin{pmatrix} s_1 & 0 & \ldots & 0 \\ 0 & s_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & s_n \end{pmatrix}$$

   Hence the matrix reprensentation of $T^* : (V, (f_1, \ldots, f_n)) \to (V, (e_1, \ldots, e_n))$ is the transpose of the above matrix M which means exactly for all $v \in V$

$$T^* v = s_1 \langle v, f_1 \rangle e_1 + \ldots + s_n \langle v, f_n \rangle e_n$$

2. The matrix representation of $T^{-1} : (V, (f_1, \ldots, f_n)) \to (V, (e_1, \ldots, e_n))$ is then the inverse of matrix M which is

$$M^{-1} = \begin{pmatrix} \frac{1}{s_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{s_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{s_n} \end{pmatrix}$$

   which means that for all $v \in V$

$$T^{-1} v = \frac{\langle v, f_1 \rangle}{s_1} e_1 + \ldots + \frac{\langle v, f_n \rangle}{s_n} e_n$$

$\square$

Problem 4 (Chapter 7 - ex 33)**.** Suppose $T \in Ł(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T. Prove that for every $v \in V$

$$\hat{s}\|v\| \leq \|Tv\| \leq s\|v\|$$

Proof. We have the singular value decomposition for T.

$$Tv = s_1\langle v, e_1\rangle f_1 + \ldots + s_n\langle v, e_n\rangle f_n$$

where $s_1, \ldots, s_n$ are singular values of T and $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are orthonormal bases of V. Since $(f_1, \ldots, f_n)$ is an orthonormal basis , we have

$$\begin{aligned}\|Tv\|^2 &= (s_1\langle v, e_1\rangle)^2 + \ldots + (s_n\langle v, e_n\rangle)^2 \\ &\leq s^2\left[(\langle v, e_1\rangle)^2 + \ldots + (\langle v, e_n\rangle)^2\right] \\ &= s^2\|v\|^2\end{aligned}$$

Similarly for the other inequality we have

$$\begin{aligned}\|Tv\|^2 &= (s_1\langle v, e_1\rangle)^2 + \ldots + (s_n\langle v, e_n\rangle)^2 \\ &\geq \hat{s}^2\left[(\langle v, e_1\rangle)^2 + \ldots + (\langle v, e_n\rangle)^2\right] \\ &= \hat{s}^2\|v\|^2\end{aligned}$$

$$\Rightarrow \hat{s}\|v\| \leq \|Tv\| \leq s\|v\|$$

□

Problem 5 (Chapter 7 - ex 34)**.** Suppose $T', T'' \in \mathcal{L}(V)$. Let s', s'', s denote the largest singular values of $T', T'', T' + T''$ correspondingly. Prove that $s \leq s' + s''$.

Proof. For all $v \in V$ by triangle's inequality and previous excercise we have

$$\begin{aligned}\|(T' + T'')(v)\| &\leq \|T'v\| + \|T''v\| \\ &\leq s'\|v\| + s''\|v\| \\ &= (s' + s'')\|v\|\end{aligned}$$

Now we can choose v to be e_i in the orthonormal basis in the singular decomposition so that $(T' + T'')e_i = s_i\langle e_i, e_i\rangle f_i$ and $s_i = s$. Thus the inequality above becomes

$$s \leq s' + s$$

□

Problem 6 (Extra problem)**.** *Proof.*

If an operator T is invertible then the smallest singular value $\hat{s}$ of T is $\dfrac{1}{s_{T^{-1}}}$, where $s_{T^{-1}}$ is the largest singular value for T^{-1}. If T is noninvertible then its smallest singular value is 0.

1. Consider the singular values of the product of two matrices.

   (a) Denote s', s'', s to be the largest singular values for A, B, AB correspondingly. We want to show that $s \leq s's''$.
   By ex 33, we have

   $$\|ABv\| = A(Bv) \leq s'\|Bv\| \leq s's''\|v\|$$

   Now pick v to be the e_i in the orthonormal basis in the singular decomposition so that $(AB)e_i = s_i\langle e_i, e_i\rangle f_i$ and $s_i = s$. Thus the inequality above becomes $s \leq s's''$.

   (b) Denote $\hat{s}', \hat{s}'', \hat{s}$ to be the smallest singular values for A, B, AB correspondingly. We want to show $s \geq s's''$. From a previous hw we have that AB is invertible if and only if A and B are both invertible.

   - If AB is not invertible then either A or B is not invertible. For an invertible operator one of its singular value has to be 0 hence the smallest singular value is also 0. Hence both sides of the inequality are 0.
   - If AB is invertible, so are both A and B. From the work shown above for the largest singular value we have

   $$s_{T^{-1}} \leq s'_{T^{-1}}s''_{T^{-1}} \Rightarrow \frac{1}{\hat{s}} \leq \frac{1}{\hat{s'}}\frac{1}{\hat{s''}} \Rightarrow \hat{s} \geq \hat{s}'\hat{s}''$$

   Hence in either case, we have $\hat{s} \geq \hat{s}'\hat{s}$

   (c) The inequality established for the largest singular value does not hold for second largest. Consider the following 2×2 matrices.

   $$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

   In this case, we have the second largest singular value of the product is 2 is greater than the product of the second largest singular values of each matrix, which is 1.

   (d) The inequality that was established for smallest singular values does not hold in the second smallestcase. Take for example

   $$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The second smallest singular value of the product is 2 while the product of the second smallest singular values for each of the matrix is 4. Hence it is not true that $2 > 2.2$.

2. Sum of matrices

   (a) The inequality established for the largest singular value does not hold in the case of second largest singular values. Consider the counter example with the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   The second largest value for the sum is 1 while the sum of second largest value is 0. Hence it is not true that $1 \leq 0 + 0$.

   (b) For the smallest singular values, we do not $\hat{s} \geq \hat{s}' + \hat{s}''$. Consider the counterexample.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   The smallest singular value of the sum is 0 while the sum of the smallest singular values is 2 hence it is false that $0 \geq 1 + 1$

   (c) For second smallest, consider the examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} ; \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 + \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 + \frac{1}{2} & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

   Here the second smallest singular value of the sum which is 4 is greater than the sum of the second smallest which is $2 + 1 + \frac{1}{2}$.
   While for the case

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} ; \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 + \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

   The second smallest singular value of the sum which is 2 is leass the sum of the second smallest which is $\frac{1}{2}$.

□