Homework 6

Ha Pham

November 12, 2008

Problem 1. Chapter 5 - ex 4: Suppose $S, T \in L(V)$ are such that $ST = TS$. Prove that $\text{Ker}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbb{F}$.

Proof. Suppose $u \in \text{Ker}(T - \lambda I)$ ie $Tu = \lambda u$.

We want to show that $Su \in \text{Ker}(T - \lambda I)$ also ie $(T - \lambda I)Su = 0$.

$$TSu = STu = S(\lambda u) = \lambda Su \Rightarrow (T - \lambda I)Su = 0$$

□

Problem 2. Chapter 5 - ex 5.

Define $T \in L(F^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

Proof.

$$T(z, z) = (z, z); T(z, -z) = (-z, z) = -(z, -z)$$

$(1, 1), (1, -1)$ are two linearly independent eigenvectors with corresponding eigenvalues 1 and -1. The vector space is of dim 2. Hence there are 2 eigenvalues 1 and -1.

The eigenvectors associated with 1 are multiples of $(1, 1)$. The eigenvectors associated with -1 are multiples are $(1, -1)$. □

Problem 3. Chapter 5 - ex 6: Define $T \in L(\mathbb{F}^3)$ by :

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$$

Find all eigenvalues and eigenvectors of T.

Proof. Denote by M the matrix representation of T with respect to the standard basis.

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

The matrix is in upper triangular form,so we can read the eigenvalues off the diagonal, thus the eigenvalues are : 0 and 5.

The eigenvectors for eigenvalue 0 are in the null space of T, which is of dimension 1. Hence they are all mulptiples of $(1, 0, 0)$.

The eigenvectors for eigenvalue 5 are in the null space of $T - 5I$, whose matrix representation is, with respect to the standard basis:

$$\begin{pmatrix} -5 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus the null space in this case is of dimension 1. And all eigenvectors associated with eigenvalue 5 are multiples of $(0, 0, 1)$.

□

Problem 4. Chapter 5 - ex 7. Suppose n is a positive integer and $T \in L(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \ldots + x_n, \ldots, x_1 + \ldots + x_n)$$

Find all eigenvectors and eigenvalues for T.

Proof. Denote by M the matrix representation of $T - \lambda I$ with respect to the standard basis:

$$\begin{pmatrix} 1-\lambda & 1 & 1 & \ldots & 1 \\ 1 & 1-\lambda & 1 & \ldots & 1 \\ 1 & 1 & 1-\lambda & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1-\lambda \end{pmatrix}$$

in other words, M is the matrix with $1 - \lambda$ along the diagonal and 1 elsewhere.

To compute the characteristic equation of T, we compute the determinant of M. First we carry out some row and column operations to put M into upper triangular form.

1. For each $i \neq 1$, replace i^{th} row with itself and -1 the first row. By this way, we obtain a matrix of the form : the first row remains the same; for each $i \neq 1$, i^{th} row consists of λ in the first entry, $-\lambda$ in the i^{th} spot, and 0 otherwise.

$$\begin{pmatrix} 1-\lambda & 1 & 1 & \ldots & 1 \\ \lambda & -\lambda & 0 & \ldots & 0 \\ \lambda & 0 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & 0 & 0 & \ldots & -\lambda \end{pmatrix}$$

2. Operating column operation on this new matrix by : replace with first column by itself with $-1 \times (C_2 + \ldots + C_n)$, which gives us the matrix :

$$\begin{pmatrix} n-\lambda & 1 & 1 & \ldots & 1 \\ 0 & -\lambda & 0 & \ldots & 0 \\ 0 & 0 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda \end{pmatrix}$$

Hence the characteristic polynomial for T is :

$$f(x) = (-1)^n (x-n) x^{n-1}$$

Thus the eigenvalues are n and 0.

The dimension of the null space for $T - nI$ is 1, hence all eigenvectors associated with eigenvalue n are multiples of $(1, \ldots, 1)$.

The dimension of the null space for T is $n-1$ and the eigenvectors associated with 0 are of the form : $(-(x_2 + \ldots + x_n), x_2, \ldots, x_n), x_i \in \mathbb{F}$, ie the null space is spanned by :

$$(-1, 1, \ldots, 0), (-1, 0, 1, \ldots, 0), \ldots, (-1, 0, \ldots, 1)$$

□

Problem 5. Chapter 5 - ex 11. V is a finite dim VS. Suppose $S, T \in L(V)$. Prove that ST, TS have the same eigenvalues.

Proof.

1. Suppose λ is a nonzero eigenvalue for ST, ie, $\exists 0 \neq u \in V, STu = \lambda u$. From this equation, note that $Tu \neq 0$. Hence apply T to both sides, we obtain :

   $$TSTu = \lambda Tu \Rightarrow TS(Tu) = \lambda(Tu)$$

   Since $Tu \neq 0$, λ is thus an eigenvalue for TS.

2. Suppose ST has 0 as an eigenvalue. WTS TS also has 0 as an eigenvalue, ie $\operatorname{Ker} TS$ is nontrivial.

   Suppose $\operatorname{Ker} TS$ is trivial. Since V is finite-dim, TS is a linear isomorphism, ie TS is invertible, which by a previous hw problem, we have both T and S are invertible. This in turn gives that ST is invertible : this contradicts ST has a 0 as one of its eigenvalue, ie nontrivial kernel.

Hence we have shown that if λ is an eigvenvalue for ST, it is also one for TS. For the other direction, simply exchange the role of T and S.

$\square$

Problem 6. Chapter 5 – ex 12. Suppose $T \in L(V)$ is such that every vector in V is an eigen vector of T. Prove that T is a scalar multiple of the identity operator.

Proof. From the hypothesis, we have $\forall u, v \in V, \exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ such that:

$$Tu = \lambda_1 u; Tv = \lambda_2 v$$

$$T(u + v) = \lambda_3(u + v)$$

Suppose that, contrary to hypothesis, T is not a multiple of the identity. This means that we may choose u, v so that $\lambda_1 \neq \lambda_2$. Then:

$$\lambda_3(u + v) = \lambda_1 u + \lambda_2 v$$

$$\Rightarrow (\lambda_3 - \lambda_1)u + (\lambda_3 - \lambda_2)v = 0$$

This means that any two vector in V are linearly dependent. Thus V is one dimensional. In this case, *every* linear transformation from V to V (without any further conditions) is a multiple of the identity operator, contradiction.

$\square$

Problem 7. Chapter 5 – ex 21. Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = \operatorname{Ker} P \oplus \operatorname{Range} P$.

Proof.

1. We have

$$v \in \operatorname{Ker} P \cap \operatorname{Range} P \Rightarrow \exists u \in V : v = Pu$$

$$\Rightarrow 0 = Pv = P^2 u = Pu = v \Rightarrow v = 0 \Rightarrow \operatorname{Ker} P \cap \operatorname{Range} P = \{0\}$$

2. For $v \in V$, we have $v = Pv + (v - Pv)$. Note that $P(v - Pv) = Pv - P^2 v = Pv - Pv = 0$. Hence $v - Pv \in \operatorname{Ker} P$

   Thus $V = \operatorname{Ker} P + \operatorname{Range} P$

Combining two steps, we have the desired result.

$\square$

Problem 8. Chapter – ex 23. Give an example of an operator $T \in L(\mathbb{R}^4)$ such that T has no real eigenvalues.

Proof. Consider T as operation defined by the multiplication on the left of column vectors by the following matrix.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then the characteristic polynomial for T is the $\det T - xI$, is the product of det of two block matrices, each has det $x^2 + 1$:

$$\begin{pmatrix} -x & -1 & 0 & 0 \\ 1 & -x & 0 & 0 \\ 0 & 0 & -x & -1 \\ 0 & 0 & 1 & -x \end{pmatrix}$$

Hence characteristic polynomial $(x^2 + 1)^2$, which has no real roots. Since if T has any eigenvalue, it has to be one of the roots of the characteristic polynomial. In this case, T must have no (real) eigenvalues. □

Problem 9. Extra problem – Suppose that the characteristic polynomial $p(x)$ of a 3×3 matrix M is $\det(xId - M) = (x-1)(x-2)(x-3)$. What is the determinant of M? What is the characteristic polynomial of M^{100}?

Proof.

We know that $\text{Det}\, M = (-1)^3 p(0) = 6$.

From the characteristic polynomial for M, we know M has 3 distinct eigenvalues $1, 2, 3$, which comes with 3 corresponding linearly independent eigenvectors, called v_1, v_2, v_3.

The matrix representation of M with respect to this basis is, denoted by D:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Note D^{100} is of the following form, thus has characteristic polynomial $(x-1)(x-2^{100})(x-3^{100})$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{pmatrix}$$

Denote A to be the change of basis matrix from $\{v_1, v_2, v_3\}$ to the standard basis, we then have : $M = ADA^{-1}$.

On the other hand,

$$\begin{aligned}\det(xM^{100} - Id) &= \det A \det(xM^{100} - Id) \det A^{-1} \\ &= \det(xAM^{100}A^{-1} - AA^{-1}) \\ &= \det(x(AMA^{-1})^{100} - Id) \\ &= \det(xD^{100} - Id)\end{aligned}$$

Hence M^{100} and D^{100} have the same characteristic polynomial, which is $(x-1)(x-2^{100})(x-3^{100})$

□