# HW 5

November 2, 2008

**Problem 1.** Suppose that $M$ is a square upper triangular matrix. Prove that the determinant of $M$ is the product of the diagonal entries.

*Proof.* Let $e_1, \ldots, e_n$ be the usual basis of $F^n$, i.e. $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0)$. Then $M(e_1) = a_{11}e_1, M(e_2) = a_{22}e_2 + a_{12}e_1, M(e_3) = a_{33}e_3 + a_{23}e_2 + a_{13}e_1, \ldots$.

Now, let $\Lambda$ be an alternating $n$-form on $F^n$. Now $M(e_1), \ldots, M(e_{n-1}), M(e_n) - a_{nn}e_n$ are contained in the span of $e_1, \ldots, e_{n-1}$ and are so linearly dependent. Therefore,

$$\begin{aligned} &\Lambda(M(e_1), M(e_2), \ldots, M(e_n)) \\ &\quad = \Lambda(M(e_1), \ldots, M(e_{n-1}), a_{nn}e_n) + \Lambda(M(e_1), \ldots, M(e_n) - a_{nn}e_n) \\ &\quad = \Lambda(M(e_1), \ldots, M(e_{n-1}), a_{nn}e_n), \end{aligned}$$

where we used the fact that an alternating $n$-form vanishes on $v_1, \ldots, v_n$ if they are not a basis.

Now, in exactly the same way, $M(e_1), \ldots, M(e_{n-2}), M(e_{n-1}) - a_{n-1,n-1}e_{n-1}$ span a space of dimension $\leq n - 2$ and so, together with $e_n$, cannot be a basis. By the same reasoning,

$$\Lambda(M(e_1), \ldots, M(e_{n-1}), a_{nn}e_n) = \Lambda(M(e_1), \ldots, M(e_{n-2}), a_{n-1,n-1}e_{n-1}, a_{nn}e_n)$$

Continuing in this way we arrive at

$$\Lambda(M(e_1), M(e_2), \ldots, M(e_n)) = \Lambda(a_{11}e_1, a_{22}e_2, a_{33}e_3, \ldots, a_{nn}e_n) = \Lambda(e_1, \ldots, e_n) \prod_{i=1}^{n} a_{ii}$$

so the determinant is $\prod a_{ii}$.

$\square$

**Problem 2.** Let $D$ be a nonzero alternating 3-form on $\mathbb{R}^3$. Describe in geometric terms when, for $v_1, v_2, v_3$ in $\mathbb{R}^3$, the sign of $D(v_1, v_2, v_3)$ is positive.

*Proof.* Suppose that $D((1,0,0),(0,1,0),(0,0,1)) > 0$. Then one may use the *right-hand rule*: if you curl your hand in the direction from $v_1$ to $v_2$, then your thumb either points in the direction of $v_3$ (in which case $D(v_1, v_2, v_3) > 0$) or in the opposite direction (in which case $D(v_1, v_2, v_3) < 0$).

(This is usually well-described in physics texts. If you have studied the *cross product*, then you should check that:

$$D(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3,$$

where $\times$ is the cross-product and $\cdot$ the dot product. The main point of the question, though, was simply to get you to think about the sign. One way to figure this out was to fix $v_1, v_2$ and then move around $v_3$ and try to think about when the sign changes from $+$ to $-$. )

□

**Problem 3.** Suppose that $D$ is a nonzero alternating n-form on an n-dimensional vector space $V$. Suppose that $e_1, \ldots, e_n$ is a basis for $V$.

1. Prove that, if we replace $e_1$ by $e_1 + \alpha e_j$, for any $\alpha \in \mathbb{F}$ and any $j > 1$, the value of $D(e_1, ..., e_n)$ remains unchanged.

2. Prove that, if $M$ is an $n \times n$ square matrix, then adding any multiple of a row of $M$ to some other row leaves $det(M)$ unchanged.

*Proof.* 1. Since $D$ is multilinear, for $j > 1$, we have :

$$D(e_1 + \alpha e_j, ..., e_n) = D(e_1, \ldots, e_j, \ldots, e_n) + \alpha D(e_j, \ldots, e_j, \ldots, e_n)$$

$D$ is alternating so $D(e_j, \ldots, e_j, \ldots, e_n) = 0$.

$$\Rightarrow D(e_1 + \alpha e_j, ..., e_n) = D(e_1, \ldots, e_j, \ldots, e_n)$$

2. Let $S$ be the matrix with all diagonal entries 1, and all other entries zero except the $(i, j)$th, which is also equal to 1. (For instance, for $n = 3$ and $i = 1, j = 2$, $S$ would look like $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, for any matrix $M$, the matrix $MS$ (respectively $SM$) is obtained from $M$ by adding the $i$th column to the $j$th column (respectively, $j$th row to the $i$th row). Now, since

$$\det(SM) = \det(MS) = \det(S)\det(M),$$

it suffices to check that $\det(S) = 1$. However, since $S(e_k) = e_k$ for all $k \neq j$ and $S(e_j) = e_j + e_i$, this follows from the first part of the question.

□

**Problem 4.** Katznelson Corollary 4.4 Katznelson IV. 5.5 .

*Proof.* Let $A$ be an $n \times n$ matrix. Denote $T : M_{n\times n} \to M_{n\times n}$, the operation the operation $B \mapsto BA$.

We claim that $\det(T) = \det(A)^n$. The space $M_{n\times n}$ is the interior direct sum of the spaces $V_i$, for $1 \le i \le n$, where

$$V_i = n \times n \text{ matrices with nonzero entries only in the } i\text{th column.}$$

Then $T(V_i) \subset V_i$ for each $i$. The determinant of $T$ (by repeated application of the Corollary 4.4) is the product of the determinants of $T$ restricted to the spaces $V_i$.

For each $1 \le i \le n$, consider the matrix $E_{ij} \in V_i$ which has a 1 in the $ij^{th}$ entry and 0 elsewhere. Then the $E_{ij}$ $(1 \le j \le n)$ form a basis for $V_i$, and the matrix of $T$ with respect to this basis is precisely $A$. So, $\det T|_{V_i} = \det(A)$ for every $i$, whence our claim.

□

**Problem 5.** Katznelson IV. 5.12 -

*Proof.* 1. As noted in the hint, $V(a_1, \dots, a_n, x)$ is a polynomial of degree $n$ in $x$, which can be seen from the cofactor expansion or the definition of the determinant. Denote this polynomial by $f_n(x)$.

If $x = a_i$, then we get a matrix with two identical rows, thus the value of the determinant is 0, ie

$$f_n(a_i) = 0 \Rightarrow f_n(x) = (x - a_1) \dots (x - a_n) g(x)$$

However degree of $f_n$ is $n$, thus for some constant $C$ , we have :

$$V(a_1, \dots, a_n, x) = f_n(x) = C(x - a_1) \dots (x - a_n),\ where\ C(-1)^n a_1 \dots a_n = f_n(0)$$

Now $f_n(0)$ equals the determinant of the matrix obtained when we plug in $x = 0$. To calculate the determinant for this matrix, use cofactors and expand along the last row, which gives $f_n(0) = (-1)^n\ det(M)$, where $M$ is the matrix as below :

$$\begin{pmatrix} a_1 & \dots & a_1^n \\ \vdots & \vdots & \vdots \\ a_n & \dots & a_n^n \end{pmatrix}$$

For $1 \le i \le n$, we can factor out $a_i$ from each $i^{th}$ row to obtain matrix $\hat{M}$

$$\begin{pmatrix} 1 & \dots & a_1^{n-1} \\ \vdots & \vdots & \vdots \\ 1 & \dots & a_n^{n-1} \end{pmatrix}$$

Hence

$$f_n(0) = (-1)^n det(M) = a_1 \dots a_n \; det(\hat{M}) = a_1 \dots a_n V(a_1, \dots, a_n)$$
$$\Rightarrow V(a_1, \dots, a_n, x) = V(a_1, \dots, a_n)(x - a_1) \dots (x - a_n)$$

2. The statement is true when $n = 2$.

   Assume this is true for $n = k$, consider the case $n = k + 1$.

   By part 1, we have

   $$V(a_1, \dots, a_k, a_{k+1}) = V(a_1, \dots, a_k)(a_{k+1} - a_1) \dots (a_{k+1} - a_k)$$

   Thus we get the result by using the inductive hypothesis $V(a_1, \dots, a_k) = \prod_{j<i}(a_i - a_j)$

3. The rank is equal to the cardinality of the set $\{a_1, \dots, a_n\}$.

□

**Problem 6.** Problem 6 and extra credit, 7.

$M$ is an $n \times n$ matrix . Suppose that, for every scalar $x$ in $\mathbb{F}$, $det\ (xM + Id) = 1$. Prove that $M^n = 0$.

*Proof.* We give a general proof by induction on the dimension. For $n = 1$, the assertion is clear. For $n = 2$ it was possible to compute everything explicitly.

The leading term of $\det(xM + Id)$ is $x^n \det(M)$, so we must have $\det(M) = 0$. Thus $M$ is not invertible.

Choose $v$ so that $M(v) = 0$ and extend it to a basis $v, v_2, \dots, v_n$. Then $V = \text{span}(v) \oplus_{int} U$, where $U = \text{span}(v_2, \dots, v_n)$. For each $u \in U$, we may write uniquely $M(u) = \lambda_u v + M'(u)$ for some $\lambda_u \in F$ and $M'(u) \in U$. (This is so by definition of interior direct sum.) In fact, $M'$ is a linear map $U \to U$.

*Claim.* $(M')^{n-1} = 0$.

Let $\Lambda$ be a nonzero alternating form on $F^n$. Since $\det(xM + Id) = 1$ for all $x$, we have:

$$\Lambda(v, v_2 + xM(v_2), \dots, v_n + xM(v_n)) = \Lambda(v, v_2 \dots, v_n).$$

Let $\Lambda'$ be the alternating $(n-1)$-form on $U$ defined by $\Lambda'(u_1, \dots, u_{n-1}) = \Lambda(v, u_1, \dots, u_{n-1})$. $\Lambda'$ is nonzero (why?) and the prior equation implies that

$$\Lambda'(v_2 + xM'(v_2), \dots, v_n + xM'(v_n)) = \Lambda'(v_2, \dots, v_n).$$

Thus also $\det(xM' + Id) = 1$ for all $x$. By the inductive hypothesis, $(M')^{n-1} = 0$. This implies (check!) that $M^n = 0$.

*Comment.* This was a tricky question! Later in the course we will cover the "Cayley-Hamilton" theorem which is a much more general statement of this type. The construction in the above proof would be more natural using the construction of a quotient of vector spaces, which, perhaps regrettably, we did not introduce at the start of the course.

□

**Problem 7.** Extra credit 8 - Let $A : V \to V$ be a linear transformation on a finite dimensional space $V$. Let $\hat{A} : V^* \to V^*$ be the adjoint. Prove $det(A) = det(\hat{A})$.

*Proof. Comment.* This corresponds to the fact, which you may have seen, that the determinant of a matrix and its transpose are the same. This is, in fact, the easiest way to prove it at this stage: you can see that the determinant of a matrix $A$ and its transpose $A^t$ are the same by induction on the size of the matrix: expand $\det(A)$ by cofactors around the first column, and $\det(A^t)$ by cofactors around the first row.

There exists a "less computational" proof, but it takes time to set up. The key fact is the following: *the space of alternating $n$-forms on $V$* is *naturally isomorphic* to the *dual space to the space of alternating $n$-forms on $V^*$.* This takes a while to explain, so come and ask me about it if you are interested!

□