HW 4

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Problem 1. Describe how the adjoint of a composition of two linear maps relates to the adjoints of each linear map separately.

Proof. Suppose V, U, W are three vector spaces. Suppose that $T \in L(V, U), S \in L(U, W)$. Then $\widehat{ST} = \hat{T}\hat{S}$.

For clarity in the proof, let us write $A \circ B$ for "A composed with B," i.e. $A \circ B(x) = A(B(x))$. Now, given $\ell \in W^*$,

$$\widehat{ST}(\ell) = \ell \circ (ST) = \ell \circ (S \circ T) = (\ell \circ S) \circ T = (\hat{S}\ell) \circ T = \hat{T}(\hat{S}\ell) = (\hat{T}\hat{S})(\ell)$$

This being so for all ℓ, we conclude that $\widehat{ST} = \hat{T}\hat{S}$.

□

Problem 2. Suppose X, Y are two vector spaces. Let Z be the direct sum of X and Y. Prove that Z^* is isomorphic to the direct sum of X^* and Y^*.

Proof. We shall define inverse maps $G : Z^* \to X^* \oplus Y^*$ and $F : X^* \oplus Y^* \to Z^*$. (In other terms, both FG and GF are the identity maps.)

In what follows, we denote by x, y, z elements of X, Y, Z respectively; and by x^*, y^*, z^* elements of X^*, Y^*, Z^* respectively. We define F, G by

$$F(x^*, y^*)(x, y) = x^*(x) + y^*(y), G(z^*) = (z^*|_X, z^*|_Y)$$

Very important. Make sure that you can decipher the line of cryptic notation above! Note that $z^*|_X$ means "z^* restricted to X"; in explicit terms, $z^*|_X(x) = z^*(x, 0)$.

Then F, G are linear maps which are inverse to each other. (At this stage in the course, you do not need to write more than this so long as you understand why it is true. Make sure that you understand the definitions of F, G as well as why they are inverses.)

□

Problem 3. In the following questions, V, W are finite dimensional vector spaces over the field $\mathbb{F}$, $T : V \to W$ is a linear map and $\hat{T} : W^* \to V^*$ be the adjoint. Prove the following:

1. There exists a basis $e_1, ..., e_n$ for V and a basis $f_1, ... f_m$ for W so that Te_i is either equal to f_i, or equal to 0. Describe the adjoint $\hat{T}$ with respect to the dual bases e_i*, f_j*.
2. If T is surjective, then $\hat{T}$ is injective.
3. If T is injective, then $\hat{T}$ is surjective.
4. The dimension of the image of T and the dimension of the image of $\hat{T}$ are the same

Proof. 1. Pick a basis for null(T), say $e_{k+1}, ..., e_n$, where $n-k =$ dim of Ker T. Extend this to a basis for V, say $e_1, ..., e_k$. Denote $f_i = Te_i$. Note $f_1, ..., f_k$ are linearly independent in W. If they are not a basis, extend this list to a basis for W, say $f_1, \ldots, f_m$. Thus we have found the wanted basis for V, and W, with $Te_i = f_i, 1 \leq i \leq k$, and 0 otherwise.

We claim that:

$$\hat{T} f_i^* = \begin{cases} e_i^*, 1 \leq i \leq k \\ 0, k+1 \leq j \leq m \end{cases} \tag{1}$$

One way to prove this is to use the theorem from class: the matrix of T and the matrix of $\hat{T}$, "with respect to dual bases", are transposes of each other. We'll prove it explicitly:

For $1 \leq i \leq k$, we have :

$$1 \leq j \leq k, \hat{T} f_i^*(e_j) = f_i^*(Te_j) = f_i^*(f_j) = \delta_j^i = e_i^*(e_j)$$

$$k+1 \leq j \leq n, \hat{T} f_i^*(e_j) = f_i^*(0) = 0 = e_i^*(e_j)$$

$$\Rightarrow \hat{T} f_i^* = e_i^*$$

For $k+1 \leq i \leq n$, we have :

$$1 \leq j \leq k, \hat{T} f_i^*(e_j) = f_i^*(Te_j) = f_i^*(f_j) = 0$$

$$k+1 \leq j \leq n, \hat{T} f_i^*(e_j) = f_i^*(0) = 0$$

$$\hat{T} f_j^* = 0$$

We shall now address the other parts of the question, making use of the result just proven: there are bases $(e_1, \ldots, e_n)$ for V and $(f_1, \ldots, f_m)$ for W so that $Te_i = f_i$ for $1 \leq i \leq k$, and $Te_i = 0$ for $i > 0$;

1. If T is surjective, then $\hat{T}$ is injective.

   Indeed, if T is surjective, then necessarily $m = k$. By the explicit description of the adjoint in (1), it carries the basis $\{f_1^*, \ldots, f_m^*\}$ to the linearly independent set $\{e_1^*, \ldots, e_m^*\}$. So it is injective.

   Alternately: If T is surjective and $l \in W^*$ satisfies $\hat{T}(l) = 0$. Since T is sujective, $\forall w \in W, l(w) = l(Tx) = 0$ for some $x \in V$. Thus $l = 0$ so $\hat{T}$ is injective.

2. If T is injective, then $\hat{T}$ is surjective.

   In this case, we must have $k = n$. By the explicit description of the adjoint in (1), its image contains the spanning set $e_1^*, \dots, e_n^*$ for V^*. So it is surjective.

3. The dimension of the image of T and the dimension of the image of $\hat{T}$ are the same.

   Indeed, the image of T is spanned by $\{f_1, \dots, f_k\}$, whereas the image of $\hat{T}$ is spanned by $\{e_1^*, \dots, e_k^*\}$. So both are equal to k.

$\square$

Problem 4. Suppose that V is a finite-dimensional vector space with basis $e_1, ...e_n$. Let $e_1^*, ..., e_n^*$ be the dual basis for $V*$. Suppose that $T : V \to V*$ has a matrix a_{ij} (with respect to these bases) that is symmetric, i.e., $a_{ij} = a_{ji}$. Prove that this is "independent of basis", i.e., given any other basis $f_1, ..., f_n$, with dual basis $f_1^*, ..., f_n^*$, the matrix of T with respect to $f_1, ..., f_n$ and $f_1^*, ..., f_n^*$ is also symmetric.

Proof. Given any $T : V \to V^*$, we may form $\hat{T} : (V^*)^* \to V^*$. We should like to say that T is symmetric if and only if $\hat{T} = T$. *This does not make sense;* T and $\hat{T}$ have different domains. However, we have defined an isomorphism $\Phi : V \to (V^*)^*$ by $\Phi(v)(w^*) = w^*(v)$. Therefore we *can* sensibly ask whether $T = \hat{T} \circ \Phi$ (here, again, $\circ$ denotes composition).

Claim. $T = \hat{T} \circ \Phi$ if and only if the matrix of T with respct to e_i, e_i^* is symmetric.

Indeed, $T = \hat{T} \circ \Phi$ if and only if $T(e_i) = \hat{T} \circ \Phi(e_i)$ for all i. (*Warning: it is almost certainly more useful to try to prove this yourself than read what follows.*) In turn, this is so if and only if

$$(T(e_i))(e_k) = (\hat{T} \circ \Phi(e_i))(e_k) \quad \text{for all } i, k$$

Write $T(e_i) = \sum_j a_{ji} e_j^*$. Then $T(e_i)(e_k) = a_{ki}$. On the other hand,

$$\left(\hat{T} \circ \Phi(e_i)\right)(e_k) = \Phi(e_i)(T(e_k)) = \Phi(e_i)(\sum_j a_{jk} e_j^*) = \sum_j a_{jk} e_j^*(e_i) = a_{ik}.$$

Therefore, $T = \hat{T} \circ \Phi$ if and only if $a_{ik} = a_{ki}$ for all i, k. We have proved the claim.

The claim immediately implies the desired result, since it is true for any basis (e_i).

$\square$