# Solutions to linear algebra, homework 1

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**Problem 1.** (Problem 8, Chapter 2, Axler). The subspace $U$ consists of all vectors of the form $(3x, x, 7y, y, z)$, where $(x, y, z) \in \mathbb{R}^3$ are arbitrary. Since

$$(3x, x, 7y, y, z) = x(3, 1, 0, 0, 0) + y(0, 0, 7, 1, 0) + z(0, 0, 0, 0, 1) \tag{1}$$

it is clear that $((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1))$ span $U$. (1) also shows that they are linearly independent. So they are a basis.

**Problem 2.** (Problem 11, Chapter 2, Axler). See Katznelson 1.2.9 below. Apologies for the repeated problem.

**Problem 3.** (Problem 13, 14 Chapter 2, Axler) Consider the linear map $T : U \oplus W \to U + W = \mathbb{R}^8$ which sends $(u, w)$ to $u + w$. The null space consists precisely of $\{(x, -x) : x \in U \cap W\}$. Thus $\dim(U \cap W) + \dim(U + W) = \dim U + \dim W$. (Or you could have quoted this result from class, or from Axler.) In the context of problem 13, this shows that $U \cap W$ is zero-dimensional, so equal to 0; in the context of problem 14, it shows that $U \cap W$ has dimension 2, so is not $\{0\}$.

**Problem 4.** (Problem 17, Chapter 2, Axler) False. Let $U_1, U_2, U_3$ be three distinct one-dimensional subspaces of $\mathbb{R}^2$. Thus $\dim U_i = 1$ for all $i$. By assumption of distinctness, $U_1 \cap U_2$ and all other intersections are zero. Thus, the sum on the right-hand equals 3, whereas $U_1 + U_2 + U_3 = \mathbb{R}^2$ has dimension 2.

**Problem 5.** (Katznelson 1.2.7)

Suppose that $\sum \lambda_i u_i + \sum \nu_i w_i = 0$. Then $\sum \lambda_i u_i = -\sum \nu_i w_i$. The left-hand side belongs to $U$; the right hand side to $W$. Thus, both sides lie in $U \cap W$, and both sides are zero. Since $u_i$ are linearly independent, all $\lambda_i$ are zero. Since $w_i$ are linearly independent, all $\nu_i$ are zero.

**Problem 6.** (Katznelson 1.2.9)

*Let V be a finite-dimensional vector space over the complex numbers. Let V' be V BUT considered as a vector space over the real numbers. Show that the dimension of V' is twice the dimension of V.*

Let $v_1, \ldots, v_n$ be a basis for $V$ over $\mathbb{C}$. Let $w_j = i.v_j$. Here $i \in \mathbb{C}$ is, as usual, an element with $i^2 = 1$.

I claim that $\mathcal{L} = v_1, \ldots, v_n, w_1, \ldots, w_n$ is a basis for $V'$.

1. $\mathcal{L}$ is LI. Indeed, suppose that $\sum a_j v_j + \sum b_j w_j = 0$ in $V'$. This equation is also valid in $V$, and it may be re-written as $\sum (a_j + ib_j) v_j = 0$. Since the $v_j$ are independent over $\mathbb{C}$, $a_j = b_j = 0$. Thus, $\mathcal{L}$ is LI.

2. $\mathcal{L}$ is spanning.

   Given $x \in V'$, we may write it as a *complex* linear combination of the $v_j$. Thus $x = \sum z_j v_j$. Write $z_j = a_j + ib_j$, where $a_j, b_j$ are real. Then $x = \sum a_j v_j + b_j w_j$. This equation makes sense in $V'$, for it involves only real scalars, and it shows that $\mathcal{L}$ is spanning.

**Problem 7.** *(Katznelson 1.2.9; previously listed incorrectly as Katznelson1.2.8) Suppose that V is a finite dimensional vector space. Show that every subspace W of V satisfies dim(W) $\leq$ dim(V), and that equality dim(W) = dim(V) holds only when W=V.*

Any linearly independent subset of $W$ is also linearly independent in $V$, so its size is $\leq \dim(V)$. Choose a maximal linearly independent list $(w_1, \ldots, w_t)$ in $W$ (it is finite, by what we just noted). By a theorem proved in class, a maximal linearly independent list is a basis, so $t = \dim(W)$. On the other hand, $w_1, \ldots, w_t$ can be extended to a basis for $V$; in particular, $t \leq \dim(V)$. If $t = \dim(V)$, then $w_1, \ldots, w_t$ *is* a basis for $V$. This means that:

$$V \subset \operatorname{span}(w_1, \ldots, w_t) = W$$

and so $V = W$.

**Problem 8.** *(Extra credit). Suppose that we are given a system of linear equations all of whose coefficients are integers (example: 2x+3y+4z=2, x-z=8) and you know that it has a solution in real numbers. Prove that it has a solution in rational numbers.*

**Mathematical rephrasing:** Given $w_1, \ldots, w_N \in \mathbb{Q}^k$ – in the example above, $N = 3, k = 2$ and $w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ – and real numbers $\lambda_i \in \mathbb{R}$ so that

$$x = \sum \lambda_i w_i \in \mathbb{Q}^k \tag{2}$$

show that there exist $\lambda_i' \in \mathbb{Q}$ so that $x = \sum \lambda_i' w_i$.

*Proof.* Let $W$ be the subspace spanned by $w_i$, for $1 \leq i \leq N$. Since $w_i$ span $W$, we can obtain a basis of $W$ by deleting certain of them; we may suppose, w.l.o.g., that $w_1, \ldots, w_t$

form a basis of $W$. Extend this to a basis $w_1, \ldots, w_t, v_1, \ldots, v_r$ of $\mathbb{Q}^k$. Consider the linear map:

$$L : \mathbb{Q}^k \to \mathbb{Q}^r, L(\sum_{i=1}^{t} a_i w_i + \sum_{i=1}^{r} b_i v_i) = (b_1, \ldots, b_r)$$

By definition,

$$\text{null}(L) = \text{span}_{\mathbb{Q}}(w_1, \ldots, w_t). \tag{3}$$

where we have written $\text{span}_{\mathbb{Q}}$ to emphasize, for later reference, that this is the span inside a $\mathbb{Q}$-vector space.

Now, $L$ extends to a linear map $\tilde{L} : \mathbb{R}^k \to \mathbb{R}^r$ of $\mathbb{R}$-vector spaces, so that[1] $\tilde{L}|\mathbb{Q}^k = L$.

Why is this true? With respect to the standard bases of $\mathbb{Q}^k, \mathbb{Q}^r$, the linear transformation $L$ is defined by a certain $r \times k$ matrix. This matrix has rational entries; thinking of the entries as real numbers, it also describes a linear transformation $\tilde{L} : \mathbb{R}^k \to \mathbb{R}^r$, which clearly extends $L$.

If $x$ is defined according to (2), then $\tilde{L}(x) = 0$, because $\tilde{L}$ is a linear map of $\mathbb{R}$-vector spaces and $\tilde{L}(w_i) = L(w_i) = 0$. Therefore, $x \in \text{null}(\tilde{L}) \cap \mathbb{Q}^k = \text{null}(L)$. So, by (3), $x$ belongs to the $\mathbb{Q}$-span of $w_i$. $\square$

**Remark.** It is also perfectly possible to solve this problem by making reference to a specific algorithm for producing a solution – e.g., some kind of row/column reduction on matrices– and observing that *if* the algorithm works over $\mathbb{R}$, it also works over $\mathbb{Q}$.

[1] This notation means that "$\tilde{L}$, when restricted to $\mathbb{Q}^k$, coincides with $L$."