Solutions to linear algebra, homework 1

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Problem 1. (Problem 6, Chapter 1, Axler)
Example of a nonempty subset $U$ of $\mathbb{R}^2$ such that $U$ is closed under addition and under taking additive inverses but $U$ is not a subspace of $\mathbb{R}^2$.

Proof. Consider the subset $\mathbb{Z}^2$. It is closed under addition; however, it is not closed under scalar multiplication. For example $\sqrt{2}(1,1) = (\sqrt{2},\sqrt{2}) \notin \mathbb{Z}^2$. \hfill $\square$

Problem 2. (Problem 7, Chapter 1, Axler)
Example of a nonempty subset $U$ of $\mathbb{R}^2$ such that $U$ is closed under scalar multiplication but $U$ is not a subspace of $\mathbb{R}^2$.

Proof. Consider $A = \{(x,y) : x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0\}$. In words, $A$ is the union of the first and third quadrants of the plane. Then $A$ is closed under scalar multiplication; however, $(1,0)$ and $(0,-1)$ both belong to $A$, but their sum $(1,-1)$ does not. \hfill $\square$

Problem 3. (Problem 13, Chapter 1, Axler)
Prove or give a counterexample: $U_1, U_2, W$ subspaces of $V$ such that $U_1 + W = U_2 + W$ then $U_1 = U_2$.

Proof. This is false. Let $V = \mathbb{R}^2$ and let $U_1, U_2, W$ be the spans of $(0,1), (1,1), (1,0)$, respectively. Geometrically, $U_1, U_2, W$ are three mutually distinct lines. Then $U_1 + W = U_2 + W = \mathbb{R}^2$ but $U_1 \neq U_2$. \hfill $\square$

Problem 4. (Problem 15, Chapter 1, Axler)
Prove or give a counterexample: If $U_1, U_2, W$ subspaces of $V$ such that $U_1 \oplus W = U_2 \oplus W = V$ then $U_1 = U_2$.

Remark. As commented in class, Axler’s use of $\oplus$ corresponded to what I called “internal direct sum” and denoted by $\oplus_{int}$ in lectures.

Proof. Again, the assertion is false, and the same counterexample works. Let $V = \mathbb{R}^2$ and, as before $U_1 = \text{span}\{(0,1)\}, U_2 = \text{span}\{(1,1)\}, W = \text{span}\{(1,0)\}$. As before, $U_1 + W = U_2 + W = \mathbb{R}^2$. Moreover, $U_1 \cap W = \{0\}, U_2 \cap W = \{0\}$, so, by a theorem proved in class, $\mathbb{R}^2 = U_1 \oplus_{int} W$ and $\mathbb{R}^2 = U_2 \oplus_{int} W$. However, $U_1 \neq U_2$. \hfill $\square$
Problem 5. (Problem 1, Chapter 2, Axler). Prove that if \((v_1, ..., v_n)\) spans \(V\) then so does \((v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)\).

Proof. Write \(y_i = v_i - v_{i+1}\) for \(1 \leq i \leq n - 1\), \(y_n = v_n\). Thus we wish to prove that \(\text{span}(v_1, ..., v_n) = \text{span}(y_1, ..., y_n)\).

Since every \(y_i\) is a linear combination of the \(v_i\), every expression \(\sum \alpha_i y_i\) may be rewritten as a sum \(\sum \beta_i v_i\). Thus \(\text{span}(y_i) \subseteq \text{span}(v_i)\).

On the other hand, we may write \(v_n = y_n, v_{n-1} = y_{n-1} + y_n, \ldots, v_1 = y_1 + \cdots + y_n\). Therefore, every \(v_i\) is a linear combination of the \(y_i\). Thus the reverse inclusion \(\text{span}(v_i) \subseteq \text{span}(y_i)\).

Problem 6. (Problem 6, Chapter 2, Axler). Prove that the real vector space consisting of all continuous real valued functions on the interval \([0, 1]\) is infinite-dimensional.

Proof. Suppose, to the contrary, that it were finite-dimensional. Let \(N\) be its dimension. The \(N + 1\) vectors \(1, x, x^2, ..., x^N\) then cannot be linearly independent, for we know that any linearly independent list in an \(N\)-dimensional space has size \(\leq N\). Thus, there must exist constants \(a_i \in \mathbb{R}\) \((0 \leq i \leq N)\), not all zero, so that:

\[
\sum a_i x^i = 0, \quad x \in [0, 1]
\]

This is ridiculous; a nonzero polynomial of degree \(N\) cannot equal zero for all \(x \in [0, 1]\) (you can see this, for example, by successively differentiating.)

Problem 7. Let \(V\) be a vector space over the field \(\mathbb{R}\) of real numbers. Prove that \(V\) is not equal to the union of a finite number of proper subspaces.

Proof. Comment. If \(V\) is finite dimensional and you are willing to use some measure theory, the problem is easy! Why? The following proof is as suggested in Katznelson’s book. The idea is that a “generic” line will intersect any subspace in at most one point (geometrically obvious in \(\mathbb{R}^3\)). However, it is a bit tricky to arrange a “generic” line in the present context.

Suppose \(V = V_1 \cup ... \cup V_n\); we choose such a union with \(n\) as small as possible. There exists \(x \in V_1\) but not in any other \(V_j, j > 1\), for otherwise we could discard \(V_1\), contradicting minimality of \(n\). Choose any \(y \notin V_1\) and consider

\[
\ell = \{y + \lambda x : \lambda \in \mathbb{R}\}.
\]

Geometrically, \(\ell\) is a line not through the origin.

Then \(\ell \cap V_j\) consists of at most one point for every \(j > 1\); if it contained two points, then there would exist \(\lambda \neq \lambda'\) so that \((\lambda - \lambda')x \in V_j\), contradicting \(x \notin V_j\). Also, by similar reasoning, \(\ell \cap V_1\) is empty. So \(\ell\) intersects \(\bigcup_{j=1}^n V_j\) in \(\leq n - 1\) points. Contradiction, for \(\ell\) is infinite.
Comment. The proof works so long when $V$ is a vector space over any field $F$ with $|F| \geq n$. Can you do any better?

Problem 8. Let $A$ be a list of vectors in a vector space. Show that $\text{span}(A)$ is the intersection of all subspaces containing $A$.

Proof. Let $B = \text{span}(A)$ and let $C$ be the intersection of all subspaces containing $A$. We will show $B = C$ by establishing separately the inclusions $B \subseteq C$ and $C \subseteq B$.

$B$ itself is a subspace, containing $A$, thus $C \subseteq B$.

Conversely, if $D$ is any subspace containing $A$, it has to contain the span of $A$, because $D$ is closed under the vector space operations. Thus $B \subseteq D$. Thus also $B \subseteq C$.

Problem 9. Can $V$ be a union of 3 proper subspaces? (Extra credit).

Proof. YES: Let $V$ be the vector space $\mathbb{F}_2^2$, where $\mathbb{F}_2$ is the finite field of size 2. It has three nonzero vectors, call them $v_1, v_2, v_3$. Then, for each $1 \leq i \leq 3$, $V_i = \{0, v_i\}$ is a proper subspace: it is closed under scalar multiplication, since $\mathbb{F}_2 = \{0, 1\}$, and it is closed under addition since $v_i + v_i = (1 + 1)v_i = 0v_i = 0$. Thus, $V = V_1 \cup V_2 \cup V_3$ gives an example.