Solutions to linear algebra, homework 1

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Problem 1. (Problem 6, Chapter 1, Axler)
Example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses but U is not a subspace of $\mathbb{R}^2$.

Proof. Consider the subset $\mathbb{Z}^2$. It is closed under addition; however, it is not closed under scalar multiplication. For example $\sqrt{2}(1,1) = (\sqrt{2}, \sqrt{2}) \notin \mathbb{Z}^2$. $\square$

Problem 2. (Problem 7, Chapter 1, Axler)
Example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication but U is not a subspace of $\mathbb{R}^2$.

Proof. Consider $A = \{(x, y) : x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0\}$. In words, A is the union of the first and third quadrants of the plane. Then A is closed under scalar multiplication; however, $(1, 0)$ and $(0, -1)$ both belong to A, but their sum $(1, -1)$ does not. $\square$

Problem 3. (Problem 13, Chapter 1, Axler)
Prove or give a counterexample: U_1, U_2, W subspaces of V such that $U_1 + W = U_2 + W$ then $U_1 = U_2$

Proof. This is false. Let $V = \mathbb{R}^2$ and let U_1, U_2, W be the spans of $(0, 1), (1, 1), (1, 0)$, respectively. Geometrically, U_1, U_2, W are three mutually distinct lines. Then $U_1 + W = U_2 + W = \mathbb{R}^2$ but $U_1 \neq U_2$. $\square$

Problem 4. (Problem 15, Chapter 1, Axler)
Prove or give a counterexample: If U_1, U_2, W subspaces of V such that $U_1 \oplus W = U_2 \oplus W = V$ then $U_1 = U_2$.

Remark. As commented in class, Axler's use of $\oplus$ corresponded to what I called "internal direct sum" and denoted by $\oplus_{int}$ in lectures.

Proof. Again, the assertion is false, and the same counterexample works. Let $V = \mathbb{R}^2$ and, as before $U_1 = \operatorname{span}\{(0, 1)\}, U_2 = \operatorname{span}\{(1, 1)\}, W = \operatorname{span}\{(1, 0)\}$. As before, $U_1 + W = U_2 + W = \mathbb{R}^2$. Moreover, $U_1 \cap W = \{0\}, U_2 \cap W = \{0\}$, so, by a theorem proved in class, $\mathbb{R}^2 = U_1 \oplus_{int} W$ and $\mathbb{R}^2 = U_2 \oplus_{int} W$. However, $U_1 \neq U_2$. $\square$

Problem 5. (Problem 1, Chapter 2, Axler). Prove that if $(v_1, ..., v_n)$ spans V then so does $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$.

Proof. Write $y_i = v_i - v_{i+1}$ for $1 \leq i \leq n-1$, $y_n = v_n$. Thus we wish to prove that $\text{span}(v_1, \ldots, v_n) = \text{span}(y_1, \ldots, y_n)$.

Since every y_i is a linear combination of the v_i, every expression $\sum \alpha_i y_i$ may be rewritten as a sum $\sum \beta_i v_i$. Thus $\text{span}(y_i) \subset \text{span}(v_i)$.

On the other hand, we may write $v_n = y_n, v_{n-1} = y_{n-1} + y_n, \ldots, v_1 = y_1 + \cdots + y_n$. Therefore, every v_i is a linear combination of the y_i. Thus the reverse inclusion $\text{span}(v_i) \subset \text{span}(y_i)$. □

Problem 6. (Problem 6, Chapter 2, Axler). Prove that the real vector space consisting of all continuous real valued functions on the interval $[0, 1]$ is infinite-dimensional.

Proof. Suppose, to the contrary, that it were finite-dimensional. Let N be its dimension. The $N + 1$ vectors $1, x, x^2, \ldots, x^N$ then *cannot* be linearly independent, for we know that any linearly independent list in an N-dimensional space has size $\leq N$. Thus, there must exist constants $a_i \in \mathbb{R}$ $(0 \leq i \leq N)$, *not all zero*, so that:

$$\sum a_i x^i = 0, \quad x \in [0, 1]$$

This is ridiculous; a nonzero polynomial of degree N cannot equal zero for all $x \in [0, 1]$ (you can see this, for example, by successively differentiating.) □

Problem 7. Let V be a vector space over the field $\mathbb{R}$ of real numbers. Prove that V is not equal to the union of a finite number of proper subspaces.

Proof. **Comment.** If V is finite dimensional and you are willing to use some measure theory, the problem is easy! Why? The following proof is as suggested in Katznelson's book. The idea is that a "generic" line will intersect any subspace in at most one point (geometrically obvious in $\mathbb{R}^3$). However, it is a bit tricky to arrange a "generic" line in the present context.

Suppose $V = V_1 \cup ... \cup V_n$; we choose such a union with n as small as possible. There exists $x \in V_1$ but not in any other V_j, $j > 1$, for otherwise we could discard V_1, contradicting minimality of n. Choose any $y \notin V_1$ and consider

$$\ell = \{y + \lambda x : \lambda \in \mathbb{R}\}.$$

Geometrically, ℓ is a line **not** through the origin.

Then $\ell \cap V_j$ consists of at most one point for every $j > 1$; if it contained two points, then there would exist $\lambda \neq \lambda'$ so that $(\lambda - \lambda')x \in V_j$, contradicting $x \notin V_j$. Also, by similar reasoning, $\ell \cap V_1$ is empty. So ℓ intersects $\bigcup_{j=1}^{n} V_j$ in $\leq n - 1$ points. Contradiction, for ℓ is infinite.

Comment. The proof works so long when V is a vector space over any field F with $|F| \geq n$. Can you do any better?

$\square$

Problem 8. Let A be a list of vectors in a vector space. Show that $\text{span}(A)$ is the intersection of all subspaces containing A.

Proof. Let $B = \text{span}(A)$ and let C be the intersection of all subspaces containing A. We will show $B = C$ by establishing separately the inclusions $B \subset C$ and $C \subset B$.

B itself is a subspace, containing A, thus $C \subseteq B$.

Conversely, if D is any subspace containing A, it has to contain the span of A, because D is closed under the vector space operations. Thus $B \subseteq D$. Thus also $B \subseteq C$. $\square$

Problem 9. Can V be a union of 3 proper subspaces ? (Extra credit).

Proof. YES: Let V be the vector space $\mathbb{F}_2^2$, where $\mathbb{F}_2$ is the finite field of size 2. It has three nonzero vectors, call them v_1, v_2, v_3. Then, for each $1 \leq i \leq 3$, $V_i = \{0, v_i\}$ is a proper subspace: it is closed under scalar multiplication, since $\mathbb{F}_2 = \{0, 1\}$, and it is closed under addition since $v_i + v_i = (1+1)v_i = 0v_i = 0$. Thus, $V = V_1 \cup V_2 \cup V_3$ gives an example. $\square$