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HW 2

§13.4

Problem 1. Ch 2 - ex 8

Find a basis for U, the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2; x_3 = 7x_4\}$$

Proof. Denote $u = (3, 1, 0, 0, 0)$, $v = (0, 0, 7, 1, 0)$, and $w = (0, 0, 0, 0, 1)$

u, v and w are linearly independent since

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0 \Rightarrow (3\lambda_1, \lambda_1, 7\lambda_2, \lambda_2, \lambda_3) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Note each u, v and w is in U. Thus U contains the span of $\{u, v, w\}$. At the same time, if $X \in U$, then

$$X = (x_1, x_2, x_3, x_4, x_5) = (3x_2, x_2, 7x_4, x_4, x_5)$$

$\Rightarrow X = x_2(3, 1, 0, 0, 0) + x_4(0, 0, 7, 1, 0) + x_5(0, 0, 0, 0, 1) = x_2 u + x_4 v + x_5 w$

Hence, $\{u, v, w\}$ spans U. They are linearly independent, hence form a basis for U.

□

Problem 2. Ch 2 - ex 11

Suppose V is finite dimensional and U is a subspace of V such that dim U = dim V. Prove that $U = V$

Proof. U has a basis of length dim$U$. Note this list contains vectors that are all linearly independent in U thus in V, and is of length dim V since dim$U$ = dim$V$. Thus by proposition 2.17 p 32, vectors in this list form a basis for V. So $U = V$.

□

Problem 3. Ch 2 - ex 13

Suppose U and W are subspaces of $\mathbb{R}_8$ such that

$$dimU = 3, dimW = 5, U + W = \mathbb{R}^8$$

Prove $U \cap W = \{0\}$

Proof. By theorem 2.18, we have

$$dimU + W = dimU + dimV - dimU \cap W$$

$$U + W = \mathbb{R}^8 \Rightarrow dimU + W = 8 \Rightarrow dimU \cap W = 0$$

Hence $U \cap W = \{0\}$

□

Problem 4. Ch 2 - ex 14

Suppose U and W are both five-dimensional subspaces of R^9. Prove that $U \cap W \neq 0$

Proof. By theorem 2.18, we have

$$dimU \cap W = dimU + dimV - dimU + W \geq dimU + dimV - dim\mathbb{R}^9 = 5 + 5 - 9 = 1$$

Hence $U \cap W \neq 0$

□

Problem 5. Ch 2 - ex 15

Prove / disprove the expression

Proof. Take $U_1 = span(1, 0), U_2 = span(0, 1), andU_3 = span(1, 1)$ $dim(U_1 + U_2 + U_3) = dim(\mathbb{R}^2) = 2; U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{0\}$

$$dimU_1 + dimU_2 + dimU_3 - dimU_1 \cap U_2 - dimU_1 \cap dimU_3 - dimU_2 \cap U_3 + dimU_1 \cap U_2 \ capU_3$$

$$= 1 + 1 + 1 - 0 - 0 - 0 + 0$$

If the two sides are equal, we would have : $2 = 3$. So the expression is not correct.

□

Problem 6. Ch 3 - ex 1

Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar.

Proof. Suppose V is a vector space of dimension 1 over the field F. Suppose T is a linear map from V to itself. We can pick a nonzero vector $v \in V$ so that $\forall x \in V \exists \lambda \in F : x = \lambda v$
$\Rightarrow T(v) = av$ for some $a \in F$ since LHS $\in V$. Then

$$\forall x \in V, T(x) = T(\lambda v) = \lambda T(v) = \lambda av = a\lambda v = ax$$

Problem 7. Ch 3 - ex 3

Suppose V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V.

Proof. Suppose U is a subspace of V and $T \in L(U, W)$. Denote $m = dimU, n = dimV$. Pick a basis say $v_1, ..., v_m$ for U and extend it to a basis for V, say $v_1, .., v_m, v_{m+1}, ..., v_n$. A linear transformation is uniquely defined by its value on a basis for V. Since we know $T(v_1), ..., T(v_m)$, we can randomly pick values for $T(v_{m+1}), ..., T(v_n)$, to turn T into an element of $L(V, W)$, for example $T(v_{m+1}) = ... = T(v_n) = 0$.

Problem 8. - Extra problem 1

Suppose that U, W are subspaces of a vector space V so that U intersects W only in the trivial vector. Suppose that $u_1, u_2, ...u_n$ is a linearly independent list in U, and $w_1, ..., w_m$ is a linearly independent list in W. Show that $u_1, u_2, ..., u_n, w_1, ..., w_m$ is linearly independent in V.

Proof. Denote F to be the scalar field. Suppose

$$a_1u_1 + a_2u_2 + ... + a_nu_n + b_1w_1 + ... + b_mw_m = 0$$

with $a_1, ..., a_n, b_1, ..., b_m \in F$

$$\Rightarrow a_1u_1 + a_2u_2 + ... + a_nu_n = -(b_1w_1 + ... + b_mw_m)$$

RHS is in U and LHS is in V. But $U \cap V = 0$, thus denote $x = a_1u_1 + a_2u_2 + ... + a_nu_n = -(b_1w_1 + ... + b_mw_m)$, then $x = 0$

$$\Rightarrow a_1u_1 + a_2u_2 + ... + a_nu_n = b_1w_1 + ... + b_mw_m = 0 \;\Rightarrow a_1 = ... = a_n = b_1 = ... = b_m = 0$$

since $u_1, u_2, ...u_n$ is a linearly independent list in U, and $w_1, ..., w_m$ is a linearly independent list in W. Thus $u_1, u_2, ..., u_n, w_1, ..., w_m$ is linearly independent in V.

Problem 9. - Extra problem 2

Suppose that V is a finite dimensional vector space. Show that every subspace W of V satisfies dimW $\leq$ dim(V), and that equality dim(W) = dim(V) holds only when $W = V$.

Proof. Since a basis of every subspace of V can be extended to a basis for V, and the length of a basis is the dimension of a vector space, dimW $\leq$ dim(V). dim(W) = dim(V) if and only if a basis for W does not need extending to get to a basis for V, ie that basis for W already span V ie $W = V$

Problem 10. - Extra problem 3

Let V be a finite-dimensional vector space over the complex numbers. Let V' be V but considered as a vector space over the real numbers. Show that the dimension of V' is twice the dimension of V.

Proof. Say $v_1, ..., v_n$ form a basis for V over $\mathbb{C}$ (*)

V is a finite-dimensional vector space over the complex numbers so $iv_1, ..., iv_n$ are elements in V. We will show $v_1, ..., v_n, iv_1, ..., iv_n$ form a basis for V' over $\mathbb{R}$, ie show they are linearly independent and span V' over $\mathbb{R}$. If $x\ inV'$, then $x\ inV$, because of (*), there exsist complex numbers $a_j + ib_j$, $a_j, b_j \in \mathbb{R}$ such that

$$x = (a_1 + ib_1)v_1 + ... + (a_n + ib_n)v_n$$

$$\Rightarrow x = a_1v_1 + b_1iv_1 + ... + a_nv_n + b_niv_n$$

Suppose $a_1v_1 + ... + a_nv_n + b_1iv_1 + ... + b_niv_n = 0$ with $a_j, b_j \in \mathbb{R}$.

Regrouping we get :

$$(a_1 + ib_1)v_1 + ... + (a_n + ib_n)v_n = 0$$

Because of (*), $(a_1 + ib_1) = ... = (a_n + ib_n) = 0 \Rightarrow a_j = b_j = 0$ for $1 \leq j \leq n$.