Problem 1 (Chapter 6 - ex 24). Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x) \, dx \quad (*)$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Proof. Use G-S on the basis $(1, x, x^2)$ to obtain an orthonormal basis denoted by $(1, p_1, p_2)$, then

$$q = \langle q, 1 \rangle 1 + \langle q, p_1 \rangle p_1 + \langle q, p_2 \rangle p_2$$

From the requirement, we have

$$\langle q, 1 \rangle = 1; \langle q, p_1 \rangle = p_1(\frac{1}{2}); \langle q, p_2 \rangle = p_2(\frac{1}{2})$$

Thus choose $q$ as

$$q = 1 + p_1(\frac{1}{2})p_1 + p_2(\frac{1}{2})p_2$$

We then have $(*)$ established for the basis elements. Since both sides are linear operators and they agree on the basis elements, they agree on all elements. Hence we have equality $(*)$ for each $p \in \mathcal{P}_2(\mathbb{R})$.

Problem 2 (Chapter 6 - ex 27). Suppose $n$ is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1})$$

Find a formula for $T^*(z_1, \ldots, z_n)$

Proof. Since the standard basis is an orthonormal basis for $\mathbb{F}^n$, we find $T^*$ by finding the matrix rep for $T$ with respect to the standard basis then take the conjugate transpose.
With respect to the standard basis, the matrix rep $M$ for $T$ is
\[
M = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

Hence $T^*(z_1, \ldots, z_n) = (z_2, \ldots, z_n, 0)$

**Problem 3** (Chapter 6 - ex 29). Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove $U$ is invariant under $T$ if and only if $U^\perp$ is invariant under $T^*$

**Proof.**

\[ \langle Tu, v \rangle = \langle u, T^*v \rangle \]  

Suppose $U$ is invariant under $T$. If $v \in U^\perp$ then by (*) $\langle u, T^*v \rangle = 0$ for all $u \in U$, hence $T^*v \in U^\perp$.

On the other hand, suppose $U^\perp$ is invariant under $T^*$. If $u \in U$ then by (*) $\langle Tu, v \rangle = 0$ for all $v \in U^\perp$, hence $Tu \in (U^\perp)^\perp = U$.

**Problem 4** (Chapter 6 - ex 30). Suppose $T \in \mathcal{L}(V, W)$. Prove that

1. $T$ is injective if and only if $T^*$ is surjective;
2. $T$ is surjective if and only if $T^*$ is injective.

**Proof.** $T : V \to W; T^* : W \to V$

1. We show that $(\text{Ker } T)^\perp = \text{Range } T^*$.

Since $\langle v, T^*w \rangle = \langle Tv, w \rangle$, $\langle v, T^*w \rangle = 0$ for all $v \in \text{Ker } T$ and $w \in W$, ie $\text{Range } T^* \subset (\text{Ker } T)^\perp$.

Also

\[
\dim \text{ Range } T^* = \dim \text{ Range } T = \dim V - \dim \ker T = \dim (\text{Ker } T)^\perp
\]

\[ \Rightarrow (\text{Ker } T)^\perp = \text{Range } T^* \]

Hence we have:

$T^*$ is surjective $\iff$ $\text{Range } T^* = V$ $\iff$ $(\text{Ker } T)^\perp = V$ $\iff$ $\text{Ker } T = 0$ $\iff$ $T$ is injective

2. Replace $T$ by $T^*$ we have $(\text{Ker } T^*)^\perp = \text{Range}(T^*)^* = \text{Range } T$

Thus

$T$ is surjective $\iff$ $\text{Range } T = V$ $\iff$ $(\text{Ker } T^*)^\perp = V$ $\iff$ $\text{Ker } T^* = 0$ $\iff$ $T^*$ is injective
Problem 5 (Chapter 7 - ex 1). Consider the inner product space $\mathcal{P}_2(\mathbb{R})$ with the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x) \, dx$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by

$$T(a_0 + a_1 x + a_2 x^2) = a_1 x$$

1. Show that $T$ is not self-adjoint.

2. Denote $M$ to be the matrix rep of $T$ with respect to the basis $(1, x, x^2)$ thus $M$ is

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

This matrix equals its conjugate transpose, even though $T$ is not self-adjoint. Explain why this is not a contradiction.

Proof.

1. With respect to an orthonormal basis, the matrix representation of $T^*$ is the conjugate transpose of that of $T$. By G-S process, we obtain an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$

$$\{1, p_2(x) = 2\sqrt{3}(x - \frac{1}{2}), p_3(x)\}$$

where $p_3$ is a quadratic polynomial. Since $T(1) = 0$, $T(p_2(x)) = 2\sqrt{3}x = p_2(x) - p_2(0)$, the matrix representation for $T$ is

$$
\begin{pmatrix}
0 & -p_2(0) & a_{13} \\
0 & 1 & a_{23} \\
0 & 0 & 0
\end{pmatrix}
$$

Hence the matrix representation for $T^*$ with respect to the same orthonormal basis is

$$
\begin{pmatrix}
0 & 0 & 0 \\
-p_2(0) & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

Since $p_2(0) \neq 0$, $T^*(1) \neq 0$ while $T(1) = 0$. Hence $T$ is not self-adjoint.

2. This is not a contradiction because basis $(1, x, x^2)$ is not orthonormal with respect to the given inner product. Thus the matrix representation of $T^*$ with respect to this basis is not necessarily the conjugate transpose of $M$. As we saw above, $T^*(1) \neq 0$, while $M^t(1) = 0$.
Problem 6 (Chapter 7 - ex 2). Prove or counterexample

Proof. Pick two self-adjoint operators that do not commute. In the case of matrices, any matrix whose columns form an orthonormal basis is self-adjoint.

Hence consider the following 2 matrices in $M_2(\mathbb{R})$.

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta \\
\end{pmatrix}
\]

When $\theta = \frac{\pi}{3}$, these two matrices do not commute since

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta \\
\end{pmatrix} = \begin{pmatrix}
\sin \theta & -\cos \theta \\
\cos \theta & \sin \theta \\
\end{pmatrix}
\]

while

\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta \\
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} = \begin{pmatrix}
\sin \theta & \cos \theta \\
-\cos \theta & \sin \theta \\
\end{pmatrix}
\]

Problem 7 (Chapter 7 - ex 3b). Show that if $V$ is a complex inner-product space, then the set of self-adjoint operators on $V$ is not a subspace of $\mathcal{L}(V)$.

Proof. Suppose $T$ is a self-adjoint operator on $V$. Pick an orthonormal basis for $V$.

Denote $M$ to the matrix rep for $T$ with respect to this basis. Since $T = T^*$, we have $M = M^t$.

Consider $iT$. Then with respect to this orthonormal basis, the matrix rep for $iT$ is $iM$.

Note that $iM^t = -M = -iM \neq iM$. Hence $iT$ is not self-adjoint. Thus the set of self-adjoint operators is not closed under scalar multiplication, hence not a subspace of $\mathcal{L}(V)$.

Problem 8 (Chapter 7 - ex 11). Suppose $V$ is a complex inner-product space. Prove that every normal operator on $V$ has a square root.

Proof. Suppose $T$ is a normal operator on $V$, ie $TT^* = T^*T$. By the complex Spectral Theorem, $V$ has an orthonormal basis consisting of eigenvectors of $T$. Hence with respect to this basis, the matrix rep for $T$ is

\[
\begin{pmatrix}
\begin{array}{ccccc}
0 & 0 & \ldots & 0 \\
0 & r_1 e^{i\theta_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & r_n e^{i\theta_n} \\
\end{array}
\end{pmatrix}
\]
with \( r_i \leq 0 \).

Define an operator \( S \) on \( V \) by specifying its matrix rep \( \tilde{M} \) with respect to this orthonormal basis by

\[
\tilde{M} = \begin{pmatrix}
\sqrt{r_1} e^{i \theta_1 / 2} & 0 & \ldots & 0 \\
0 & \sqrt{r_2} e^{i \theta_2 / 2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \sqrt{r_n} e^{i \theta_n / 2}
\end{pmatrix}
\]

Hence \( \tilde{M}^2 = M \). Thus \( S^2 = T \), ie \( S \) is a square root of \( T \).

\[\square\]

Problem 9 (Chapter 7 - Ex 14). Suppose \( T \in \mathcal{L}(V) \) is self-adjoint, \( \lambda \in \mathbb{F} \) and \( \epsilon > 0 \). Prove that if there exists \( v \in V \) such that \( \|v\| = 1 \), and \( \|Tv - \lambda v\| < \epsilon \), then \( T \) has an eigenvalue \( \lambda' \) such that \( |\lambda - \lambda'| < \epsilon \)

Proof. \( T \) is self-adjoint thus there exists an orthonormal basis containing eigenvectors for \( T \), say \((v_1, \ldots, v_n)\). Denote \( M \) to be the matrix rep for \( T \) with respect to this basis, thus \( M \) is a diagonal matrix.

Denote \( w = (a_1, \ldots, a_n) \in \mathbb{F}^n \), where \( a_i = \langle v, v_i \rangle \). Hence \( \|w\|_{\mathbb{F}^n} = 1 \) since

\[
1 = \|v\|^2 = \sum_{i=1}^{n} |\langle v, v_i \rangle|^2 = \sum_{i=1}^{n} |a_i|^2 = \|w\|^2_{\mathbb{F}^n}
\]

Also \( \|Tv - \lambda v\|_{V} = \|Mw - \lambda w\|_{\mathbb{F}^n} \)

Suppose all eigenvalues of \( T \) are such that \( |\lambda_i - \lambda| > \epsilon \) for all \( 1 \leq i \leq n \), then

\[
\|Mw - \lambda w\|^2_{\mathbb{F}^n} = \|((\lambda_1 - \lambda)a_1, \ldots, (\lambda_n - \lambda)a_n)\|^2_{\mathbb{F}^n} = \sum_{i=1}^{n} |\lambda_i - \lambda|^2 |a_i|^2
\]

With the assumption above,

\[
\|Tv - \lambda v\|^2_{V} = \|Mw - \lambda w\|^2_{\mathbb{F}^n} \geq \epsilon^2 \sum_{i=1}^{n} |a_i|^2 = \epsilon
\]

which contradicts the given hypothesis.

Hence \( T \) has an eigenvalue \( \lambda' \) such that \( |\lambda - \lambda'| < \epsilon \)

\[\square\]