Homework 8

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Problem 1 (Chapter 6 - ex 24)**.** Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)\,dx(*)$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Proof. Use G-S on the basis $(1, x, x^2)$ to obtain an orthonormal basis denoted by $(1, p_1, p_2)$, then

$$q = \langle q, 1\rangle 1 + \langle q, p_1\rangle p_1 + \langle q, p_2\rangle p_2$$

From the requirement, we have

$$\langle q, 1\rangle = 1; \langle q, p_1\rangle = p_1(\frac{1}{2}); \langle q, p_2\rangle = p_2(\frac{1}{2})$$

Thus choose q as

$$q = 1 + p_1(\frac{1}{2})p_1 + p_2(\frac{1}{2})p_2$$

We then have (*) established for the basis elements. Since both sides are linear operators and they agree on the basis elements, they agree on all elements. Hence we have equality (*) for each $p \in \mathcal{P}_2(\mathbb{R})$

□

Problem 2 (Chapter 6 - ex 27)**.** Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1})$$

Find a formular for $T^*(z_1, \ldots, z_n)$

Proof. Since the standard basis is an orthonormal basis for $\mathbb{F}^n$, we find T^* by finding the matrix rep for T with respect to the standard basis then take the conjugate tranpose. □

With respect to the standard basis, the matrix rep M for T is

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Hence $T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0)$

Problem 3 (Chapter 6 - ex 29)**.** Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove U is invariant under T if and only if $U^\perp$ is invariant under $T*$

Proof.

$$\langle Tu, v\rangle = \langle u, T^*v\rangle \, (*)$$

Suppose U is invariant under T. If $v \in U^\perp$ then by (*) $\langle u, T^*v\rangle = 0$ for all $u \in U$, hence $T^*v \in U^\perp$.

On the other hand, suppose $U^\perp$ is invariant under T^*. If $u \in U$ then by (*) $\langle Tu, v\rangle = 0$ for all $v \in U^\perp$, hence $Tu \in (U^\perp)^\perp = U$. □

Problem 4 (Chapter 6 - ex 30)**.** Suppose $T \in \mathcal{L}(V, W)$. Prove that

1. T is injective if and only if T^* is surjective;
2. T is surjective if and only if T^* is injective.

Proof. $T : V \to W; T^* : W \to V$

1. We show that $(\operatorname{Ker} T)^\perp = \operatorname{Range} T^*$.

   Since $\langle v, T^*w\rangle = \langle Tv, w\rangle$, $\langle v, T^*w\rangle = 0$ for all $v \in \operatorname{Ker} T$ and $w \in W$, ie $\operatorname{Range} T^* \subset (\operatorname{Ker} T)^\perp$.

   Also
   $$\dim \operatorname{Range} T^* = \dim \operatorname{Range} T = \dim V - \dim \ker T = \dim(\operatorname{Ker} T)^\perp$$
   $$\Rightarrow (\operatorname{Ker} T)^\perp = \operatorname{Range} T^*$$

   Hence we have :

   $$T^* \text{ is surjective} \Leftrightarrow \operatorname{Range} T^* = V \Leftrightarrow (\operatorname{Ker} T)^\perp = V \Leftrightarrow \operatorname{Ker} T = 0 \Leftrightarrow T \text{ is injective}$$

2. Replace T by T^* we have $(\operatorname{Ker} T^*)^\perp = \operatorname{Range}(T^*)^* = \operatorname{Range} T$

   Thus

   $$T \text{ is surjective} \Leftrightarrow \operatorname{Range} T = V \Leftrightarrow (\operatorname{Ker} T^*)^\perp = V \Leftrightarrow \operatorname{Ker} T^* = 0 \Leftrightarrow T^* \text{ is injective}$$

□

Problem 5 (Chapter 7 - ex 1)**.** Consider the inner product space $\mathcal{P}_2(\mathbb{R})$ with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by

$$T(a_0 + a_1x + a_2x^2) = a_1x$$

1. Show that T is not self-adjoint.
2. Denote M to be the matrix rep of T with respect to the basis $(1, x, x^2)$ thus M is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Proof.

1. With respect to an orthonormal basis, the matrix representation of T^* is the conjugate transpose of that of T. By G-S process, we obtain an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$

$$\{1, p_2(x) = 2\sqrt{3}(x - \frac{1}{2}), p_3(x)\}$$

   where p_3 is a quadratic polynomial. Since $T(1) = 0, T(p_2(x)) = 2\sqrt{3}x = p_2(x) - p_2(0)$, the matrix representation for T is

$$\begin{pmatrix} 0 & -p_2(0) & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

   Hence the matrix representation for T^* with respect to the same orthonormal basis is

$$\begin{pmatrix} 0 & 0 & 0 \\ -p_2(0) & 1 & 0 \\ a_{13} & a_{23} & 0 \end{pmatrix}$$

   Since $p_2(0) \neq 0$, $T^*(1) \neq 0$ while $T(1) = 0$.Hence T is not self-adjoint

2. This is not a contradiction because basis $(1, x, x^2)$ is not orthonormal with respect to the given inner product. Thus the matrix representation of T^* with respect to this basis is not necessarily the conjugate transpose of M. As we saw above, $T^*(1) \neq 0$, while $\overline{M}^t(1) = 0$.

□

Problem 6 (Chapter 7 - ex 2)**.** Prove or counterexample

Proof. Pick two self-adjoint operators that do not commute. In the case of matrices, any matrix whose columns form an orthonormal basis is self-adjoint.

Hence consider the following 2 matrices in $M_2(\mathbb{R})$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

When $\theta = \frac{\pi}{3}$, these two matrices do not commute since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

while

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

□

Problem 7 (Chapter 7 - ex 3b)**.** Show that if V is a complex inner-product space, then the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Suppose T is a self-adjoint operator on V. Pick an orthonormal basis for V.

Denote M to the matrix rep for T with respect to this basis . Since $T = T^*$, we have $M = \overline{M}^t$.

Consider iT. Then with respect to this orthonormal basis, the matrix rep for iT is iM.

Note that $\overline{iM}^t = -i\,\overline{M}^t = -iM \neq iM$. Hence iT is not self-adjoint. Thus the set of self-adjoint operators is not closed under scalar multiplication, hence not a subspace of $\mathcal{L}(V)$.

□

Problem 8 (Chapter 7 - ex 11)**.** Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root.

Proof. Suppose T is a normal operator on V, ie $TT^* = T^*T$. By the complex Spectral Theorem, V has an orthonormal basis consisting of eigenvectors of T. Hence with respect to this basis, the matrix rep for T is

$$\begin{pmatrix} r_1 e^{i\theta_1} & 0 & \dots & 0 \\ 0 & r_2 e^{i\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & r_n e^{i\theta_n} \end{pmatrix}$$

with $r_i \leq 0$.

Define an operator S on V by specifing its matrix rep $\tilde{M}$ with respect to this orthonormal basis by

$$\tilde{M} = \begin{pmatrix} \sqrt{r_1}e^{i\theta_1/2} & 0 & \dots & 0 \\ 0 & \sqrt{r_2}e^{i\theta_2/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \sqrt{r_n}e^{i\theta_n/2} \end{pmatrix}$$

Hence $\tilde{M}^2 = M$. Thus $S^2 = T$, ie S is a square root of T.

□

Problem 9 (Chapter 7 - Ex 14)**.** Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$ and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$, and $\|Tv - \lambda v\| < \epsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$

Proof. T is self-adjoint thus there exists an orthonormal basis containing eigenvectors for T, say $(v_1, \dots, v_n)$. Denote M to be the matrix rep for T with respect to this basis, thus M is a diagonal matrix.

Denote $w = (a_1, \dots, a_n) \in \mathbb{F}^n$, where $a_i = \langle v, v_i \rangle$. Hence $\|w\|_{\mathbb{F}^n} = 1$ since

$$1 = \|v\|^2 = \sum_{i=1}^{n} |\langle v, v_i \rangle|^2 = \sum_{i=1}^{n} |a_i|^2 = \|w\|^2_{\mathbb{F}^n}$$

Also $\|Tv - \lambda v\|_V = \|Mw - \lambda w\|_{\mathbb{F}^n}$

Suppose all eigenvalues of T are such that $|\lambda_i - \lambda| > \epsilon$ for all $1 \leq i \leq n$, then

$$\|Mw - \lambda w\|^2_{\mathbb{F}^n} = \|((\lambda_1 - \lambda)a_i, \dots, (\lambda_n - \lambda)a_n)\|^2_{\mathbb{F}^n} = \sum_{i=1}^{n} |\lambda_i - \lambda|^2 |a_i|^2$$

With the assumption above,

$$\|Tv - \lambda v\|^2_V = \|Mw - \lambda w\|^2_{\mathbb{F}^n} \geq \epsilon^2 \sum_{i=1}^{n} |a_i|^2 = \epsilon$$

which contradicts the given hypothesis.

Hence T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$

□