Math 113: Linear Algebra
Adjoints of Linear Transformations

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1 Recap

Last time, we discussed the Gram-Schmidt process.

For example, if we take V to be the space of polynomials of degree $\leq N$ from $[-1, 1]$ to $\mathbb{R}^n$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, and start with basis $e_1 = 1, e_2 = x, e_3 = x^2, \ldots$, then by the Gram-Scmidt process, we get the orthonormal basis

$$\begin{aligned} f_1 &= \frac{e_1}{\|e_1\|} \\ &= \frac{1}{\sqrt{2}} \\ f_2' &= e_2 - \langle e_2, f_1\rangle f_1 \\ f_2 &= \frac{f_2'}{\|f_2'\|} \\ &= x \\ f_3 &= \alpha(3x^2 - 1) \\ f_4 &= \beta(5x^3 - 3x^2) \end{aligned}$$

(where α and β are constants that scale the polynomials to the appropriate length). These f_i's are called Legendre polynomials. They arise often in studying systems with spherical symmetry. $f_i(\cos\theta)$ are examples of "spherical harmonics".

More generally, given any interval $[A, B]$ and a function $w(x) > 0$ for $x \in [A, B]$, we can apply the Gram-Schmidt process to polynomials on $[A, B]$ with respect to inner product

$$\langle f, g\rangle = \int_A^B w(x)f(x)g(x)dx.$$

This gives "orthogonal polynomials"; these arise very often (for various A, B, w) in math and physics.

2 The Adjoint of a Linear Transformation

We will now look at the adjoint (in the inner-product sense) for a linear transformation. A *self-adjoint* linear transformation has a basis of orthonormal eigenvectors $v_1, \ldots, v_n$.

Earlier, we defined for $T : V \to W$ the adjoint $\widehat{T} : W^* \to V^*$. If V and W are inner product spaces, we can "reinterpret" the adjoint as a map $T^* : W \to V$. The motivation for this construction is something like the following: Earlier we saw that a bilinear pairing $X \times Y \to F$ (where X, Y are vector spaces over F) induces maps $X \to Y^*$ and $Y \to X^*$. In the case that $F = \mathbb{R}$, then an inner product on V — which gives a bilinear map on $V \times V \to \mathbb{R}$ — gives an isomorphism of V and V^*. Roughly, an inner product gives a way to equate V and V^*.

Definition 1 (Adjoint)**.** If V and W are finite dimensional inner product spaces and $T : V \to W$ is a linear map, then the **adjoint** T^* is the linear transformation $T^* : W \to V$ satisfying for all $v \in V, w \in W$,

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle .$$

Lemma 2.1 (Representation Theorem)**.** *If V is a finite dimensional inner product space and $\ell : V \to F$ ($F = \mathbb{R}$ or $\mathbb{C}$) is a linear functional, then there exists a unique $w \in V$ so that $\ell(v) = \langle v, w\rangle$ for all $v \in V$.*

Proof. Proof left as an exercise (use an orthonormal basis). □

This theorem is called a "representational theorem" because it shows that you can represent a linear functional $\ell \in V^*$ by a vector $w \in V$.

Why does T^* (as in the definition of an adjoint) exist? For any $w \in W$, consider $\langle T(v), w\rangle$ as a function of $v \in V$. It is linear in v. By the lemma, there exists some $y \in V$ so that $\langle T(v), w\rangle = \langle v, y\rangle$. Now we define $T^*(w) = y$. This gives a function $W \to V$; we need only to check that it is linear.

Properties of T^*:

1. If e_1 is an orthonormal basis for V and f_j is an orthonormal basis for W, then the matrix of T with respect to e_i, f_j is the **conjugate transpose** of the matrix of T^* with respect to f_j, e_i.

   For example, if $V = \mathbb{C}^2$, $W = \mathbb{C}^2$, the inner product is $\langle(z_1, w_1), (z_2, w_2)\rangle = z_1\overline{z_2} + w_1\overline{w_2}$, and T is defined by $\begin{bmatrix} 1 & i \\ 0 & i \end{bmatrix}$, then T^* is defined by $\begin{bmatrix} 1 & 0 \\ -i & -i \end{bmatrix}$.

2. If $T_1, T_2 \in \mathcal{L}(V, W)$ and $\lambda_1, \lambda_2 \in F$, then

   $$(\lambda_1 T_1 + \lambda_2 T_2)^* = \overline{\lambda_1} T_1^* + \overline{\lambda_2} T_2^* .$$

3. $(T^*)^* = T$ (follows directly from the definition).

4. The range (image) of T is the perpendicular space of the nullspace of its adjoint, and vice-versa:

$$\begin{aligned} \text{image}(T) &= \text{null}(T^*)^\perp \\ \text{null}(T) &= \text{image}(T^*)^\perp \end{aligned}$$

and similarly exchanging T and T^*.

Most of the proofs are very similar to previous ones about $\widehat{T}$ (see Axler, end of Chapter 6). For example,

Proposition 1. $\text{image}(T) = \text{null}(T^*)^\perp$.

Proof. Note that both $\text{image}(T), \text{null}(T^*) \subseteq W$.

We will actually show that $\text{image}(T)^\perp = \text{null}(T^*)$; this implies the desired result, since $(\text{image}(T)^\perp)^\perp = \text{image}(T)$.

Recall that $w \in \text{image}(T)^\perp$ if and only if $\langle T(v), w \rangle = 0$ for all $v \in V$. By the definition of the adjoint, this is if and only if $\langle v, T^*(w) \rangle = 0$ for all $v \in V$, i.e. $T^*(w) = 0$, so $w \in \text{null}(T^*)$.

Note that at the last step, we used the fact that if $v_0 \in V$ satisfies $\langle v, v_0 \rangle = 0$ for all v, then $v_0 = 0$. Indeed, this implies that $\langle v_0, v_0 \rangle = 0$, and hence by an axiom of inner products, $v_0 = 0$. □

3 Self-Adjoint

Recall that we want:

Theorem 3.1. *If $T : V \to V$ (where V is a finite dimensional inner product space over F) so that $T = T^*$ ("self-adjoint"), then there is an orthonormal basis of eigenvectors and all eigenvalues are real.*

Proof. Why are all eigenvalues real? Given v an eigenvector with eigenvalue λ, i.e. $T(v) = \lambda v$, we can consider

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle T(v), v \rangle = \langle v, T(v) \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \langle v, v \rangle .$$

Note that $\langle T(v), v \rangle = \langle v, T(v) \rangle$ because T is self-adjoint. From the above, we see that $\lambda = \overline{\lambda}$, i.e. λ is real.

Now, let's prove the rest of the theorem for $F = \mathbb{C}$. We know that there exists an eigenvector $v \in V$. Let U be $\text{span}(v)^\perp$. Then, T preserves U: if $u \in U$, then $T(u) \in U$. Now, by induction on the dimension, we can find an orthonormal basis of eigenvectors on U. □

This is perhaps the most important result in the course.