# Math 113: Linear Algebra Norms and Inner Products 

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## 1 Norms and Inner Products

We want to have a precise way of describing the "size" of a vector in a vector space $V$ is over $\mathbb{R}$ or $\mathbb{C}$. For example, if we want to approximate $T \in \mathcal{L}(V, W)$ by a "simpler" one, then we need some way to describe " $a$ is close to $b$ ".

### 1.1 Norms

The most general way of describing "size" is via a norm.
Definition 1 (Norm). A norm is a function $N: V \rightarrow \mathbb{R}$ so that

1. $N(v) \geq 0$ with equality if and only if $v=0$.
2. $N(\lambda v)=|\lambda| N(v)$
3. $N(v+w) \leq N(v)+N(w)$

For example, if $v=\mathbb{R}^{2}, N_{1}(x, y)=\sqrt{x^{2}+y^{2}}$ (the usual length), $N_{2}(x, y)=|x|+|y|$, and $N_{3}=\max (|x|,|y|)$ are all norms. A norm gives a notion of "closeness": $v$ is "close" to $w$ when $N(v-w)$ is small. $N_{1}, N_{2}, N_{3}$ all give (slightly) different notions of "closeness".

### 1.2 Inner Products

Among all norms, there's one class-the inner product-which are easiest to work with; e.g. one can quantify the error in $N(v+w) \leq N(v)+N(w)$. The motivation here is to come up with a generalization of the dot product in $\mathbb{R}^{n}$, defined by

$$
\left(x_{1}, \ldots, x_{n}\right) \cdot\left(y_{1}, \ldots, y_{n}\right)=\sum_{i=1}^{n} x_{i} y_{i}
$$

Also, the length of $v$ is

$$
\|v\|=\sqrt{v \cdot v}
$$

which is a norm on $\mathbb{R}^{n}$.
Definition 2 (Inner Product). An inner product on a vector space $V$ over a field $F$ (which is either $\mathbb{R}$ or $\mathbb{C})$ is a function $V \times V \rightarrow F$, denoted $(v, w) \mapsto\langle v, w\rangle$, such that

Linear It is linear in the first variable: $\left\langle\lambda_{1} v_{1}+\lambda_{2} v_{2}, w\right\rangle=\lambda_{1}\left\langle v_{1}, w\right\rangle+\lambda_{2}\left\langle v_{2}, w\right\rangle$.
Positive $\langle v, v\rangle \in \mathbb{R}$, and $\langle v, v\rangle \geq 0$ with equality if and only if $v=0$.
Symmetric $\langle v, w\rangle=\overline{\langle w, v\rangle}$ (this notation denotes the conjugate)
In the case that $F=\mathbb{R}$, an inner product is a symmetric $(\langle v, w\rangle=\langle w, v\rangle)$ bilinear positive map $V \times V \rightarrow \mathbb{R}$.

In the case that $F=\mathbb{C}$, if $\langle v, w\rangle$ were bilinear, then for any $v \in V$ and any scalar $\lambda \in \mathbb{C}$, then we would have

$$
\langle\lambda v, \lambda v\rangle=\lambda^{2}\langle v, v\rangle
$$

Here there's no guarantee that $\lambda^{2} \in \mathbb{R}$, or $\lambda^{2} \geq 0$. On the other hand, as we actually defined the inner product,

$$
\langle\lambda v, \lambda v\rangle=\lambda \bar{\lambda}\langle v, v\rangle=\underbrace{|\lambda|^{2}}_{\text {nonnegative real }}\langle v, v\rangle .
$$

### 1.3 Examples

Suppose $V=\mathbb{R}^{n}$. We could then define an inner product as

$$
\left\langle\left(x_{1}, \ldots, x_{n}\right),\left(y_{1}, \ldots, y_{n}\right)\right\rangle=\sum x_{i} y_{i}
$$

Suppose $V=\mathbb{C}^{n}$. We could then define an inner product as

$$
\left\langle\left(x_{1}, \ldots, x_{n}\right),\left(y_{1}, \ldots, y_{n}\right)\right\rangle=\sum x_{i} \overline{y_{i}}
$$

so

$$
\left\langle\left(x_{1}, \ldots, x_{n}\right),\left(x_{1}, \ldots, x_{n}\right)\right\rangle=\sum\left|x_{i}\right|^{2}
$$

If $V$ is the space of continuous functions $[0,1] \rightarrow \mathbb{R}$ over $\mathbb{R}$, we can define an inner product

$$
\langle f, g\rangle=\int_{0}^{1} f(x) g(x) d x
$$

(check the axioms! For example, $\langle f, f\rangle=\int_{0}^{1} f(x)^{2} d x \geq 0$ ). If $V$ is the space of continuous functions $[0,1] \rightarrow \mathbb{C}$ over $\mathbb{C}$, we can define an inner product

$$
\langle f, g\rangle=\int_{0}^{1} f(x) \overline{g(x)} d x
$$

We will generally use the notation

$$
\|v\|=\sqrt{\langle v, v\rangle}
$$

to denote the length of $v$ according to a given inner product.

### 1.4 Like dot

The below is a sequence of facts that tell us essentially that inner products behave like the dot product.

Proposition 1 (Cauchy-Schwarz Inequality).

$$
|\langle v, w\rangle|^{2} \leq\langle v, v\rangle\langle w, w\rangle
$$

Proof. (when $F=\mathbb{R}$ )
We'll use the fact that $\|v+\lambda\|^{2} \geq 0$. If we expand this out, we find that

$$
\begin{aligned}
0 & \leq\|v+\lambda w\|^{2} & & \\
& =\langle v+\lambda w, v+\lambda w\rangle & & \\
& =\langle v, v\rangle+\lambda(\langle w, v\rangle+\langle v, w\rangle)+\lambda^{2}\langle w, w\rangle & & \text { by bilinearity } \\
& =\lambda^{2}\langle w, w\rangle+2 \lambda\langle v, w\rangle+\langle v, v\rangle & & \text { by symmetry. }
\end{aligned}
$$

Note that in general, if $a x^{2}+b x+c \geq 0$ for all real $x$, it cannot have two real roots. Therefore, $b^{2}-4 a c \leq 0$. Thus,

$$
\begin{aligned}
4\langle v, w\rangle^{2} & \leq 4\langle v, v\rangle\langle w, w\rangle \\
|\langle v, w\rangle|^{2} & \leq\langle v, v\rangle\langle w, w\rangle
\end{aligned}
$$

Exercise: repeat this proof for $F=\mathbb{C}$. You should get that the real part of the inner product is bounded by the lengths: $|\operatorname{Re}\langle v, w\rangle| \leq\|v\| \cdot\|w\|$. The exercise is to show that $|\langle v, w\rangle| \leq\|v\|\|w\|$.

Corollary 1.1. For all $v, w$,

$$
|\langle v, w\rangle| \leq\|v\|\|w\|
$$

Proposition 2 (Pythagoras's law). We say $v, w$ are are orthogonal (perpendicular) if $\langle v, w\rangle=$ 0 . If so, $\|v+w\|^{2}=\|v\|^{2}+\|w\|^{2}$.

Proof. Left as an exercise; use linearity properties of the inner product.
Proposition 3 (Parallelogram law). For any $v, w$,

$$
\|v+w\|^{2}+\|v-w\|^{2}=2\left(\|v\|^{2}+\|w\|^{2}\right)
$$

Proof. Left as an exercise; use linearity properties of the inner product.
Proposition 4 (Triangle inequality). For any $v, w$,

$$
\|v+w\| \leq\|v\|+\|w\|
$$

Proof.

$$
\begin{aligned}
\|v+w\|^{2} & =\langle v+w, v+w\rangle \\
& =\underbrace{\langle v, v\rangle}_{\|v\|^{2}}+\underbrace{\langle w, w\rangle}_{\|v\|^{2}}+\underbrace{\langle v, w\rangle}_{\leq\|v\|\|w\| \text { by Cauchy-Schwarz }}+\underbrace{\langle w, v\rangle}_{\leq\|v\|\|w\| \text { by Cauchy-Schwarz }} \\
& \leq\|v\|^{2}+\|w\|^{2}+2\|v\|\|w\| \\
& =(\|v\|+\|w\|)^{2} \\
\|v+w\| & \leq\|v\|+\|w\|
\end{aligned}
$$

Exercise: Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ with inner product. Find the maximum length of a list $v_{1}, \ldots, v_{k}, \ldots, v_{n}$ so that $\left\langle v_{i}, v_{j}\right\rangle \leq 0$ if $i \neq j$.

