# Math 113: Linear Algebra Norms and Inner Products

Ilya Sherman

November 5, 2008

# 1 Norms and Inner Products

We want to have a precise way of describing the "size" of a vector in a vector space V is over  $\mathbb{R}$  or  $\mathbb{C}$ . For example, if we want to *approximate*  $T \in \mathcal{L}(V, W)$  by a "simpler" one, then we need some way to describe "a is close to b".

#### 1.1 Norms

The most general way of describing "size" is via a norm.

**Definition 1** (Norm). A norm is a function  $N: V \to \mathbb{R}$  so that

1.  $N(v) \ge 0$  with equality if and only if v = 0.

2. 
$$N(\lambda v) = |\lambda| N(v)$$

3.  $N(v+w) \le N(v) + N(w)$ 

For example, if  $v = \mathbb{R}^2$ ,  $N_1(x, y) = \sqrt{x^2 + y^2}$  (the usual length),  $N_2(x, y) = |x| + |y|$ , and  $N_3 = \max(|x|, |y|)$  are all norms. A norm gives a notion of "closeness": v is "close" to w when N(v - w) is small.  $N_1, N_2, N_3$  all give (slightly) different notions of "closeness".

#### **1.2** Inner Products

Among all norms, there's one class—the inner product—which are easiest to work with; e.g. one can quantify the error in  $N(v+w) \leq N(v) + N(w)$ . The motivation here is to come up with a generalization of the dot product in  $\mathbb{R}^n$ ,  $\cdot$  defined by

$$(x_1,\ldots,x_n)\cdot(y_1,\ldots,y_n)=\sum_{i=1}^n x_iy_i.$$

November 5, 2008

Also, the length of v is

$$\|v\| = \sqrt{v \cdot v}$$

which is a norm on  $\mathbb{R}^n$ .

**Definition 2** (Inner Product). An **inner product** on a vector space V over a field F (which is either  $\mathbb{R}$  or  $\mathbb{C}$ ) is a function  $V \times V \to F$ , denoted  $(v, w) \mapsto \langle v, w \rangle$ , such that

**Linear** It is linear in the first variable:  $\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle.$ 

**Positive**  $\langle v, v \rangle \in \mathbb{R}$ , and  $\langle v, v \rangle \ge 0$  with equality if and only if v = 0.

Symmetric  $\langle v, w \rangle = \overline{\langle w, v \rangle}$  (this notation denotes the conjugate)

In the case that  $F = \mathbb{R}$ , an inner product is a symmetric  $(\langle v, w \rangle = \langle w, v \rangle)$  bilinear positive map  $V \times V \to \mathbb{R}$ .

In the case that  $F = \mathbb{C}$ , if  $\langle v, w \rangle$  were bilinear, then for any  $v \in V$  and any scalar  $\lambda \in \mathbb{C}$ , then we would have

$$\langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle.$$

Here there's no guarantee that  $\lambda^2 \in \mathbb{R}$ , or  $\lambda^2 \ge 0$ . On the other hand, as we actually defined the inner product,

$$\langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = \underbrace{|\lambda|^2}_{\text{nonnegative real}} \langle v, v \rangle.$$

### 1.3 Examples

Suppose  $V = \mathbb{R}^n$ . We could then define an inner product as

 $\langle (x_1,\ldots,x_n), (y_1,\ldots,y_n) \rangle = \sum x_i y_i.$ 

Suppose  $V = \mathbb{C}^n$ . We could then define an inner product as

$$\langle (x_1,\ldots,x_n), (y_1,\ldots,y_n) \rangle = \sum x_i \overline{y_i},$$

 $\mathbf{SO}$ 

$$\langle (x_1,\ldots,x_n), (x_1,\ldots,x_n) \rangle = \sum |x_i|^2$$

If V is the space of continuous functions  $[0,1] \to \mathbb{R}$  over  $\mathbb{R}$ , we can define an inner product

$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx.$$

(check the axioms! For example,  $\langle f, f \rangle = \int_0^1 f(x)^2 dx \ge 0$ ). If V is the space of continuous functions  $[0,1] \to \mathbb{C}$  over  $\mathbb{C}$ , we can define an inner product

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

We will generally use the notation

$$\|v\| = \sqrt{\langle v, v \rangle}$$

to denote the length of v according to a given inner product.

## 1.4 Like dot

The below is a sequence of facts that tell us essentially that inner products behave like the dot product.

Proposition 1 (Cauchy-Schwarz Inequality).

$$|\langle v, w \rangle|^2 \le \langle v, v \rangle \langle w, w \rangle.$$

*Proof.* (when  $F = \mathbb{R}$ )

We'll use the fact that  $||v + \lambda||^2 \ge 0$ . If we expand this out, we find that

$$\begin{split} 0 &\leq \|v + \lambda w\|^2 \\ &= \langle v + \lambda w, v + \lambda w \rangle \\ &= \langle v, v \rangle + \lambda (\langle w, v \rangle + \langle v, w \rangle) + \lambda^2 \langle w, w \rangle \qquad by \ bilinearity \\ &= \lambda^2 \langle w, w \rangle + 2\lambda \langle v, w \rangle + \langle v, v \rangle \qquad by \ symmetry. \end{split}$$

Note that in general, if  $ax^2 + bx + c \ge 0$  for all real x, it cannot have two real roots. Therefore,  $b^2 - 4ac \le 0$ . Thus,

$$4\langle v, w \rangle^2 \le 4\langle v, v \rangle \langle w, w \rangle$$
$$|\langle v, w \rangle|^2 \le \langle v, v \rangle \langle w, w \rangle \qquad \Box$$

Exercise: repeat this proof for  $F = \mathbb{C}$ . You should get that the real part of the inner product is bounded by the lengths:  $|\operatorname{Re}\langle v, w\rangle| \leq ||v|| \cdot ||w||$ . The exercise is to show that  $|\langle v, w\rangle| \leq ||v|| ||w||$ .

Corollary 1.1. For all v, w,

 $|\langle v, w \rangle| \le \|v\| \, \|w\| \, .$ 

**Proposition 2** (Pythagoras's law). We say v, w are are **orthogonal** (perpendicular) if  $\langle v, w \rangle = 0$ . If so,  $||v + w||^2 = ||v||^2 + ||w||^2$ .

Proof. Left as an exercise; use linearity properties of the inner product.

**Proposition 3** (Parallelogram law). For any v, w,

$$||v + w||^2 + ||v - w||^2 = 2(||v||^2 + ||w||^2)$$

Proof. Left as an exercise; use linearity properties of the inner product.

**Proposition 4** (Triangle inequality). For any v, w,

$$\|v + w\| \le \|v\| + \|w\|.$$

Proof.

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w \rangle \\ &= \underbrace{\langle v,v \rangle}_{\|v\|^2} + \underbrace{\langle w,w \rangle}_{\|v\|^2} + \underbrace{\langle v,w \rangle}_{\leq \|v\|\|w\|} \underbrace{\langle v,w \rangle}_{\text{by Cauchy-Schwarz}} + \underbrace{\langle w,v \rangle}_{\leq \|v\|\|w\|} \underbrace{\langle w,v \rangle}_{\text{by Cauchy-Schwarz}} \\ &\leq \|v\|^2 + \|w\|^2 + 2 \|v\| \|w\| \\ &= (\|v\| + \|w\|)^2 \\ \|v+w\| \leq \|v\| + \|w\| \end{aligned}$$

*Exercise:* Let V be a finite-dimensional vector space over  $\mathbb{R}$  with inner product. Find the maximum length of a list  $v_1, \ldots, v_k, \ldots, v_n$  so that  $\langle v_i, v_j \rangle \leq 0$  if  $i \neq j$ .