1 a) S_3 b) $\langle (12) \rangle$ in S_3 c) $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_4$, $\varphi(0) = 0$, $\varphi(1) = 2$. d) $\{e, (12)(34), (13)(24), (14)(23)\}$

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2. a) $x = 2$ is a generator: enough to prove order $(x) = 12$.
By Lagrange's theorem, if order $(x) \neq 12$, then order$(x) =$ 1, 2, 3, 4 or 6. But (with $x = 2$):

$$x^1 = 2, \quad x^2 = 4, \quad x^3 = 8, \quad x^4 = 3, \quad x^6 = 12$$

So order $(x) = 12$.

b) 6. [Reason: the homomorphism are exactly those given by $\varphi(x^i) = a^i$, $1 \leq i \leq 12$ for any $a \in \mathbb{Z}_6$, and there are 6 possibilities for a]

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3. a) Take $g \in G$, $g \neq e$. Then $\langle g \rangle$ is a subgroup of G and is not $\{e\}$, so $G = \langle g \rangle$. So G is cyclic.
Let N be the order of g. If N is not prime, $N = pq$ with $1 < p, q < N$. Then g^p (call it y) satisfies $y^q = e$ (since $y^q = g^{pq} = e$), and $y \neq e$.
So ~~any~~ y has order $\leq q$, so $\langle y \rangle$ is a subgroup of G besides $\{e\}$, strictly smaller than G. This is a contradiction — so N is prime.

b) - AB contains e: $e = \underbrace{e}_{\text{in } A} \cdot \underbrace{e}_{\text{in } B}$

- AB is closed under inversion: $(ab)^{-1} = b^{-1}a^{-1} = \underbrace{(b^{-1}a^{-1}b)}_{\text{in } A \text{, since } A \text{ is normal}} \underbrace{b}_{\text{in } B}$

- AB is closed under multiplication:

~~[illegible]~~

for $a, a' \in A$, $b, b' \in B$

$$(ab)(a'b') = \underbrace{a\,\underbrace{ba'b^{-1}}_{\substack{\text{in } A,\\ \text{since}\\ A \text{ is normal}}}}_{\text{in } A}\ \underbrace{bb'}_{\text{in } B} \in AB.$$

4 a) omitted

b) The "group generated by x, y" is the intersection of all subgroups containing x, y:

$$\underbrace{\langle x, y\rangle}_{\text{group generated by } x,y} = \bigcap_{\substack{H \text{ is a subgroup}\\ \text{of } G, \text{ such}\\ \text{that } x,y \in H}} H$$

if x, y are as in (a), then

order(x) divides order $\langle x, y\rangle$ and
order(y) divides order $\langle x, y\rangle$ } Lagrange's theorem.

So order $\langle x, y\rangle$ is divisible by 7 and 3,
so 21 divides order $(\langle x, y\rangle)$.

5 a) Euclidean algorithm:

$$\begin{aligned} 96 &= 2\cdot 37 + 22 \\ 37 &= 1\cdot 22 + 15 \\ 22 &= 1\cdot 15 + 7 \\ 15 &= 2\cdot 7 + 1 \end{aligned}$$

$$\begin{aligned} 1 &= 15 - 2\cdot 7 \\ &= 15 - 2(22-15) = 3\cdot 15 - 2\cdot 22 \\ &= 3\cdot(37-22) - 2\cdot 22 = 3\cdot 37 - 5\cdot 22 \\ &= 3\cdot 37 - 5(96 - 2\cdot 37) = 13\cdot 37 - 5\cdot 96 \end{aligned}$$

So $13\cdot 37 - 5\cdot 96 = 1$

So 13 is an inverse.

b) $x = 2^{13}$ will do: then $x^{37} = 2^{13\cdot 37} = 2^{1+5\cdot 96} = 2$ since $2^{96} = 1$ in G.

(In G): Now $2^6 = 64$, $2^7 = 31$, $2^8 = 62$, $2^9 = 27$, $2^{10} = 54$, $2^{11} = 11$
$2^{13} = \boxed{44}$

6 a) omitted ; subgroup is not normal

6) $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -b/a \\ 0 & 1 \end{bmatrix}$, so

$$(*) \quad \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{-1} & -b/a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} xa & xb+y \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x & ya^{-1}+(x-1)b/a \\ 0 & 1 \end{bmatrix}.$$

This shows that, for every $x \neq 1$,

$$C_x = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \mid y \in \mathbb{R} \right\} \text{ is}$$

a single conjugacy class. Indeed, our computation $(*)$, shows that if $g \in C_x$, then any conjugate of g lies in C_x, too.

Moreover, $(*)$ shows – by taking $b = \frac{-y}{(x-1)}$ – that any element of C_x is conjugate to $\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$, so any two element of C_x are conjugate.

For $x = 1$, we see similarly that

$$C_1 = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} / \, y \in \mathbb{R}, \, y \, \underline{\text{nonzero}} \right\}$$

is a conjugacy class. Finally, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a conjugacy class ; together with the C_x, this gives all conjugacy classes.

7a) Let Z be the center of G

Firstly Z is a subgroup:

- $e \in Z$, since $eg = ge = g$ for all $g \in G$
- if $x \in Z$, then $x^{-1} \in Z$, since for all $g \in G$

$$x^{-1}g = (g^{-1}x)^{-1} \underset{\substack{\uparrow \\ x \in Z}}{=} (xg^{-1})^{-1} = gx^{-1}.$$

- if $x, y \in Z$ then $xy \in Z$; for all $g \in G$

$$(xy)g = x(yg) \underset{\substack{\uparrow \\ y \in Z}}{=} x(gy) \underset{\substack{\uparrow \\ x \in Z}}{=} (gx)y = g(xy).$$

Then, Z is normal:

if $g \in G$ and $z \in Z$,

$$gzg^{-1} \underset{\substack{\uparrow \\ z \in Z}}{=} (zg)g^{-1} = z \in Z.$$

b) Center (D_n) is $\{e\}$ if n odd
and $\{e, \text{rotn by } 180^\circ\}$ if n even.

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8. omitted

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9 a) If A, B are groups, the <u>direct product group</u> $A \times B$ has as elements pairs (a, b) with $a \in A, b \in B$ and has group law

$$(a, b) \circ (a', b') = (aa', bb')$$

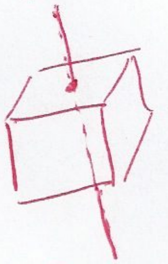
S_5 isn't isomorphic to $A_5 \times \mathbb{Z}_2$:
In $A_5 \times \mathbb{Z}_2$, the element $((12345), 1)$ has order 10. But S_5 has no elements of order 10,

9b) orbit $= T$, stabilizer $= \{e, (12), (34), (12)(34)\}$

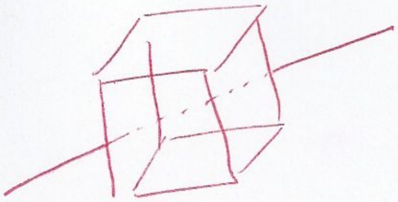
10. ⓐ They are:

$$\overbrace{\{e\}}^{1}, \quad \overbrace{\left\{\begin{array}{c}\text{rotns through}\\ \text{a face axis}\\ \text{by } 90^\circ\end{array}\right\}}^{6}, \quad \overbrace{\left\{\begin{array}{c}\text{rotns through}\\ \text{face axis}\\ \text{by } 180^\circ\end{array}\right\}}^{3}$$

$$\underbrace{\left\{\begin{array}{c}\text{rotns through}\\ \text{edge-axis}\\ \text{by } 180^\circ\end{array}\right\}}_{6}, \quad \underbrace{\left\{\begin{array}{c}\text{rotations through}\\ \text{vertex axis by } 120^\circ\\ \text{(diagonal)}\end{array}\right\}}_{8}$$



face-axis



edge-axis

ⓑ Recall we have proved that $G \cong S_4$ (via permuting diagonals) and S_4 is generated by transpositions. Under the isomorphism of S_4 with G, transpositions correspond to edge-axis rotns by 180°. These act on the eight vertices as a product of 4 transpositions, so have even sign.

Since G is generated by these, ~~are~~ all elements of G ~~has~~ act with even sign.