1. a) \( S_3 \)
   b) \( \langle(12)\rangle \) in \( S_3 \)

2. a) \( x = 2 \) is a generator; enough to prove \( \text{order}(x) = 12 \).
   b) \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \)
      \( \phi(0) = 0 \), \( \phi(1) = 2 \)

   By Lagrange's theorem, if \( \text{order}(x) \neq 12 \), then \( \text{order}(x) = 1, 2, 3, 4, 6, 12 \).
   So \( \text{order}(x) = 12 \).
   Then \( \phi(2) = \phi(1) = 2 \).

3. a) Take \( g \in G \), \( g \neq e \).
    b) Let \( N \) be the order of \( g \).
    c) \( \langle g \rangle \) is a subgroup of \( G \).
   d) \( g^k \) is the order of \( g \) for \( 1 \leq k \leq N \).
    e) \( g^k = e \) if and only if \( k \) is a multiple of \( N \).

4. a) \( \phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6 \)
    b) \( \phi(2) = \phi(1) = 2 \).
    c) \( \phi(3) = \phi(1) = 2 \).

5. a) \( \phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6 \)
    b) \( \phi(2) = \phi(1) = 2 \).
    c) \( \phi(3) = \phi(1) = 2 \).

6. a) \( \phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_6 \)
    b) \( \phi(2) = \phi(1) = 2 \).
    c) \( \phi(3) = \phi(1) = 2 \).
for $a, a' \in A$, $b, b' \in B$

$$(a \cdot b) (a' \cdot b') = \frac{a \cdot b'}{\text{in } A} \cdot \frac{b' \cdot a}{\text{in } B} \in AB$$

since $A$ is normal in $A$

4 a) omitted

6) The "group generated by $x, y$" is the intersection of all subgroups containing $x, y$:

$$\langle x, y \rangle = \bigcap H$$

$H$ is a subgroup of $G$, such that $x, y \in H$

if $x, y$ are as in (a), then

order ($x$) divides order ($\langle x, y \rangle$) and $|$ Lagrange's theorem.

So order ($\langle x, y \rangle$) is divisible by 7 and 3,
so 21 divides order ($\langle x, y \rangle$).

5 a) Euclidean algorithm:

$96 = 2 \cdot 37 + 22$
$37 = 1 \cdot 22 + 15$
$22 = 1 \cdot 15 + 7$
$15 = 2 \cdot 7 + 1$

$1 = 15 - 2(7) = 15 - 2(22 - 15) = 3 \cdot 15 - 2 \cdot 22$
$= 3(37 - 22) - 2 \cdot 22 = 3 \cdot 37 - 5 \cdot 22$
$= 3 \cdot 37 - 5(96 - 2 \cdot 37) = 13 \cdot 37 - 5 \cdot 96$

So $13 \cdot 37 - 5 \cdot 96 = 1$

So 13 is an inverse.

6) $x = 2^{13}$ will do: then $x^{37} = 2^{13 \cdot 37} = 2^{1 + 5 \cdot 96} = 2$

since $2^6 = 1$ in $G$,

(In $G$): Now $6 = 64$, $7 = 31$, $8 = 62$, $10 = 27$, $11 = 54$; $2^{13} = 44$
6a) omitted; subgroup is not normal.

6) \[
\begin{bmatrix}
0 & 6 \\
1 & 0
\end{bmatrix}^{-1} = \begin{bmatrix}
a^{-1} & -6a \\
0 & 1
\end{bmatrix}, \text{ so}
\]

\[
(*) \quad \begin{bmatrix}
a & 6 \\
0 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
x & y \\
0 & 1
\end{bmatrix} \begin{bmatrix}
a & 6 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
a^{-1} & -6a \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
x & \frac{ya^{-1} + \frac{1}{2}a(2x)}{a} \\
0 & 1
\end{bmatrix}.
\]

This shows that, for every \( x \neq 1 \),

\[ C_x = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \mid y \in \mathbb{R} \right\} \]

is a single conjugacy class. Indeed, our computation shows that if \( g \in C_x \),
then any conjugate of \( g \) lies in \( C_x \), too.

Moreover, \( (*) \) shows — by taking

\[ 6 = \frac{-y}{(x-1)} \]

that any element of \( C_x \) is conjugate to \( \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \), so any two elements of \( C_x \) are conjugate.

For \( x = 1 \), we see similarly that

\[ C_1 = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \mid y \in \mathbb{R}, y \text{ nonzero} \right\} \]

is a conjugacy class. Finally, \( \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \)

is a conjugacy class; together with the \( C_x \),
this gives all conjugacy classes.
7a) Let \( Z \) be the center of \( G \).

Firstly, \( Z \) is a subgroup:
- \( e \in Z \), since \( eg = ge = g \) for all \( g \in G \);
- if \( x \in Z \), then \( x^{-1} \in Z \), since
  \[ x^{-1}g = (g^{-1}x)^{-1} = (xg^{-1})^{-1} = gx^{-1} \]
- if \( x, y \in Z \) then \( xy \in Z \) for all \( g \in G \)
  \[ (xy)g = x(yg) = x(gy) = (gx)y \]
  \[ y \in Z \quad x \in Z = g(xy). \]

Then, \( Z \) is normal:
- if \( g \in G \) and \( z \in Z \),
  \[ gzg^{-1} = (zg)g^{-1} = z \in Z, \]
  \[ z \in Z \]

6) Center \( (D_n) \) is \( \{e\} \) if \( n \) odd
   and \( \{e, \text{rotn by 180°}\} \) if \( n \) even.

8. omitted

9a) If \( A, B \) are groups, the direct product group \( A \times B \) has as elements pairs \( (a, b) \) with \( a \in A, b \in B \)
    and has group law:
    \[ (a, b) \circ (a', b') = (aa', bb') \]
$S_5$ isn't isomorphic to $A_5 \times \mathbb{Z}_2$:

In $A_5 \times \mathbb{Z}_2$, the element $(12345, 1)$ has order 10. But $S_5$ has no elements of order 10.

orbital = $T$, stabilizer = \{e, (12), (34), (2)(34)\}

10. @ They are: 6
   \{e\},
   \{rotations through a face axis by 90°\},
   \{rotations through face axis by 180°\},
   \{rotations through edge-axis by 180°\},
   \{rotations through vertex axis by 120° (diagonal)\}

Since $G$ is generated by these all elements, we have proved that $G \cong S_4$ (via permuting diagonals) and $S_4$ is generated by transpositions.

Under the isomorphism of $S_4$ with $G$, transpositions correspond to edge-axis rotations by 180°. These act on the eight vertices as a product of 4 transpositions, so have even sign.