1. Math 109: practice midterm

A group $(G, \circ)$ is *abelian* if $x \circ y = y \circ x$ for every $x, y \in G$.

(1) Let $G = \{0, \ldots, 14\}$ with group law addition modulo 15 (e.g. $7 +_{15} 11 = 1$).
(a) Describe all the subgroups of G. No proof is necessary.
(b) Prove that G is isomorphic to a subgroup of S_8.

(2) Let G, H be groups, and let $f : G \to H$ be a function.
(a) Explain what it means for f to be a homomorphism.
(b) Prove that, if f is a homomorphism, the set $\{x \in G : f(x) = e_H\}$ is a subgroup. Here e_H denotes the identity element of H.

(3) Let G be a group of order 100. Prove that "every element is a cube": for every element $g \in G$ there exists $x \in G$ such that $g = x^3$.

Hint: Try taking x to be a suitable power of g.

(4) (a) Prove that an abelian group cannot be isomorphic to a non-abelian group.
(b) Give an example of an abelian group G and a non-abelian group H which have the same order.

(5) Let $\alpha = (123)(456)$ and $\beta = (325)$, both considered as elements of S_6.
(a) Compute $\alpha\beta$ and $\beta\alpha$. You do not need to justify your answers.
(b) Explain briefly why α and β **do not** generate S_6.