*Revised version:* The first version had a few questions that were a bit too hard.
Notation: $\mathbf{Z}_m$ denotes the integers modulo $m$ under addition.

(1) Give examples of each of the following. No proof or justification is necessary:
  (a) A group which is not cyclic;
  (b) A subgroup which is not normal;
  (c) A homomorphism which is not an isomorphism;
  (d) A normal subgroup of $S_4$, besides $A_4$.

(2) Let $G = \{1, 2, \ldots, 12\}$ with group law multiplication modulo 13.
  (a) Prove that $G$ is cyclic, and find a generator.
  (b) How many homomorphisms from $G$ to the group $\mathbf{Z}_6$? No proof is necessary.

(3) (a) Suppose that $G$ is a finite group with the property that the only subgroups of $G$ are $\{e\}$ and $G$. Prove that $G$ is cyclic of prime order. (Hint: To prove that $G$ is cyclic, consider the subgroup generated by an element $g \in G$.)
  (b) Suppose $A$ is a normal subgroup and $B$ any subgroup. Prove that $AB = \{ab : a \in A, b \in B\}$ is a subgroup of $G$.

(4) Consider the permutations $x = (1234567)$ and $y = (124)(365)$, considered as elements of $S_7$.
  (a) Show that $yxy^{-1} = x^2$, that $x$ has order 7, and $y$ has order 3.
  (b) Define "the group generated by $x$ and $y$." Show that this group has order $\geq 21$.

(5) (a) Compute the inverse of 37 in the multiplicative group of integers modulo 96.
  (b) Let $G$ be the group of integers $\{1, 2, 3, \ldots, 96\}$ under multiplication modulo 97. Find $x \in G$ such that $x^{37} = 2$ in $G$.
  Clearly describe your method: you will get liberal partial credit even if there are computational errors.

(6) Consider the collection of all $2 \times 2$ matrices of the form $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a$ and $b$ are real numbers and $a \neq 0$.
  (a) Prove that $A$ forms a subgroup of $\mathrm{GL}_2$. Is this subgroup normal? (no proof is needed for the second question)
  (b) Describe the conjugacy classes of $A$.

(7) The center of a group $G$ is the set $\{z \in G : zg = gz\}$ for every $g \in G$.
  (a) Prove that the center of $G$ is a normal subgroup.
  (b) What is the center of $D_n$? (No proof is needed.)

(8) Let $\alpha = (142356)$ and $\beta = (765432)$, both considered as elements of $A_7$.
  (a) Compute $\alpha\beta$ and $\beta\alpha$.
  (b) Prove that $\alpha$ is conjugate to $\beta$ inside $A_7$.

(9) (a) Define the direct product of two groups. Prove that $S_5$ isn't isomorphic to $A_5 \times \mathbf{Z}/2$.

(b) Let $T$ be the set of transpositions in $S_4$, and let $S_4$ act on $T$ via the homomorphism $\varphi : S_4 \to \mathrm{Sym}(T)$ given by

$$\varphi(g)(t) = gtg^{-1},$$

where $g \in S_4$ and $t$ is a transposition. Find the orbit and stabilizer of (12).

(10) (a) Let $G$ be the group of rotations that preserve a cube. Describe the conjugacy classes of $G$.

(b) Each element of $g$ permutes the eight vertices of the cube. Prove that the sign of the resulting permutation is always 1.