Revised version: The first version had a few questions that were a bit too hard.

Notation: $\mathbb{Z}_m$ denotes the integers modulo $m$ under addition.

1. Give examples of each of the following. No proof or justification is necessary:
   (a) A group which is not cyclic;
   (b) A subgroup which is not normal;
   (c) A homomorphism which is not an isomorphism;
   (d) A normal subgroup of $S_4$, besides $A_4$.

2. Let $G = \{1, 2, \ldots, 12\}$ with group law multiplication modulo 13.
   (a) Prove that $G$ is cyclic, and find a generator.
   (b) How many homomorphisms from $G$ to the group $\mathbb{Z}_6$? No proof is necessary.

3. (a) Suppose that $G$ is a finite group with the property that the only subgroups of $G$ are \{e\} and $G$. Prove that $G$ is cyclic of prime order. (Hint: To prove that $G$ is cyclic, consider the subgroup generated by an element $g \in G$.)
   (b) Suppose $A$ is a normal subgroup and $B$ any subgroup. Prove that $AB = \{ab : a \in A, b \in B\}$ is a subgroup of $G$.

4. Consider the permutations $x = (1234567)$ and $y = (124)(365)$, considered as elements of $S_7$.
   (a) Show that $yxy^{-1} = x^2$, that $x$ has order 7, and $y$ has order 3.
   (b) Define “the group generated by $x$ and $y$.” Show that this group has order $\geq 21$.

5. (a) Compute the inverse of 37 in the multiplicative group of integers modulo 96.
   (b) Let $G$ be the group of integers $\{1, 2, 3, \ldots, 96\}$ under multiplication modulo 97. Find $x \in G$ such that $x^{37} = 2$ in $G$.
      Clearly describe your method: you will get liberal partial credit even if there are computational errors.

6. Consider the collection of all $2 \times 2$ matrices of the form $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a$ and $b$ are real numbers and $a \neq 0$.
   (a) Prove that $A$ forms a subgroup of $GL_2$. Is this subgroup normal? (no proof is needed for the second question)
   (b) Describe the conjugacy classes of $A$.

7. The center of a group $G$ is the set $\{z \in G : zg = gz\}$ for every $g \in G$.
   (a) Prove that the center of $G$ is a normal subgroup.
   (b) What is the center of $D_n$? (No proof is needed.)

8. Let $\alpha = (142356)$ and $\beta = (765432)$, both considered as elements of $A_7$.
   (a) Compute $\alpha \beta$ and $\beta \alpha$.
   (b) Prove that $\alpha$ is conjugate to $\beta$ inside $A_7$.

9. (a) Define the direct product of two groups. Prove that $S_5$ isn’t isomorphic to $A_5 \times \mathbb{Z}/2$.  

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(b) Let $T$ be the set of transpositions in $S_4$, and let $S_4$ act on $T$ via the homomorphism $\varphi : S_4 \to \text{Sym}(T)$ given by
\[ \varphi(g)(t) = gtg^{-1}, \]
where $g \in S_4$ and $t$ is a transposition. Find the orbit and stabilizer of $(12)$.

(10) (a) Let $G$ be the group of rotations that preserve a cube. Describe the conjugacy classes of $G$.
   
   (b) Each element of $g$ permutes the eight vertices of the cube. Prove that the sign of the resulting permutation is always $1$. 