Homework 8 solutions.

Problem 16.1. Which of the following define homomomorphisms from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$?

Answer. a) $f_1 : z \to z^*$

Yes, f_1 is a homomorphism. We have that z^* is the complex conjugate of z. If z_1, z_2 are two complex numbers, then $(z_1 z_2)^* = z_1^* z_2^*$. This is exactly the statement that $f_1(z_1 z_2) = f_1(z_1) f_1(z_2)$.

b) $f_2 : z \to z^2$

This is also a homomorphism. Note that if $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ then $(z_1 z_2)^2 = z_1^2 z_2^2$, which is exactly the statement that $f_2(z_1 z_2) = f_2(z_1) f_2(z_2)$.

c) $f_3 : z \to iz$

This is not a homomorphism. If $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ then $f_3(z_1 z_2) = iz_1 z_2$ while $f_3(z_1) f_3(z_2) = iz_1 iz_2 = -z_1 z_2$.

d) $f_4 : z \to |z|$

This is a homomorphism. If $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ then $|z_1 z_2| = |z_1||z_2|$, which is exactly the statement that $f_4(z_1 z_2) = f_4(z_1) f_4(z_2)$.

□

Problem 16.2. Do any of the following determine homomorphisms from $GL_n(\mathbb{C})$ to $GL_n(\mathbb{C})$?

Proof. a) $A \to A^t$

This is not a homomorphism because if $A, B \in GL_n(\mathbb{C})$, then $(AB)^t = B^t A^t$, and matrices in $GL_n(\mathbb{C})$ do not commute. So if we choose any two matrices for which $AB \neq BA$, we would get that $A^t B^t \neq B^t A^t$.

b) $A \to (A^{-1})^t$

This is a homomorphism because if $A, B \in GL_n(\mathbb{C})$, then $((AB)^{-1})^t = (B^{-1} A^{-1})^t = (A^{-1})^t (B^{-1})^t$.

c) $A \to A^2$

This is not a homomorphism because matrices don't commute. That is, if $A, B \in GL_n(\mathbb{C})$, then $(AB)^2 = ABAB$. This is the same as $A^2 B^2$ iff $AB = BA$. So for any choice of non-commuting A, B we see that the map fails to satisfy the homomorphism condition.

d) $A \to A^*$

This is a homomorphism. To see this, note that if a, b, c, d are four complex numbers, then $(ab + cd)^* = a^* b^* + c^* d^*$. Thus if $A, B \in GL_n(\mathbb{C})$, then $(AB)^* = A^* B^*$ because the entries of AB are just sums and multiples of the entries of A and B. And this is exactly the formula that means the map is a homomorphism.

□

Problem 16.8. Show that a function $\phi : G \to G'$ is a homomorphism if and only if $\{(g, \phi(g)) \mid g \in G\}$ is a subgroup of $G \times G'$.

Proof. Let G, G' be two groups. Suppose some function $\phi : G \to G'$ is a homomorphism, and consider the subset $H = \{(g, \phi(g)) \mid g \in G\}$ of $G \times G'$. We show that H is a subgroup by verifying the following properties.

Closed under multiplication: Suppose $(g, \phi(g))$ and $(h, \phi(h))$ are elements of H. Then

$$(g, \phi(g))(h, \phi(h)) = (gh, \phi(g)\phi(h))$$

But ϕ is a homomorphism, so $\phi(g)\phi(h) = \phi(gh)$. So in fact

$$(g, \phi(g))(h, \phi(h)) = (gh, \phi(gh))$$

Thus $(g, \phi(g))(h, \phi(h)) \in H$, so H is closed under multiplication.

Identity: Since ϕ is a homomorphism, $\phi(e) = e'$ where e is the identity of G and e' is the identity of G'. Thus (e, e') is in H. Since (e, e') is the identity in $G \times G'$, the identity is in H.

Inverses: Since ϕ is a homomorphism, $\phi(g^{-1}) = (\phi(g))^{-1}$. Thus if $(g, \phi(g)) \in H$ then $(g^{-1}, (\phi(g))^{-1}) \in H$. But $(g^{-1}, (\phi(g))^{-1})$ is the inverse of $(g, \phi(g))$ in $G \times G'$. So H has inverses.

Associativity: Since H is a subset of the group $G \times G'$, its elements are associative because elements of $G \times G'$ are associative.

Therefore if ϕ is a homomorphism then H is a group.

Now suppose $\phi : G \to G'$ is just some map for which the set $H = \{(g, \phi(g)) \mid g \in G\}$ is a subgroup of $G \times G'$. We need to show that ϕ is actually a homomorphism. To check this, we need to show that for any two elements $g, h \in G$ we have that $\phi(gh) = \phi(g)\phi(h)$.

So choose two elements g, h in G. From the way that H is defined, we know that $(g, \phi(g))$ and $(h, \phi(h))$ are in H. Since H is a group, we know that their product is also in H. So the element $(g, \phi(g))(h, \phi(h)) = (gh, \phi(g)\phi(h))$ is in H. But the only element of H with gh in the first coordinate is the element $(gh, \phi(gh))$. That is because each element of G appears as the first entry in exactly one element of H. So we must have that $(gh, \phi(gh)) = (gh, \phi(g)\phi(h))$. But that means exactly that $\phi(gh) = \phi(g)\phi(h)$. Therefore, if H is a subgroup of $G \times G'$ then ϕ is a homomorphism. So we are done. □

Problem 17.1. Let G be the subgroup of S_8 generated by $(123)(45)$ and (78). Then G acts as a group of permutations of the set $X = \{1, 2, \ldots, 8\}$. Calculate the orbit and stabilizer of every integer in X.

Proof. Note that the permutations $\alpha = (123)(45)$ and $\beta = (78)$ are disjoint. In fact, α permutes the numbers 1 though 5 amongst themselves, and β permutes the numbers 7 and 8. Since neither α nor β move the number 6, the orbit of 6 is just $\{6\}$. Since β takes 7 to 8 and 8 back to 7, the orbit of both 7 and 8 is the set $\{7, 8\}$. Finally, α is the product of two disjoint cycles. So the set $\{1, 2, 3\}$ is one orbit, and the set $\{4, 5\}$ is another orbit.

To compute stabilizers, note that since α and β are disjoint, every element of G can be written as a power of α times a power of β. An element $\alpha^n\beta^m$ leaves a number i fixed iff $\alpha^n(i) = i$ and $\beta^m(i) = i$. So every element of G leaves 6 fixed. Only powers of β leave the numbers 1 through 5 fixed, and only powers of α leave the numbers 6 and 7 fixed. So the stabilizer of the numbers 1 though 5 is the subgroup generated by β, the stabilizer of 6 is G and the stabilizer of the numbers 6 and 7 is the subgroup generated by α. □

Problem 17.2. The infinite dihedral group D_∞ acts on the real line in a natural way (see Chapter 5). Work out the orbit and the stabilizer of each of the points 1, $\frac{1}{2}$, $\frac{1}{3}$.

Answer. Recall that D_∞ is generated by the functions $t : \mathbb{R} \to \mathbb{R}$ s.t. $t(x) = x + 1$ and $s : \mathbb{R} \to \mathbb{R}$ s.t. $s(x) = -x$. Every element of D_∞ can be written as $s^i t^n$ where $i = 0, 1$ and $n \in \mathbb{Z}$. Recall further that $t^n(x) = x + n$ (for any $n \in \mathbb{Z}$, so n can be negative) and $st^n(x) = -x - n$. Translations of the form t^n move every number, so they cannot be in the stabilizer of any number. The reflection st^n moves every number except for the x that satisfies $x = st^n(x)$. That is, st^n moves every number except the x s.t. $x = -x - n$. We compute that this x is $-n/2$.

Orbit of 1: The set of all integers. Since elements of D_∞ send integers to integers, the orbit of 1 is at most the set of all integers. Since $t^n(1) = n+1$, all integers are in fact in the orbit of 1.

Stabilizer of 1: From the general discussion at the beginning of this solution, we see that the only elements that fix 1 are the identity e and the reflection st^{-2}. So the stabilizer of 1 is $< st^{-2} >= \{e, st^{-2}\}$.

Orbit of 1/2: All numbers of the form $n + 1/2$ for $n \in \mathbb{Z}$. To see this, note that $t^n(1/2) = 1/2 + n$ and $st^n(1/2) = -1/2 - n$. Both of these can be written as $m + 1/2$ for some integer m.

Stabilizer of 1/2: Again, referring to the paragraph at the beginning of the answer, we see that the only elements of D_∞ that fix 1/2 are the identity e and the reflection st^{-1}. So the stabilizer of 1/2 is the group $< s^{t-1} >= \{e, st^{-1}\}$.

Orbit of 1/3: All number of the form $n + 1/3$ and $n + 2/3$ for $n \in \mathbb{Z}$. To see this, note that $t^n(1/3) = 1/3 + n$ and $st^n(1/3) = -1/3 - n$ which can be rewritten as $st^n(1/3) = -n - 1 + 2/3$ where n and $-n - 1$ can be any integer.

Stabilizer of 1/3: The only element of D_∞ that fixes 1/3 is the identity e since 1/3 cannot be written as $n/2$ for any integer n. So the stabilizer of 1/3 is $\{e\}$.

□

Problem 17.3. Identify S_4 with the rotational symmetry group of a cube as in Chapter 8, and consider the action of A_4 on the set of vertices of the cube. Find the orbit and the stabilizer of each vertex.

Proof. Let G be the rotational symmetry group of a cube. We identify G with S_4 as follows. Given vertex v on a cube C, let v' be the vertex of C farthest away from v. Then the line segment with endpoints v and v' is called a principal diagonal. C has four such principal diagonals. Since the rotations of a cube preserve distance, a rotations r will send the farthest vertex from v to the farthest vertex from $r(v)$. So rotations permute the principal diagonals. Numbering the diagonals 1 through 4, we get that each permutation of the diagonals corresponds to a permutation of the numbers 1 through 4.

So we get the isomorphism

$$\phi : G \to S_4$$

that maps a rotation to the corresponding permutation of the numbers 1 through 4.

Let d_i be the i^{th} diagonal, and number its endpoints i and i', so that the vertices of the cube are either numbered i or i' for some $i = 1, 2, 3, 4$.

Now a rotation of a cube can have one of three types of axes of symmetry. There are axes that go through the centers of opposite faces of C, axes that go through the centers of opposite edges, and axes that go through the centers of opposite vertices. (Here, “opposite” means “farthest away”.)

Rotations through axes going through opposite faces correspond to powers of 4-cycles. A 4-cycle is not in A_4, but the square of a 4-cycle is a permutation of type (2,2), which is in A_4. Rotations through axes going through opposite edges correspond to 2-cycles. So these rotations do not get mapped to A_4. The rotations which axes that go through opposite vertices correspond to 3-cycles, which are in A_4.

So we just need to consider the action of two types of rotations. Type 1: rotations of 180^o about axes going through pairs of faces, and Type 2: rotations of 120^o and 240^o about axes going through vertices.

There are 3 axes that go through opposite faces. See Figure 1. Rotations of 180^o about these axes correspond to the elements $(12)(34), (13)(24)$ and $(14)(23)$. We see that $(12)(34)$ (which is rotation about the axes on the left) corresponds to the permutation $(12')(1'2)(34')(3'4)$ of the vertices. Next, $(13)(24)$ corresponds to the permutation $(13)(24)(1'3')(2'4')$. Finally, $(14)(23)$ corresponds to $(1'4)(14')(23')(2'3)$.

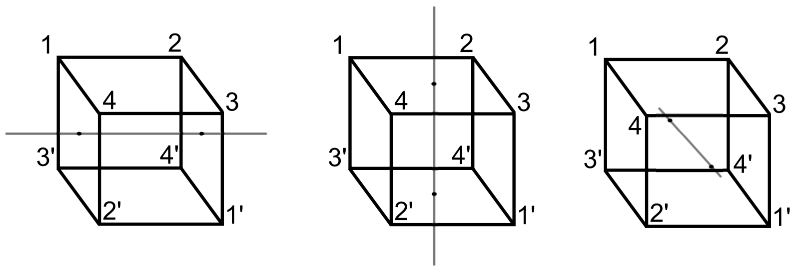

Figure 1. Three axes of type 1. Rotations by 180^o correspond to elements of A_4.

Next, there are 4 axes of type 2, and each one corresponds to a distinct 3-cycle in A_4. In the permutation group of the vertices, they correspond to products of 2 disjoint 3-cycles. For example, the 3-cycle (123) in A_4 corresponds to $(12'3)(1'23)$. The 3-cycle (132) is just the square of (123), so the corresponding element in the permutation group of the vertices is just the square of $(12'3)(1'23)$. Thus we only need to give the elements in the permutation group of the vertices that correspond to the 3-cycles (134), (234) and (124). The remaining 3-cycles are just the squares of these, so the corresponding permutations of the vertices will just be squares as well. The permutation corresponding to (134) is $(134')(1'34)$, the permutation corresponding to (234) is $(23'4)(2'34')$ and the permutation corresponding to (124) is $(1'24)(12'4')$. See Figure 2 below for an illustration of the axes of rotation of type 2.

□

Problem 17.4. Given the action of G on a set, show that every point of some orbit has the same stabilizer if and only if this stabilizer is a normal subgroup of G.

Proof. Suppose G acts on a set X. Let $O = \{x_1, \ldots, x_n\}$ be an orbit of the action. Suppose the stabilizer of each of the x_i is the same group H. We need to show that H is normal.

So let h be an element of H, and let g be any other element of G. Consider the action of ghg^{-1} on an element x_i. We know that $g^{-1} \cdot x_i$ is some other element

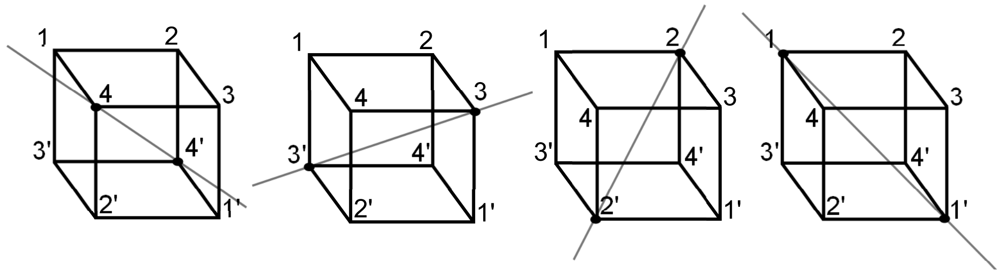

FIGURE 2. Four axes of type 2. Rotations by $120^o, 240^o$ correspond to elements of A_4.

x_j in O. Then h is in the stabilizer of every element of O, so $h \cdot x_j = x_j$. Finally, $g^{-1} \cdot x_j = x_i$ by the properties of an action. So $ghg^{-1} \cdot x_i = x_i$. Therefore if h is in the stabilizer of x_i, so is ghg^{-1}. Since the stabilizer of x_i is H, we have that $gHg^{-1} = H$, so H is normal.

Now suppose $O = \{x_1, \ldots, x_n\}$ is an orbit of the action of G on X. Let $S(x_i)$ be the stabilizer of x_i. We need to show that if $S(x_i)$ is normal for all i then all of the $S(x_i)$ are the same.

Let g_i be an element of G for which $g_i \cdot x_i = x_1$. There is such an element of g since x_1, x_i are in the same orbit. Suppose $h \in S(x_1)$. Then $g_i h g_i^{-1}$ is in $S(x_1)$ since $S(x_1)$ is normal. So

$$g_i h g_i^{-1} \cdot x_1 = x_1$$

But $g_i^{-1} \cdot x_1 = x_i$ by the properties of an action. So that formula really means that

$$g_i h \cdot x_i = x_1.$$

Multiplying both sides by g_i^{-1} we get that $h \cdot x_i = g_i^{-1} \cdot x_i$. But this just means that

$$h \cdot x_i = x_i$$

So h, which was in $S(x_1)$, is also an element of $S(x_i)$. Thus $S(x_1) \leq S(x_i)$. Since we only used that $S(x_1)$ is normal, and since $S(x_i)$ is also normal, the same reasoning also gives us that $S(x_i) \leq S(x_1)$. So all of the $S(x_i)$ are equal. □

Problem 17.5. If G acts on X and H acts on Y prove that $G \times H$ acts on $X \times Y$ via

$$(g, h)(x, y) = (g(x), h(y))$$

Check that the orbit of (x, y) is $G(x) \times H(y)$ and that its stabilizer is $G_x \times H_y$. We shall call this action the **product action** of $G \times H$ on $X \times Y$.

Proof. $G \times H$ acts on $X \times Y$ if for any two elements $(g, h), (g', h) \in G \times H$ and any element $(x, y) \in X \times Y$ we have that

$$\big((g, h)(g', h)\big) \cdot (x, y) = (g, h) \cdot \big((g', h) \cdot (x, y)\big)$$

and if the identity (e, f) acts on any $(x, y) \in X \times Y$ trivially, so $(e, f) \cdot (x, y) = (x, y)$.

The first condition holds because

$$\begin{aligned}\big((g,h)(g',h)\big)\cdot(x,y) &= (gg',hh')\cdot(x,y)\\ &= ((gg')\cdot x,(hh')\cdot y)\\ &= (g\cdot(g'\cdot x),h\cdot(h'\cdot y)) \text{ because } G,H \text{ act on } X,Y\\ &= (g,h)\cdot(g'\cdot x,h'\cdot y)\\ &= (g,h)\cdot\big((g',h)\cdot(x,y)\big)\end{aligned}$$

The second condition holds because the identity in $G\times H$ is (e,f) where e is the identity in G and f is the identity in H. So $(e,f)\cdot(x,y)=(e\cdot x,f\cdot y)$ and this is just (x,y) since G,H act on X,Y. Thus we have shown that $G\times H$ acts on $X\times Y$.

Let $G(x)$, $H(y)$ be the orbits of x,y in G,H, respectively. Then the orbit of (x,y) in $G\times H$ is the set $G\times H(x,y)=\{(g,h)\cdot(x,y)\}$. This is exactly the set $\{(g\cdot x,h\cdot y)\mid g\in G,h\in H\}$. Thus $G\times H(x,y)=G(x)\times H(y)$.

Let G_x,H_y be the stabilizers of x,y in G,H, respectively. Then the stabilizer in $G\times H$ of (x,y) is just the set $G\times H_{(x,y)}=\{(g,h)\mid(g,h)\cdot(x,y)=(x,y)\}$. This is exactly the set $\{(g,h)\mid(g\cdot x,h\cdot y)=(x,y)\}$. Note that $(g\cdot x,h\cdot y)=(x,y)$ iff $gcdotx=x$ and $h\cdot y=y$. Thus the stabilizer of (x,y) is just $G_x\times H_y$. □

Problem 17.10. Let x be an element of a group G. Show that the elements of G which commute with x form a subgroup of G. This subgroup is called the *centralizer* of x and written $C(x)$. Prove that the size of the conjugacy class of x is equal to the index of $C(x)$ in G. If some conjugacy class contains precisely two elements, show that G cannot be a simple group.

Proof. Let $C(x)$ denote the set of elements of G that commute with x. We first need to show that this is a group. The set is closed under multiplication because if g,h commute with x then $(gh)x=g(xh)=x(gh)$, so gh commutes with x. The identity is in $C(x)$ because the identity commutes with everything. If $hx=xh$, then multiplying both sides by h^{-1} on the right and on the left gives us that $xh^{-1}=h^{-1}x$, so $h^{-1}\in C(x)$. Lastly, $C(x)$ is associative because any subset of the group G is associative. Therefore $C(x)$ is a subgroup of G.

The index of $C(x)$ is the number of cosets of $C(x)$. Suppose $\{C(x),g_1C(x),\dots,g_nC(x)\}$ is the set of distinct cosets. Say $C(x)=g_0C(x)$ where g_0 is the identity. That way, all the cosets are of the form $g_iC(x)$ for some i. Now suppose g_i,g_i' are two elements of $g_iC(x)$. Then $g_i=g_i'h$ where $h\in C(x)$. So

$$\begin{aligned}g_ixg_i^{-1} &= (g_i'h)x(h^{-1}g_i^{-1})\\ &= g_i'xg_i'^{-1}\end{aligned}$$

since h commutes with x. So all elements of $g_iC(x)$ conjugate x to the same number. Thus x has at most as many elements in its conjugacy class as $C(x)$ has cosets.

We will show that each g_i conjugates x to a different number. Suppose not. That is, suppose $g_iC(x)\neq g_jC(x)$ but $g_ixg_i^{-1}=g_jxg_j^{-1}$. Then manipulating this equation gives us that $g_j^{-1}g_ix=xg_j^{-1}g_i$. But this means exactly that $g_j^{-1}g_i\in C(x)$. Recall from a previous problem set that in this case, $g_iC(x)=g_jC(x)$. This contradicts our assumption to the contrary, so we have that all elements of the form $g_ixg_i^{-1}$ are distint. Therefore, $C(x)$ has exactly as many cosets as x has elements in its conjugacy class. That means that the size of the conjugacy class of x is the index of its centralizer.

Suppose that there is an x whose conjugacy class has exactly two elements. That means the $C(x)$ has index 2. But all index 2 subgroups of a group are normal. So G has a proper normal subgroup (that is neither $\{e\}$ nor G.) That means G cannot be simple. $\square$